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ON THE DWORK TRACE FORMULA

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We prove a generalization of Dwork's trace formula for certain completely continuous operators on p-adic Banach spaces. This generalization makes it simpler to apply Dwork's theory to the study of certain exponential sums involving both additive and multiplicative characters. As an example, we treat the case of Gauss sums and give a new proof of the Gross-Koblitz formula.

0. Introduction. The Dwork Trace Formula is a basic tool for applying the techniques of p-adic analysis to the study of exponential sums with an additive character. Let p be a prime and let $\mathbf{F}_q$ be a finite field with $q = p^f$ elements. Let $\Psi\colon \mathbf{F}_q \to \mathbf{C}^\times$ be an additive character. For $f \in \mathbf{F}_q[x_1,\ldots,x_n]$, define an exponential sum

$$(0.1)\qquad S(f) = \sum_{x_1,\ldots,x_n \in \mathbf{F}_q} \Psi(f(x_1,\ldots,x_n)).$$

Bombieri [1] has used the Dwork Trace Formula to study such exponential sums and their associated L-functions. The purpose of this article is to prove a generalization of the Dwork Trace Formula (Theorem 1) which will allow one to treat in a straightforward manner sums of the form

$$(0.2)\qquad \sum_{x_1,\ldots,x_n \in \mathbf{F}_q} \chi_1(x_1)\cdots\chi_n(x_n)\Psi(f(x_1,\ldots,x_n)),$$

where $\chi_1,\ldots,\chi_n\colon \mathbf{F}_q^\times \to \mathbf{C}^\times$ are multiplicative characters. Such sums can be handled by the earlier trace formula at the expense of certain technical complications, i.e., change of variable in the polynomial f, which results in changes in the Frobenius operator and the differential operators with which Frobenius commutes (see for example [4, eqs. (6.47), (6.48), and (6.49)]). Our point here is that by enlarging the space on which Frobenius operates, one obtains the sums (0.1) and (0.2) from the same Frobenius operator, hence the commuting differential operators are unchanged also. This enables one to apply the other elements of Dwork's theory more directly.

As an example, in §2 we give another proof of the Gross-Koblitz formula. We follow the ideas of [2], although we simplify by avoiding any appeal to the dual theory. We hope that the ideas of this paper will lead to an interpretation of the Gauss sum relations of [**2**, §8, Remark 2] in terms of Dwork cohomology.

We use the standard notation for binomial-type coefficients: for n a non-negative integer,

$$(z)_n = \begin{cases} z(z+1)\cdots(z+n-1) & \text{if } n>0, \\ 1 & \text{if } n=0, \end{cases}$$

$$\binom{z}{n} = \begin{cases} z(z-1)\cdots(z-n+1)/n! & \text{if } n>0, \\ 1 & \text{if } n=0. \end{cases}$$

We denote by $\mathbf{C}_p$ a completion of an algebraic closure of the p-adic numbers $\mathbf{Q}_p$.

1. Trace formula. Let p be a prime and d a positive integer with $(p, d) = 1$. Let $\mathbf{Q}_p$ denote the p-adic numbers and let K be a discretely-valued extension field of $\mathbf{Q}_p$. We assume the valuation on K normalized so that ord $p = 1$, and we let $|\ |$ denote the corresponding absolute value. In this section we shall use multi-index notation: $i = (i_1, \ldots, i_m)$ and $j = (j_1, \ldots, j_n)$ are sequences of non-negative integers, and

$$x^{i/d}y^j = x_1^{i_1/d} \cdots x_m^{i_m/d} y_1^{j_1} \cdots y_n^{u_n}.$$

Let $\beta \in K$ and put $b = \operatorname{ord} \beta \in \mathbf{R}$. Let $L(b; d)$ denote the set of all formal series

$$\eta = \sum_{i,j>0} a(i, j) x^{i/d} y^j, \tag{1.1}$$

where $a(i, j) \in K$ satisfy

$$\operatorname{ord} a(i, j) - b((i_1 + \cdots + i_m)/d + j_1 + \cdots + j_n) \geq c \tag{1.2}$$

for some $c \in \mathbf{R}$ and all $i, j \geq 0$. We are treating $x^{i/d}$ as a formal expression only and hence do not regard η as a function. The vector space $L(b; d)$ is made into a Banach space by the following norm:

$$|\eta| = \sup_{i,j\geq 0} |a(i, j)| |\beta|^{-((i_1 + \cdots + i_m)/d + j_1 + \cdots + j_n)}. \tag{1.3}$$

This sup exists by (1.2).

