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DOUBLE BRANCHED COVERS AND PRETZEL KNOTS

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Given a knot K we describe a modification of K which leaves the double branched cover of S^3 branched along K unchanged. We then modify certain pretzel knots in this way to produce arbitrarily large families of distinct knots having the property that all of the associated double branched covers are homeomorphic.

1. Introduction. This paper concerns the relationship between a knot and its associated double branched cover. A brief review of the history of this problem will indicate some of the known results.

Given a tame knot K in S^3 the unique representation of $\Pi_1(S^3 - K)$ onto Z_2 yields a unique closed orientable 3-manifold $M(K)$ called the double branched cover of S^3 branched along K. By Waldhausen [**17**] K is trivial if and only if $M(K) \approx S^3$. This leads one to ask whether knot types and the homeomorphism types of their branched covers are in one-to-one correspondence. Birman and Hilden [**2**] give an affirmative answer in the case where the Heegaard genus of $M(K)$ is 2 and where homeomorphism type is replaced by Heegaard splitting class.

Unfortunately (or possibly fortunately) when the above restriction on genus is removed, counter-examples appear in abundance. Montesinos [**8**] and Viro [**16**] independently give examples of distinct composite links for which the double covers are homeomorphic. Birman, Gonzalez-Acuna and Montesinos [**3**] remove the restrictions "composite" and "link" by producing pairs of distinct prime knots such that for each pair the double covers are the same. These examples are also described in different ways by Takahashi [**14**] and Bedient [**1**]. Montesinos [**7**] has also given examples of arbitrarily large families of distinct *links* such that within each family all of the associated double covers are homeomorphic.

In this paper we will show that such families of *knots* exist. Boileau and Siebenmann [**4**] have obtained similar examples.

This result can be interpreted in a different manner as follows. We note that the manifold constructed is a Seifert fibered manifold which then admits n distinct involutions where distinct here means non-conjugate in the automorphism group. For more on this see Plotnick [**11**].

2. The modification. As a first step in the construction we exhibit a modification discussed in detail by Montesinos which may be made on any suitable knot (or link) which will not change the homeomorphism type of the associated double branched cover. It may or may not change the knot type.

Suppose that a knot K contains a section like the one shown in Figure 1 where A and B represent "rational tangles" as described by Conway [5]. If A and B are interchanged the resulting knot K' is shown in Figure 2. Montesinos [7] shows that $M(K) \approx M(K')$.

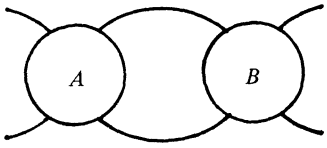

FIGURE 1

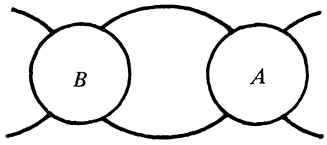

FIGURE 2

3. Pretzel knots. We now extend the technique to construct the examples promised in the introduction. We start with a generalization of the well-known pretzel knots.

We will denote the knot in Figure 3 by $K(p_1,\ldots,p_n)$ where the $p_i \in Z$ represent the number of half twists.

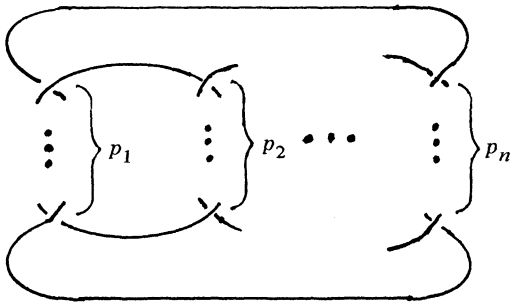

FIGURE 3

REMARK. If more than one of the p_i is even or if n is even and none of the p_i is even then K is a link. In all other cases however K is a knot. This does not affect the results which follow.

