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$*$ PRODUCTS AND REPRESENTATIONS OF NILPOTENT GROUPS

D. ARNAL

On each orbit W of the coadjoint representation of a nilpotent, connected and simply connected Lie group G, there exist $*$ products which are relative quantizations for the Lie algebra $\mathfrak{g}$ of G. Choosing one of these $*$ products, we first define a $*$-exponential for each X in $\mathfrak{g}$. These $*$-exponentials are formal power series and, with the $*$ product, they form a group. Thanks to that, we are able to define a representation of G in a "$*$ polarization" and to intertwine it with the unitary irreducible one associated to W. Finally, we study the uniqueness of our construction.

1. Introduction. The mathematical signification of quantization was specified by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer with the theory of deformations of the associative algebra of C^∞ functions on the symplectic manifold W defined by the classical system [**3**]. (The principal results of the theory will be given in §2 for completeness.)

Previously, some other methods of geometrical quantization were considered by Kirillov [**7, 9, 16, 8**]. This last approach had a very important link with questions of finding and classifying unitary irreducible representations of a group G. The easiest and the most complete case is of course when G is nilpotent. Let us suppose G is nilpotent, connected and simply connected. We know all its unitary irreducible representations [**8, 15**]. There exists a one-to-one mapping between classes of these representations U and orbits W of the coadjoint representation of G. On the other hand, the geometrical quantization of W, i.e. the construction of fibre bundles with base W and fibre a circle, is unique, the de Rham cohomology of W being trivial. Moreover, the representation U can be canonically defined with that quantization and a polarization [**7**].

It is tempting to "test" the method of quantization by deformation ($*$ quantization) on the problem of constructing and classifying unitary irreducible representations of connected, simply connected, nilpotent groups.

The goal of this work is to canonically find the unitary irreducible representation associated to an arbitrary orbit W by means of $*$ products defined on W. We first recall the principal definitions and results of the theory of $*$ products (which are formal deformations with parameter λ of

the algebra $C^\infty(W)$). We consider the invariance and covariance of $*$ products with respect to the action of G. Then we give an example showing that a too strong invariance property cannot be imposed. We thus consider only $\mathfrak{g}$-relative quantizations, i.e. $*$ products for which the relation

$$\tilde{X} * \tilde{Y} - \tilde{Y} * \tilde{X} = 2\lambda\widetilde{[X, Y]} \qquad \forall X, Y \in \mathfrak{g}$$

holds (here $\tilde{X}$ is the function defined on W by $\tilde{X}(\xi) = \langle \xi, X\rangle$, $\xi \in W$).

We prove the existence of relative quantizations on each orbit W: the so-called Moyal $*$ product defined in a particular global chart on W.

Now we study the representations property of that $*$ product. We remain in the frame of deformation theory as far as possible: we fix the value of λ only in the last step. Thus our approach is entirely distinct from the method of Fronsdal and Lugo [**4, 11**]. Moreover, we do not limit the structure of G and W.

For each X in $\mathfrak{g}$, we can define a formal power series in $1/\lambda$:

$$e^{*\tilde{X}/2\lambda} = \sum_{n>0} \frac{1}{n!}\left(\frac{1}{2\lambda}\right)^n (\tilde{X})^{*n}.$$

The set G^* of such series is a group for the law $*$. The map

$$\Phi\colon G \to G^*, \qquad \Phi(\exp X) = e^{*\tilde{X}/2\lambda},$$

is a group homomorphism.

Let $\mathfrak{n}$ be a subalgebra of $\mathfrak{g}$ subordinate to ξ_0 in W. Following a method of Fronsdal, we solve, in a space of formal power series in $1/\lambda$, the equation

$$x * \tilde{X} = \xi_0(X)x \qquad \forall X \in \mathfrak{n}.$$

The space S of solutions ($*$ polarization) carries a representation π of G:

$$\pi(g)x = \Phi(g) * x.$$

We define an intertwining operator between π and the unitary irreducible representation U associated to W. This operator fixes the value of our parameter λ and sums the formal series. That gives us each class of unitary irreducible representations for arbitrary nilpotent G.

In the last part we prove that G^* and Φ are unique up to an automorphism and we study the uniqueness of π.