Define an operator ψ by

$$\psi(\eta) = \sum_{i,j\geq 0} a(pi, pj) x^{i/d} y^j, \tag{1.4}$$

where η is as in (1.1). Note that ψ is a linear map of $L(b; d)$ into $L(pb; d)$.

Let $\delta = (\delta_1, \ldots, \delta_m)$ be an ordered m-tuple of integers with $0 \le \delta_1, \ldots, \delta_m \le d - 1$, and let $L(b; d, \delta)$ be the set of all $\eta \in L(b; d)$, η as in (1.1), satisfying $a(i, j) = 0$ unless

$$i_1 \equiv \delta_1 \pmod{d}, \ldots, i_m \equiv \delta_m \pmod{d}.$$

Then $L(b; d)$ decomposes as a direct sum of d^m subspaces

$$L(b; d) = \bigoplus_{\delta} L(b; d, \delta). \tag{1.5}$$

If we put for $k = 1, \ldots, m$,

$$\delta_k' = \text{least non-negative residue of } p\delta_k \text{ modulo } d,$$

then ψ maps $L(b; d, \delta')$ into $L(pb; d, \delta)$.

For f a positive integer, $q = p^f$, define $\psi_q = (\psi)^f$. Since $(d, p) = 1$, there exists f such that $d \mid (p^f - 1)$, in which case ψ_q maps $L(b; d, \delta)$ into $L(qb; d, \delta)$. For $F = \sum_{k,l \ge 0} A(k, l) x^k y^l \in L(b; d, 0)$, multiplication by F is stable on each $L(b; d, \delta)$. Note that if $\beta' \in K$ with $\operatorname{ord} \beta' = b' > b$, then $L(b'; d)$ is a subspace of $L(b; d)$, and the canonical injection i: $L(b'; d) \to L(b; d)$ is completely completely continuous ([**6**, §9]). Now suppose $b > 0$ and let α_F: $L(qb; d, \delta) \to L(qb; d, \delta)$ be the composition

$$L(qb; d, \delta) \xrightarrow{i} L(b; d, \delta) \xrightarrow{F} L(b; d, \delta) \xrightarrow{\psi_q} L(qb; d, \delta).$$

Then α_F is completely continuous ([**6**, §3]). By [**6**, §5], the trace $\operatorname{tr} \alpha_F$ and Fredholm determinant $\det(I - t\alpha_F)$ are well defined, and

$$\det(I - t\alpha_F) = \exp\left(- \sum_{r=1}^{\infty} \operatorname{tr}(\alpha_F)^r t^r / r \right) \tag{1.6}$$

is a p-adic entire function.

THEOREM 1.

$$\begin{aligned}(q-1)^{m+n} &\operatorname{tr}(\alpha_F \mid L(qb; d, \delta)) \\ &= \sum_{\substack{x^{q-1}=1 \\ y^{q-1}=1}} x_1^{-(q-1)\delta_1/d} \cdots x_m^{-(q-1)\delta_m/d} F(x_1, \ldots, x_m; y_1, \ldots, y_n).\end{aligned}$$

Proof. By [**6**, Prop. 7(a) and §9], the trace of α_F on $L(qb;\, d, \delta)$ may be computed by summing the coefficient of $x^{i/d}y^j$ in $\alpha_F(x^{i/d}y^j)$ over all $(i, j) \geq 0$ with $i \equiv \delta \pmod d$:

$$\alpha_F(x^{i/d}y^j) = \psi_q\Bigl(\sum_{k,l\geq 0} A(k, l)x^{k+(i/d)}y^{l+j}\Bigr)$$
$$= \sum_{k,l\geq 0} A(qk + (q-1)(i/d), ql + (q-1)j)x^{k+(i/d)}y^{l+j}.$$

The coefficient of $x^{i/d}y^j$ in this expression is $A((q-1)i/d, (q-1)j)$, hence

$$\operatorname{tr}\alpha_F = \sum_{\substack{i,j\geq 0\\ i\equiv\delta \,(\mathrm{mod}\, d)}} A((q-1)i/d, (q-1)j). \tag{1.7}$$