To construct the double branched cover of K, we apply the techniques of Montesinos [6] and [7], removing and replacing the balls $B_1, B_2, \ldots, B_{n+1}$ as shown in Figure 4. Note the special role played by the ball B_{n+1}. Let φ be any permutation of the numbers $1, 2, \ldots, n$. Then by the previous discussion,

$$M(K(p_1, \ldots, p_n)) \approx M(K(p_{\varphi(1)}, \ldots, p_{\varphi(n)}))$$

since we may interchange the first n balls at will. It is also clear that if φ is a cyclic permutation of $1, 2, \ldots, n$ or if φ simply reverses the order of $1, 2, \ldots, n$ or is any combination of these, then

$$K(p_1, \ldots, p_n) \sim K(p_{\varphi(1)}, \ldots, p_{\varphi(n)}).$$

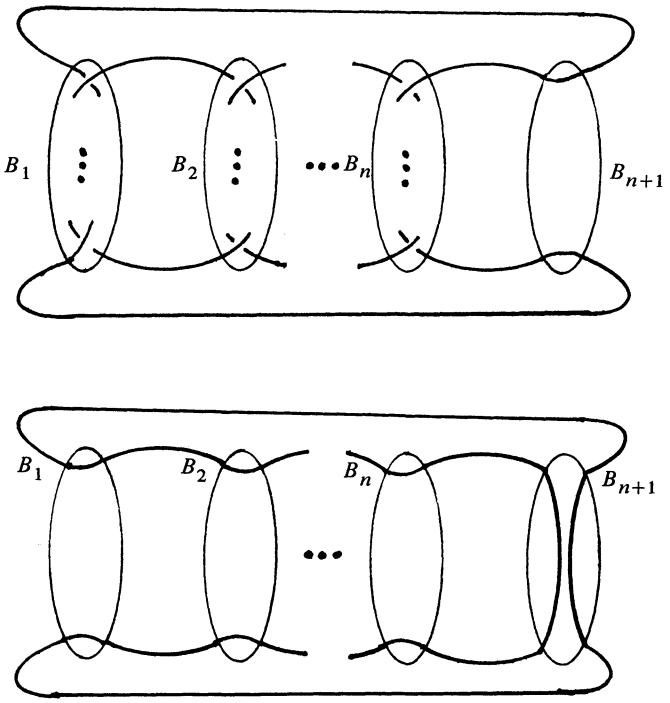

FIGURE 4

The above conditions turn out to be sufficient. We state this as the following theorem:

THEOREM. *Let $p_i > 1 \in Z$ such that the p_i are distinct, $i = 1, \ldots, n$, and such that $\Sigma 1/p_i < 1$. Let $K(p_1, \ldots, p_n)$ be as in Figure 6. Then*

$$M(K(p_1, \ldots, p_n)) \approx M(K(p_{\varphi(1)}, \ldots, p_{\varphi(n)}))$$

for any permutation φ *of* $1,\ldots,n$. *However,*

$$K(p_1,\ldots,p_n) \sim K(p_{\varphi(1)},\ldots,p_{\varphi(n)})$$

if and only if φ *is a cyclic permutation, an order reversing permutation or a combination of these.*

Before proceeding to the proof of the theorem, we will state and prove the following corollary.

COROLLARY. *There exist arbitrarily large families of distinct knots with homeomorphic double branched covers.*

Proof of Corollary. For any given n there are $n!$ distinct permutations of $1,\ldots,n$ and only $2n$ permutations which satisfy the conditions of the theorem. Thus, there are $n!/2n = (n-1)!/2$ distinct pretzel knots for each choice of n. □

Proof of Theorem. The first part of the theorem is clear from the discussion preceding the statement of the theorem. Thus we must show the equivalence of the conditions on the permutation φ and the isotopy between $K(p_1,\ldots,p_n)$ and $K(p_{\varphi(1)},\ldots,p_{\varphi(n)})$. We start with a method first used by Riedemeister [**12**]. Let $\Pi = \Pi_1(S^3 - K(p_1,\ldots,p_n))$. Next let H be the normal closure in Π of the square of any meridian (and hence all meridians). Form $G = \Pi/H$. Now let $F = [G, G]$ the commutator subgroup of G. Let U be the center of F and form $W = G/U$. Make this same construction for $K' = K(p_{\varphi(1)},\ldots,p_{\varphi(n)})$ to form Π', G', and W'. We want to show the relationship between the properties of φ and those of K and K'. This will be done in two steps, denoted Lemmas 1 and 2.

LEMMA 1. *If* $K \sim K'$ *then* $W \approx W'$.

LEMMA 2. $W \approx W'$ *if and only if* φ *has the properties noted in the theorem.*

Proof of Lemma 1. If $K \sim K'$ then there exists an isotopy of the pair (S^3, K) to the pair (S^3, K') which will induce an isomorphism from Π to Π'. We will see in what follows that the generators used to describe Π (respectively Π') are meridians. Geometrically these are curves which are null-homotopic in a tubular neighborhood of K (K'). All such curves represent conjugate elements of Π (Π'). Thus the elements whose squares generate H must map to the elements whose squares generate H'. It then follows that $G \approx G'$.