2. $*$ Products. $*$ products are introduced in [**3**] in order to define the quantization(s) of a problem of classical mechanics. In fact we suppose that quantum mechanics can be described as a deformation (in

the sense of Gerstenhaber [5]) of classical mechanics with parameter λ related to use $\hbar$ (in general $\lambda = i\hbar/2$). We thus deform the structure of the set of observables without modifying the observables themselves. More precisely:

The classical problem is described by a symplectic manifold W; on the space $N = C^\infty(W, \mathbf{R})$ of observables, we consider the structures of Lie algebras (for the Poisson bracket $\{\ ,\ \}$) and associative algebras (for usual product). A $*$ product is an associative deformed structure

$$u * v = uv + \sum_{r>0} \lambda^r C^r(u, v) \qquad \forall u, v \in N$$

($u * v$ is a formal power series). We suppose that each C^r is a bidifferential operator vanishing on constants. Thus the relation

$$1 * u = u * 1 = u, \qquad \forall u \in N,$$

holds. Of course we do not impose commutativity for $*$ since the $*$ products of observables will correspond to composition of operators in the usual formalism of quantum mechanics. Gerstenhaber defined a cohomology associated to that deformation problem:

If C is an n-differential operator, vanishing on constants, its coboundary δC will be: $\forall u_0, \ldots, u_n \in N$

$$\begin{aligned}(\delta C)&(u_0, u_1, \ldots, u_n)\\ &= u_0 C(u_1, \ldots, u_n) - C(u_0 u_1, u_2, \ldots, u_n) + C(u_0, u_1 u_2, \ldots, u_n)\\ &\quad + \cdots + (-1)^n C(u_0, u_1, \ldots, u_{n-1} u_n) + (-1)^{n+1} C(u_0, \ldots, u_{n-1}) u_n.\end{aligned}$$

Now if ω is an n-form on W, we define an n-differential operator C_ω by

$$C_\omega(u_1, \ldots, u_n) = \omega(X_{u_1}, \ldots, X_{u_n}), \qquad (u_1, \ldots, u_n) \in N,$$

where X_u is the symplectic gradient of u, i.e.,

$$X_u v = \{u, v\} \qquad \forall v \in N.$$

J. Vey [17] determined the cohomology groups $H^n(\delta)$:

$$H^n(\delta) = \{[C_\omega], \omega\ n\text{-form on } W\} \simeq \text{space of } n\text{-forms on } W.$$

The obstruction to deformation is in $H^3(\delta)$, the equivalence of deformations is given (in the theory of Gerstenhaber) by $H^2(\delta)$. These groups are in general very large. Now we want to simultaneously deform the Lie algebra structure of N with the $*$ commutator because usually the commutator of operators corresponds (up to an $i\hbar$ factor) to the Poisson

bracket for classical observables. Thus we impose

$$\begin{aligned} C^1(u,v) &= \{u,v\}, \\ C^n(u,v) &= (-1)^n C^n(v,u), \end{aligned} \qquad \forall u, v \in N.$$

Then

$$\frac{1}{2\lambda}(u * v - v * u) = [u,v] = \{u,v\} + \sum_{r>0} \lambda^{2r} C^{2r+1}(u,v)$$

is a deformation of the Poisson bracket. The first condition is in fact admissible and the obstruction to deformation appears now as an element of $H^3(W)$, the third cohomology group in the de Rham cohomology of the manifold W ([**13**]). Similarly, equivalences are described by $H^2(W)$ ([**13**]); $*$ and $*'$ are equivalents if there exists a series

$$H = \mathrm{Id} + \sum_{r>0} \lambda^r H_r,$$

where the H_r are differential operators without constant terms such that

$$H(u * v) = Hu *' Hv.$$

3. Vey $*$ products. We know a $*$ product on $\mathbf{R}^{2k}$, the so called Moyal $*$ product ([**3**]) associated to the Weyl-Wigner quantization

$$u * v = uv + \sum_{n>0} \frac{\lambda^n}{n!} P^n(u,v),$$

where the operators P^n are defined by:

$$\begin{aligned} P^1(u,v) &= P(u,v) = \Lambda^{ij}\partial_i u \partial_j v = \{u,v\}, \\ P^n(u,v) &= \Lambda^{i_1 j_1} \cdots \Lambda^{i_n j_n} \partial_{i_1 \cdots i_n} u \partial_{j_1 \cdots j_n} v. \end{aligned}$$