On the other hand,

$$\sum_{\substack{x^{q-1}=1\\ y^{q-1}=1}} x^{-(q-1)\delta/d}F(x, y) = \sum_{k,l\geq 0}\ \sum_{\substack{x^{q-1}=1\\ y^{q-1}=1}} A(k, l)x^{k-(q-1)(\delta/d)}y^l,$$

and

$$\sum_{\substack{x^{q-1}=1\\ y^{q-1}=1}} x^{k-(q-1)(\delta/d)}y^l = \begin{cases} (q-1)^{m+n} & \text{if there exist } i, j \geq 0 \text{ such that}\\ & k-(q-1)(\delta/d) = (q-1)i,\\ & l = (q-1)j,\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\sum_{\substack{x^{q-1}=1\\ y^{q-1}=1}} x^{-(q-1)\delta/d}F(x, y)$$
$$= (q-1)^{m+n}\sum_{i,j\geq 0} A((q-1)i + (q-1)(\delta/d), (q-1)j)$$
$$= (q-1)^{m+n}\sum_{\substack{i,j\geq 0\\ i\equiv\delta \,(\mathrm{mod}\, d)}} A((q-1)i/d, (q-1)j).$$

The theorem now follows from eq. (1.7). □

COROLLARY.

$$(q^r-1)^{m+n}\operatorname{tr}(\alpha_F^r \mid L(qb;\, d, \delta))$$
$$= \sum_{\substack{x^{q^r-1}=1\\ y^{q^r-1}=1}} \Bigl(\prod_{i=1}^m x_i^{-(q^r-1)\delta_i/d}\Bigr) F(x;\, y)F(x^q;\, y^q)\cdots F(x^{q^{r-1}};\, y^{q^{r-1}}).$$

2. Application. Fix $\bar{\lambda} \in \mathbf{F}_q^x$, where $d \mid (q-1)$ and $q = p^f$, and consider the exponential sum

$$S(\bar{\lambda}, d) = \sum_{\bar{x} \in \mathbf{F}_q} \exp\left(\frac{2\pi i}{p} \operatorname{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(\bar{\lambda}\bar{x}^d)\right).$$

Let G be the group of dth roots of unity in $\mathbf{F}_q$, and let $\hat{G}$ be its character group. Then

$$S(\bar{\lambda}, d) = 1 + \sum_{\bar{x} \in \mathbf{F}_q^x} \sum_{\chi \in \hat{G}} \chi(\bar{x}^{(q-1)/d}) \exp\left(\frac{2\pi i}{p} \operatorname{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(\bar{\lambda}\bar{x})\right)$$

$$= 1 + \sum_{\chi \in \hat{G}} \chi(\bar{\lambda}^{-(q-1)/d}) \sum_{\bar{x} \in \mathbf{F}_q^x} \chi(\bar{x}^{(q-1)/d}) \exp\left(\frac{2\pi i}{p} \operatorname{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(\bar{x})\right).$$

Put

$$g(\chi) = \sum_{\bar{x} \in \mathbf{F}_q^x} \chi(\bar{x}^{(q-1)/d}) \exp\left(\frac{2\pi i}{p} \operatorname{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(\bar{x})\right).$$

By [**2**, eq. (4.4)], the Gauss sum $g(\chi)$, considered p-adically, factors in a natural way into a product of f factors. The Gross-Koblitz formula describes these factors in terms of the p-adic gamma function. We give a proof of the Gross-Koblitz formula for the factorization of $\chi(\bar{\lambda}^{-(q-1)/d})g(\chi)$, in which we also describe how each of the f factors depends on $\bar{\lambda}$.

To apply the trace formula to exponential sums, we need p-adic analytic lifting of the additive character. Consider the function of two variables on $\mathbf{C}_p$ (where now $\pi \in \mathbf{C}_p$ is such that $\pi^{p-1} = -p$),

$$F(\lambda, x) = \exp \pi(\lambda x - \lambda^p x^p) = \sum_{r=0}^{\infty} A_r \lambda^r x^r. \tag{2.1}$$