The commutator subgroup F of G and its center U must be carried to F' and U' respectively by any isomorphism of G to G'. Hence we have induced an isomorphism from W to W'. □

Proof of Lemma 2. We will compute nice presentations for W and W' and then show that they are recognizable as groups of isometries of the hyperbolic plane. Here we are following Trotter [**15**].

We start by finding $\Pi = \Pi_1(S^3 - K(p_1,\ldots,p_n))$. We will use the Wertinger generators shown in Figure 5. Let m_i be defined by $p_i = 2m_i + 1$, $i = 1,\ldots,n$. Each two stranded braid in Figure 5 can be used to write the generators x_i' in terms of the generators x_i and also to eliminate all other generators as follows: Π is generated by $x_1,\ldots,x_n$, $x_1',\ldots,x_n'$ with relations

$$x_1' = \left(x_n x_1^{-1}\right)^{m_n} x_n \left(x_1 x_2^{-1}\right)^{-m_n}$$

$$x_1' = \left(x_1 x_2^{-1}\right)^{m_1+1} x_2 \left(x_1 x_2^{-1}\right)^{-m_1-1}$$

$$x_2' = \left(x_1 x_2^{-1}\right)^{m_1} x_1 \left(x_1 x_2^{-1}\right)^{-m_1}$$

$$x_2' = \left(x_2 x_3^{-1}\right)^{m_2+1} x_3 \left(x_2 x_3^{-1}\right)^{-m_2-1}$$

and similar pairs for $x_3',\ldots,x_n'$. We may use these to eliminate the x_i' so that Π is now generated by $x_1,\ldots,x_n$ with n relations of the form

$$\left(x_n x_1^{-1}\right)^{m_n} x_n \left(x_n x_1^{-1}\right)^{-m_n} = \left(x_1 x_2^{-1}\right)^{m_1+1} x_2 \left(x_1 x_2^{-1}\right)^{-m_1-1}.$$

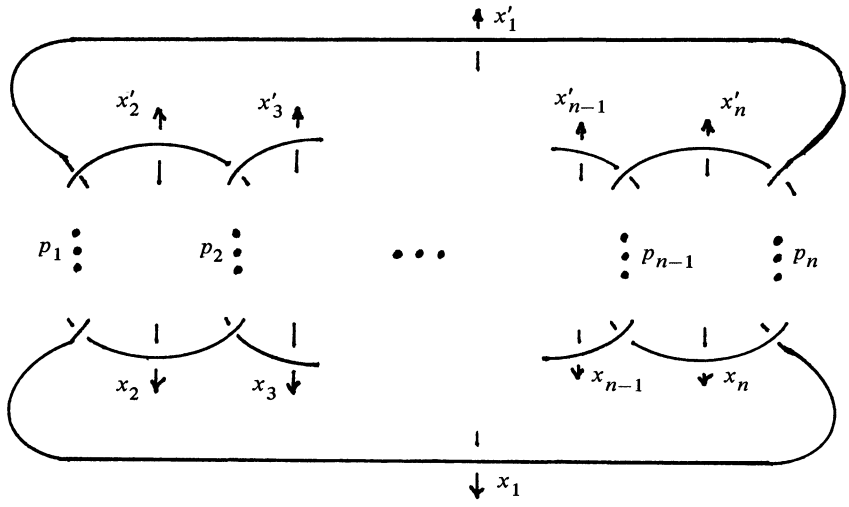

FIGURE 5

Recall that H is the normal subgroup of Π generated by the squares of the x_i. To form Π/H we just add the relations $x_i^2 = 1$, $i = 1, 2, \ldots, n$ to those of Π. Replacing x_i^{-1} by x_i and simplifying, the above relation becomes

$$(x_n x_1)^{p_n} = (x_1 x_2)^{p_1}.$$

We do this for each of the remaining relations to get

$$\Pi/H = G = \{x_1, \ldots, x_n | (x_1 x_2)^{p_1} = (x_2 x_3)^{p_2} = \cdots = (x_n x_1)^{p_n}, \\ x_1^2 = x_2^2 = \cdots = x_n^2 = 1\}.$$