If W is a symplectic manifold and Γ a symplectic connection on W, we can define operators

$$P_\Gamma^n(u,v) = \Lambda^{i_1 j_1} \cdots \Lambda^{i_n j_n} \nabla_{i_1 \cdots i_n} u \nabla_{j_1 \cdots j_n} v.$$

But the series $\sum \lambda^n P_\Gamma^n / n!$ defines an associative product only if Γ is flat ([**3**]). By extension, we shall call a $*$ product a *Vey $*$ product if*

$$C^n(u,v) = \frac{1}{n!} Q^n(u,v) \quad \text{for all } n,$$

where P_Γ^n and Q^n have the same principal symbols (they do not depend on Γ). Lichnerowicz [**13**] has shown:

THEOREM 3.1. *Each $*$ product is equivalent to a Vey $*$ product.*

4. Invariant * product. Let us now introduce a symplectic action of a Lie group G on W. We have a notion of invariance.

DEFINITION 4.1. G acts on N by

$$(g \cdot u)(\xi) = u(g^{-1}\xi) \qquad \forall u \in N, g \in G, \xi \in W.$$

We extend, with the same notation, this action to the space of formal power series with coefficients in N.

A $*$ product is invariant if

$$g(u * v) = (g \cdot u) * (g \cdot v) \qquad \forall u, v \in N, g \in G.$$

Two $*$ products, $*$ and $*'$, are equivariantly equivalents if they are equivalents:

$$\left(\mathrm{Id} + \sum_{r>0} \lambda^r H_r\right)(u * v) = \left(\mathrm{Id} + \sum_{r>0} \lambda^r H_r\right) u *' \left(\mathrm{Id} + \sum_{r>0} \lambda^r H_r\right) v$$

and the H_r are invariants.

In fact, the theory of invariant $*$ products is well known only if there exists on W an invariant connection. In the hermitian case S. Gutt [**6**] has shown the following theorem, whose generalization was given by Molin [**12**].

THEOREM 4.1. *If there exists on W an invariant connection, then the obstruction to constructing an invariant $*$ product is in $H^3_{\mathrm{inv}}(W)$, the group of closed invariant 3-forms modulo exact invariant 3-forms, and the classification up to an invariant equivalence is described by $H^2_{\mathrm{inv}}(W)$.*

On the other hand, the proof of Theorem 3.1 can be rewritten in the invariant case. We obtain

PROPOSITION 4.1. *If there exists on W an invariant connection, then each invariant $*$ product is equivariantly equivalent to an invariant Vey $*$ product.*

Finally, studying $C^2(u, v)$ it is easy to prove

PROPOSITION 4.2. ([**10**]). *If there exists on W an invariant Vey $*$ product, then there exists an invariant connection.*

5. A counterexample in the nilpotent case. From now on G is a nilpotent, connected and simply connected Lie group, $\mathfrak{g}$ its Lie algebra, and W one of their orbits in the coadjoint representation in $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. W is a symplectic manifold on which G acts. It is natural to ask for invariant $*$ products on W. Unfortunately this is generally impossible, as the following example shows:

PROPOSITION 5.1. *Let $\mathfrak{g}$ be the nilpotent Lie algebra with basis $X_0, X_1, \ldots, X_n, Y$ $(n \geq 3)$ and commutation relations*

$$[Y, X_i] = X_{i-1}, \qquad i = 1, \ldots, n,$$

all the remaining brackets vanishing. Let G be the corresponding connected and simply connected Lie group. Then the generic orbits W (the orbits such that $\langle \xi, X_0 \rangle \neq 0$) are two dimensional, and there does not exist an invariant connection, neither an invariant $*$ product, on W.