By [**3**, §4] one has $F(\lambda, x) \in L((p-1)p^{-2} + \operatorname{ord} \lambda; d, 0)$, where $L((p-1)p^{-2} + \operatorname{ord} \lambda; d, 0)$ is a space as in §1 with $m = 1$, $n = 0$. Furthermore, $F(1, 1)$ is a primitive pth root of unity, and if $\lambda^{p^r} = \lambda$, $x^{p^r} = x$, $\lambda, x \neq 0$, then

$$\prod_{i=0}^{r-1} F(\lambda^{p^i}, x^{p^i}) = F(1, 1)^{\operatorname{Tr}_r(\bar{\lambda}\bar{x})}, \tag{2.2}$$

where $\bar{\lambda}, \bar{x} \in \mathbf{F}_{p^r}$ are the reductions of λ, $x \bmod p$, and

$$\operatorname{Tr}_r \colon \mathbf{F}_{p^r} \to \mathbf{F}_p$$

is the trace map. Put

$$G(\lambda, x) = \prod_{i=0}^{f-1} F(\lambda^{p^i}, x^{p^i}) = \exp \pi(\lambda x - \lambda^q x^q).$$

For $0 \leq j < d$, define

$$-g_q((q-1)j/d) = \sum_{x^{q-1}=1} x^{-(q-1)j/d} G(1, x).$$

By (2.2) this is an imbedding of a Gauss sum $g(\chi)$ into $\mathbf{C}_p$. By (2.2) and a simple argument, if $\lambda^{q-1} = 1$, then

$$-\lambda^{(q-1)j/d} g_q((q-1)j/d) = \sum_{x^{q-1}=1} x^{-(q-1)j/d} G(\lambda, x),$$

which is an imbedding of a $\chi(\bar{\lambda}^{(q-1)/d}) g(\chi)$ into $\mathbf{C}_p$.

We assume from now on that $\operatorname{ord} \lambda > -(p-1)/p^2$. For notational convenience, we abbreviate $L((p-1)/p + \operatorname{ord} \lambda; d)$ (resp: $L((p-1)/p + \operatorname{ord} \lambda; d, j))$ by $L(\lambda)$ (resp: $L(\lambda; j)$). Let α_λ: $L(\lambda; j') \to L(\lambda^p; j)$ denote the composition

$$L(\lambda; j') \overset{F(\lambda, x)}{\to} L\left(\frac{p-1}{p^2} + \operatorname{ord} \lambda; d, j'\right) \overset{\psi}{\to} L(\lambda^p, j).$$

Suppose $\lambda^{q-1} = 1$. Since $d \mid (q-1)$, the operator β_λ defined by

$$\beta_\lambda = \alpha_{\lambda^{q/p}} \circ \cdots \circ \alpha_{\lambda^p} \circ \alpha_\lambda \quad \left(= \psi_q \circ G(\lambda, x)\right) \tag{2.3}$$

is stable on $L(\lambda; j)$ and, by Theorem 1,

$$\begin{aligned} \operatorname{tr}(\beta_\lambda \mid L(\lambda; j)) &= (q-1)^{-1} \sum_{x^{q-1}=1} x^{-(q-1)j/d} G(\lambda, x) \\ &= -(q-1)^{-1} \lambda^{(q-1)j/d} g_q((q-1)j/d). \end{aligned} \tag{2.4}$$

The factorization of $\lambda^{(q-1)j/d} g_q((q-1)j/d)$ is derived from (2.3) by studying the differential operator that commutes with α_λ. Formally one has

$$\alpha_\lambda = \exp(-\pi\lambda x) \circ \psi \circ \exp(\pi\lambda x). \tag{2.5}$$

This factorization is a priori valid only for $|\lambda x| < 1$ (where $\exp(\pi\lambda x)$ converges), but by analytic continuation it describes the action of α_λ on elements of $L(\lambda)$. From (2.5) it is easy to check that

$$\alpha_\lambda \circ D_\lambda = p D_{\lambda^p} \circ \alpha_\lambda, \tag{2.6}$$

where

$$D_\lambda = \exp(-\pi\lambda x) \circ x\frac{d}{dx} \circ \exp(\pi\lambda x) = x\frac{d}{dx} + \pi\lambda x \tag{2.7}$$

is an endomorphism of $L(\lambda)$. Put

$$\mathcal{W}(\lambda) = L(\lambda)/D_\lambda L(\lambda).$$

Then (2.6) implies that α_λ induces a map

$$\bar{\alpha}_\lambda \colon \mathcal{W}(\lambda) \to \mathcal{W}(\lambda^p).$$

The operator D_λ respects the decomposition $L(\lambda) = \bigoplus_{j=0}^{d-1} L(\lambda; j)$, hence

$$D_\lambda L(\lambda) = \bigoplus_{j=0}^{d-1} D_\lambda L(\lambda; j).$$

Thus if we put $\mathcal{W}(\lambda; j) = L(\lambda; j)/D_\lambda L(\lambda; j)$, then

$$\mathcal{W}(\lambda) = \bigoplus_{j=0}^{d-1} \mathcal{W}(\lambda; j),$$

and $\bar{\alpha}_\lambda$ maps $\mathcal{W}(\lambda; j')$ into $\mathcal{W}(\lambda^p; j)$.