Algebraic manipulations show that the commutator subgroup F of G is generated by $(x_1 x_2), (x_2 x_3), \ldots, (x_n x_1)$. Thus

$$F = \{(x_1 x_2), \ldots, (x_n x_1) | (x_1 x_2)^{p_1} = (x_2 x_3)^{p_2} = \cdots = (x_n x_1)^{p_n}, \\ x_1^2 = x_2^2 = \cdots = x_n^2 = 1\}.$$

The next step is to show that the subgroup U of F generated by $(x_1 x_2)^{p_1}$ is the center of F. The fact that U is contained in the center of F follows from the relations in F. To show that U is in fact the entire center of F we show that F/U is centerless. First note that

$$F/U = \{(x_1 x_2), \ldots, (x_n x_1) | (x_1 x_2)^{p_1} = (x_2 x_3)^{p_2} = \cdots = (x_n x_1)^{p_n} \\ = x_1^2 = x_2^2 = \cdots = x_n^2 = 1\}.$$

This however, is isomorphic to the group denoted by $Q/\langle h \rangle$ and shown to be centerless by Raymond and Orlik [**9**]. Finally we have

$$W = G/U = \{x_1, \ldots, x_n | (x_1 x_2)^{p_1} = (x_2 x_3)^{p_2} = \cdots = (x_n x_1)^{p_n} \\ = x_1^2 = x_2^2 = \cdots = x_n^2 = 1\}.$$

Now consider a permutation φ of $1, \ldots, n$ and construct $\Pi' = \Pi_1(S^3 - K(p_{\varphi(1)}, \ldots, p_{\varphi(n)}))$. We follow exactly the same procedure as above to get

$$W' = \{x_1, \ldots, x_n | (x_1 x_2)^{p_{\varphi(1)}} = (x_2 x_3)^{p_{\varphi(2)}} = \cdots = (x_n x_1)^{p_{\varphi(n)}} \\ = x_1^2 = x_2^2 = \cdots = x_n^2 = 1\}.$$

We then need to prove $W \not\approx W'$ except for the special classes of permutations allowed in the theorem. Still following Trotter we construct an n-gon P in the hyperbolic plane with vertex angles $\pi/p_1, \pi/p_2, \ldots, \pi/p_n$ as in Figure 6. Let y_i be reflection in the lines shown in Figure 6. Note that

$(y_i y_{i+1})$ is rotation through an angle $2\pi/p_i$ about the vertex with angle π/p_i. Thus, the map $x_i \to y_i$ induces an isomorphism of W to the full group of isometries of the plane with fundamental region P. Referring to Macbeath [**8**] we find that these are exactly the non-Euclidean crystallographic (NEC) groups which he classifies in Theorems 1 and 1a. Using his notation

$$W = (0, +, [\,], \{(p_1, \ldots, p_n)\})$$

and

$$W' = \left(0, +, [\,], \left\{\left(p_{\varphi(1)}, \ldots, p_{\varphi(n)}\right)\right\}\right).$$

Then by the theorems just mentioned $W \approx W'$ if and only if φ is a cyclic permutation, an order reversing permutation, or a combination of these. □

This together with Lemma 1 concludes the proof of the theorem. □

It seems clear from the nature of the construction as well as from the work of Montesinos that this phenomenon is fairly common. That is, if the balls which are being permuted are more complicated than those in the pretzel knots the same results would be expected.

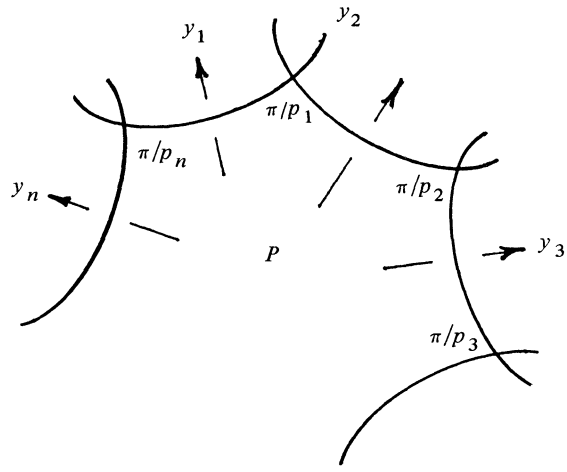

FIGURE 6

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