Proof. We easily show that W can be parametrized by $(p, q) \in \mathbf{R}^2$ in such a manner that the vector fields differentials of the action of G are

$$q^i \frac{\partial}{\partial p}, \quad i = 0, 1, \ldots, n-1; \qquad \frac{\partial}{\partial q}$$

(see [**2**] and a general proof in Proposition 6.1). If Γ is an invariant connection,

$$\left[\partial_p, \nabla_{\partial_q} \partial_q\right] = 0 = \left[\partial_q, \nabla_{\partial_q} \partial_q\right]$$

gives us

$$\nabla_{\partial_q} \partial_q = a\partial_p + b\partial_q \quad \text{with } a \text{ and } b \text{ constants.}$$

Moreover the following relations are incompatible:

$$\left[q\partial_p, \nabla_{\partial_q} \partial_q\right] = -\nabla_{\partial_p} \partial_q - \nabla_{\partial_q} \partial_p = -b\partial_p,$$

$$\left[q^2\partial_p, \nabla_{\partial_q} \partial_q\right] = -2q\left(\nabla_{\partial_p} \partial_q + \nabla_{\partial_q} \partial_p\right) - 2\partial_p = -2bq\partial_p.$$

Let G_0 be the subgroup of G, exponential of $\partial/\partial q$, $\partial/\partial p$ and $q(\partial/\partial p)$. Clearly there exist G_0-invariant connections on W. Moreover a differential operator H is G_0-invariant if and only if it is G-invariant.

Then if $*$ is a G-invariant $*$ product, it is a G_0-invariant $*$ product; there exists a G_0-invariant Vey $*$ product equivariantly equivalent to $*$. The equivalence being G-invariant, our Vey $*$ product is G-invariant and there should be a G-invariant connection on W.

6. Relative quantizations. The notion of relative quantization was given in [3]. Let W be a coadjoint orbit of a Lie group G. Thus elements X of $\mathfrak{g}$ appear as functions $\tilde{X}$ on W:

$$\tilde{X}(\xi) = \langle \xi, X \rangle \qquad \forall \xi \in W, \forall X \in \mathfrak{g}.$$

A $\mathfrak{g}$-*relative quantization* on W is a $*$ product such that

$$[\tilde{X}, \tilde{Y}] = \frac{1}{2\lambda}(\tilde{X} * \tilde{Y} - \tilde{Y} * \tilde{X}) = \{\tilde{X}, \tilde{Y}\} \left(= \widetilde{[X, Y]}\right) \qquad \forall X, Y \in \mathfrak{g}.$$

In [2] we proved that each relative quantization is a *covariant* $*$ product. That means there exists a representation ρ of G, by automorphisms of $*$, which is a deformation of the action defined in Definition 4.1:

$$\begin{aligned} &\rho(g)(u * v) = \rho(g)u * \rho(g)v, \\ &\rho(gg') = \rho(g)\rho(g'), \\ &\rho(g) = \left(\mathrm{Id} + \sum_{s \geq 1} a_s(g)\right) \circ g, \end{aligned}$$

where the $a_s(g)$ are differential operators without constant terms. Thus relative quantizations give rise to representations. In the nilpotent case, we proved in [2] the existence of relative quantizations on each coadjoint orbit. For completeness and because it will be our starting point, we give this construction.

PROPOSITION 6.1. ([15], [2]). *Let W be an orbit of the coadjoint representation of a nilpotent, connected and simply connected Lie group G. Then there exists on W a global chart*

$$W \to \mathbf{R}^{2k}, \qquad \xi \mapsto (p, q),$$

such that:

(a) *The canonical 2-form on W is $\sum_{i=1}^{k} dp_i \wedge dq_i$;*

(b) *Each $\tilde{X}$, $X \in \mathfrak{g}$ takes the form*

$$\tilde{X}(p, q) = \sum_{i=1}^{k} \alpha_i(q) p_i + \alpha_0(q)$$

where the $\alpha_i(q)$ $(i \geq 0)$ are polynomial functions in $q_{i+1}, \ldots, q_k$.