Suppose $\lambda^{q-1} = 1$. Since $d \mid (q-1)$, the operator $\bar{\beta}_\lambda = \bar{\alpha}_{\lambda^{q/p}} \circ \cdots \circ \bar{\alpha}_{\lambda^p} \circ \bar{\alpha}_\lambda$ is an endomorphism of $\mathcal{W}(\lambda; j)$. It is easily checked from the definition that D_λ is injective on $L(\lambda)$, hence for each j there is, by (2.6), a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & L(\lambda; j) & \xrightarrow{D_\lambda} & L(\lambda; j) & \to & \mathcal{W}(\lambda; j) & \to & 0 \\ & & \downarrow q\beta_\lambda & & \downarrow \beta_\lambda & & \downarrow \bar{\beta}_\lambda & & \\ 0 & \to & L(\lambda; j) & \xrightarrow{D_\lambda} & L(\lambda; j) & \to & \mathcal{W}(\lambda; j) & \to & 0. \end{array}$$

It follows from [**6**, Prop. 9] that

$$\begin{aligned} \det\big(I - t\bar{\beta}_\lambda \mid \mathcal{W}(\lambda; j)\big) \\ = \det\big(I - t\beta_\lambda \mid L(\lambda; j)\big)/\det\big(I - qt\beta_\lambda \mid L(\lambda; j)\big). \end{aligned} \tag{2.8}$$

LEMMA 1. $\operatorname{tr} \bar{\beta}_\lambda = \lambda^{(q-1)j/d} g_q((q-1)j/d)$.

Proof. Take the logarithm of both sides of (2.8) and use (1.6) and (2.4). □

Put

$$g_{q^r}((q^r-1)j/d) = \sum_{x^{q^r-1}=1} x^{-(q^r-1)j/d} G(1,x)G(1,x^q)\cdots G(1,x^{q^{r-1}}).$$

A similar argument, using the corollary to Theorem 1 to evaluate $\mathrm{tr}\,\beta_\lambda^r$, shows that

$$\mathrm{tr}(\bar{\beta}_\lambda)^r = \lambda^{(q^r-1)j/d} g_{q^r}((q^r-1)j/d). \tag{2.9}$$

LEMMA 2. $\dim \mathcal{W}(\lambda; j) = 1$.

Proof. Let $\eta = \sum_{n=0}^\infty a_{j+nd} x^{(j+nd)/d} \in L(\lambda; j)$. An inductive argument using the relation

$$x^{(j+nd)/d} = -\left(\frac{j}{d}+n-1\right)x^{(j+(n-1)d)/d} + \frac{1}{\pi\lambda} D_\lambda\left(x^{(j+(n-1)d)/d}\right)$$

shows that

$$x^{(j+nd)/d} = \frac{(-1)^n (j/d)_n}{(\pi\lambda)^n} x^{j/d} + D_\lambda(\xi_n),$$

where

$$\xi_n = \sum_{i=0}^{n-1} \frac{(-1)^i (j/d+n-i)_i}{(\pi\lambda)^{i+1}} x^{j/d+n-i-1}.$$

Hence

$$\eta = \left(\sum_{n=0}^\infty a_{j+nd} \frac{(-1)^n (j/d)_n}{(\pi\lambda)^n}\right) x^{j/d} + D_\lambda\left(\sum_{n=0}^\infty a_{j+nd}\xi_n\right). \tag{2.10}$$

A straightforward calculation using the growth condition satisfied by the a_{j+nd} (inequality (1.2)) shows that the first series on the right-hand side of (2.10) converges and that the second series lies in $L(\lambda; j)$. Hence $\dim \mathcal{W}(\lambda; j) \le 1$.

Suppose $j \ne 0$. The equation

$$D_\lambda\left(\sum_{n=0}^\infty b_{j+nd} x^{(j+nd)/d}\right) = x^{j/d}$$

gives a recursion relation which determines the b_{j+nd}:

$$b_{j+nd} = \frac{(-1)^n \pi^n \lambda^n}{(j/d+1)_n}.$$

Thus $\operatorname{ord} b_{j+nd} \leq n \operatorname{ord} \lambda + s_n/(p-1)$, where s_n is the sum of the digits in the p-adic expansion of n. This estimate shows

$$\sum_{n=0}^{\infty} b_{j+nd} x^{(j+nd)/d} \notin L(\lambda; j).$$

The image of D_λ does not contain any series with a non-zero constant term, so the result is valid when $j = 0$ also. □

REMARK. Lemmas 1 and 2 imply

$$\operatorname{tr}(\bar{\beta}_\lambda)^r = \left(\lambda^{(q-1)j/d} g_q((q-1)j/d)\right)^r.$$

Comparing this with (2.9) and using the equality $\lambda^{(q-1)jr/d} = \lambda^{(q^r-1)j/d}$ (which follows from $q \equiv 1 \pmod d$ and $\lambda^{q-1} = 1$), we get

$$g_{q^r}((q^r-1)j/d) = g_q((q-1)j/d)^r,$$

a classical formula of Hasse and Davenport.

Fix j, $0 < j < d$, and let $j_0, j_1, \ldots, j_{f-1}$ be the minimal positive residues mod d of $j, pj, \ldots, p^{f-1}j$. Put $\nu' = f - 1 - \nu$ and define γ_ν, $\nu = 0, 1, \ldots, f-1$, by

$$\alpha_{\lambda^{p^{\nu'}}}(x^{j_{\nu+1}/d}) \equiv \gamma_\nu x^{j_\nu/d} \quad \left(\operatorname{mod} D_{\lambda^{p^{\nu'+1}}} L\left(\lambda^{p^{\nu'+1}}; j_\nu\right)\right). \tag{2.11}$$

By Lemma 2, γ_ν is well defined. By the definition of $\bar{\beta}_\lambda$, Lemmas 1 and 2 imply

$$\lambda^{(q-1)j/d} g_q((q-1)j/d) = \prod_{\nu=0}^{f-1} \gamma_\nu. \tag{2.12}$$

The Gross-Koblitz formula expresses the γ_ν in terms of values of Morita's p-adic gamma function Γ_p.

Let i be a positive integer, $i \not\equiv 0 \pmod d$. Define a function G on fractions i/d by

$$\alpha_\lambda(x^{pi/d}) \equiv G(i/d) x^{i/d} \quad (\operatorname{mod} D_{\lambda^p} L(\lambda^p; i)). \tag{2.13}$$

The function G is well defined: The same argument as in the proof of Lemma 2 shows that $x^{i/d}$ (resp: $x^{pi/d}$) is a basis for $\mathfrak{W}(\lambda^p; i)$ (resp: $\mathfrak{W}(\lambda; pi)$). In fact, we have for n a non-negative integer,

$$x^{(i/d)+n} \equiv \frac{(-1)^n (i/d)_n}{(\pi\lambda^p)^n} x^{i/d} \quad (\operatorname{mod} D_{\lambda^p} L(\lambda^p; i)). \tag{2.14}$$

This leads to a formula for $G(i/d)$:

$$\alpha_\lambda(x^{pi/d}) = \psi\left(\sum_{n=0}^{\infty} A_n\lambda^n x^{(pi/d)+n}\right) = \sum_{n=0}^{\infty} A_{pn}\lambda^{pn}x^{(i/d)+n}$$

$$\equiv \left(\sum_{n=0}^{\infty}(-1)^n A_{pn}(i/d)_n/\pi^n\right)x^{i/d} \pmod{D_{\lambda^p}L(\lambda^p;i)}$$

by (2.14). Hence

$$G(i/d) = \sum_{n=0}^{\infty}\frac{(-1)^n A_{pn}(i/d)_n}{\pi^n}. \tag{2.15}$$

Note that although both sides of (2.13) depend on λ, G itself is independent of λ.

Extend G by defining

$$G(z) = \sum_{n=0}^{\infty}(-1)^n A_{pn}(z)_n/\pi^n. \tag{2.16}$$

Since $\operatorname{ord} A_{pn} \geq n(p-1)/p$, equation (2.16) defines an analytic function on the set

$$\operatorname{ord} z > -\left(\frac{p-1}{p} - \frac{1}{p-1}\right).$$

LEMMA 3. *Assume $p \geq 3$. For $z \in \mathbf{Z}_p$, $G(z) = \Gamma_p(pz)$.*

Proof. By definition, Γ_p is the unique continuous function on $\mathbf{Z}_p$ satisfying

$$\Gamma_p(r) = (-1)^r \prod_{\substack{1\leq i\leq r-1\\ p\nmid i}} i$$

for positive integers r. It satisfies the functional equation

$$\Gamma_p(z+1) = \Gamma_p(z)\cdot\begin{cases}-1 & \text{if } z \in p\mathbf{Z}_p,\\ -z & \text{if } z \notin p\mathbf{Z}_p.\end{cases} \tag{2.17}$$

Hence for positive integers r,

$$\Gamma_p(-r) = (-1)^r \prod_{\substack{-r\leq i<0\\ p\nmid i}} i^{-1}.$$

In particular,

$$\Gamma_p(-pr) = (-1)^r p^r r!/(pr)! \tag{2.18}$$

By (2.1),

$$A_{pn} = (-1)^n \pi^n \sum_{i=0}^{n} p^i/(pi)!\,(n-i)!$$

Observe also that

$$(z)_n = (-1)^n n! \binom{-z}{n}.$$

Hence by (2.16),

$$G(-r) = \sum_{n=0}^{r} \sum_{i=0}^{n} (-1)^n p^i i! \binom{n}{i}\binom{r}{n} \Big/ (pi)!.$$

By (2.18) and the fact that

$$\binom{n}{i}\binom{r}{n} = \binom{r}{i}\binom{r-i}{n-i},$$

$$G(-r) = \sum_{n=0}^{r} \sum_{i=0}^{n} (-1)^{n+i} \Gamma_p(-pi) \binom{r}{i}\binom{r-i}{n-i}.$$

Interchanging the order of summation:

$$G(-r) = \sum_{i=0}^{r} (-1)^i \Gamma_p(-pi) \binom{r}{i} \sum_{n=i}^{r} (-1)^n \binom{r-i}{n-i} = \Gamma_p(-pr),$$

since the inner sum collapses. We are now done by the continuity of G and Γ_p. □

Let j, j_ν $(\nu = 0, 1, \ldots, f-1)$, and ν' be as above. Put $k_\nu = (pj_\nu - j_{\nu+1})/d$. Then $0 \le k_\nu \le p-1$; in fact, these are the digits in the p-adic expansion of $(q-1)j/d$:

(2.19) $$(q-1)j/d = k_{f-1} + k_{f-2}p + \cdots + k_1 p^{f-2} + k_0 p^{f-1}.$$

By (2.14),

(2.20) $$x^{pj_\nu/d} \equiv \frac{(-1)^{k_\nu}(j_{\nu+1}/d)_{k_\nu}}{(\pi\lambda^{p^{\nu'}})^{k_\nu}} x^{j_{\nu+1}/d} \quad \left(\operatorname{mod} D_{\lambda^{p^{\nu'}}} L\left(\lambda^{p^{\nu'}};\, j_{\nu+1}\right)\right).$$

Using (2.6) and (2.11),

(2.21) $$\alpha_{\lambda^{p^{\nu'}}}(x^{pj_\nu/d}) \equiv \frac{(-1)^{k_\nu}(j_{\nu+1}/d)_{k_\nu}\gamma_\nu}{(\pi\lambda^{p^{\nu'}})^{k_\nu}} x^{j_\nu/d} \quad \left(\operatorname{mod} D_{\lambda^{p^{\nu'+1}}} L\left(\lambda^{p^{\nu'+1}};\, j_\nu\right)\right).$$

By (2.13) and Lemma 3,

$$\gamma_\nu = (-1)^{k_\nu}(\pi\lambda^{p^{\nu'}})^{k_\nu}\Gamma_p(pj_\nu/d)/(j_{\nu+1}/d)_{k_\nu}.$$

Repeated use of the functional equation (2.17) gives

$$\gamma_\nu = (\pi\lambda^{p^{\nu'}})^{k_\nu}\Gamma_p(j_{\nu+1}/d).$$

The Gross-Koblitz formula then follows from (2.12) (the powers of λ cancel by (2.19)):

$$g_q((q-1)j/d) = \prod_{\nu=0}^{f-1} \pi^{k_\nu}\Gamma_p(j_\nu/d). \tag{2.22}$$

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