Proof. We prove it by induction on dim $\mathfrak{g}$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. If the kernel of the form $\mathfrak{w} \in \mathfrak{z}^*$ defined by

$$\mathfrak{w}(Z) = \langle \xi, Z \rangle \qquad \forall \xi \in W, Z \in \mathfrak{z},$$

is nontrivial, W is isomorphic to an orbit $\overline{W}$ of $\bar{\mathfrak{g}} = \mathfrak{g}/\mathrm{Ker}\,\mathfrak{w}$ and the proposition holds. If it is not the case, $\mathfrak{z}$ is one dimensional, and we can write

$$\mathfrak{g} = \mathbf{R}X_0 + \mathfrak{g}_1,$$

where $\mathfrak{g}_1$ is an ideal ([**15**]) and W is isomorphic to $W_1 \times \mathbf{R}^2$, where W_1 is an orbit in $\mathfrak{g}_1^* \simeq X_0^\perp$, the isomorphism being

$$(\xi_1, q_k, p_k) \mapsto \exp -q_k X_0 \xi_1 + p_k \xi_{X_0},$$

$$\xi_{X_0} \in \mathfrak{g}_1^\perp \quad \text{and} \quad \langle \xi_{X_0}, X_0 \rangle = 1.$$

This isomorphism gives us the chart of Proposition 6.1. A direct computation proves (a) and (b). The following corollary is an immediate consequence.

COROLLARY 6.1. *On each orbit W of the coadjoint representation of G, the Moyal $*$ product in the chart of Proposition* 6.1 *defines a relative quantization.*

7. The $*$ exponential. The relation of relative quantization is formally

$$\left(\frac{1}{2\lambda}\tilde{X}\right) * \left(\frac{1}{2\lambda}\tilde{Y}\right) - \left(\frac{1}{2\lambda}\tilde{Y}\right) * \left(\frac{1}{2\lambda}\tilde{X}\right) = \frac{1}{2\lambda}\widetilde{[X, Y]}.$$

Thus it is tempting to consider the functions $\tilde{X}/2\lambda$ in order to define a representation of $\mathfrak{g}$ and of G by exponentiation. Let us be more precise. We now fix the chart of Proposition 6.1, put $p_0 = 1$ by convention and define the spaces:

$$A = \Bigg\{\text{polynomial functions } x = \sum_{l=0}^{L} \left(\frac{1}{2\lambda}\right)^l x_l, \\ \text{where } x = \sum_{i_1 \cdots i_l} \alpha_{i_1 \cdots i_l}(q) p_{i_1} \cdots p_{i_l}, \\ \alpha_{i_1 \cdots i_l} \text{ being a polynomial function} \\ \text{in the variables } q_j, j > \inf(i_1 \cdots i_l)\Bigg\};$$

$$B = \Bigg\{\text{formal power series } x = \sum_{l \geq 0} \left(\frac{1}{2\lambda}\right)^l x_l \\ \text{with the same conditions on } x_l\Bigg\}.$$

Now we take the Moyal $*$ product on our chart. First we prove the existence of a $*$ exponential.

THEOREM 7.1. (a) *For each* $X_1,\ldots,X_n$ *in* $\mathfrak{g}$, $(1/2\lambda)^n\tilde{X}_1 * \cdots * \tilde{X}_n$ *belongs to* A.

(b) *The coefficient of* $(1/2\lambda)^l$ *of this expression vanishes if* n *is sufficiently large.*

(c) *For each* X *in* $\mathfrak{g}$, *there exists a formal power series in* B:

$$e^{*\tilde{X}/2\lambda} = \sum_{n\geq 0} \frac{1}{n!}\left(\frac{1}{2\lambda}\right)^n \tilde{X}^{*n}.$$

Proof. (a) Starting from the relation

$$\frac{1}{2\lambda}\tilde{X} * x = \sum_{s\geq -1}\sum_{l=0}^{L}(2\lambda)^{-l+s}\frac{1}{2^{s+1}(s+1)!}P^{s+1}$$
$$\times\left(\sum_{i=0}^{k}\alpha_i(q)p_i,\ \sum_{i_1\cdots i_l}\alpha_{i_1\cdots i_l}(q)p_{i_1}\cdots p_{i_l}\right),$$

where X belongs to $\mathfrak{g}$ and x to A, we see that the only remaining terms satisfy $s < l$ and are of the form $b_{i_1\cdots i_l}\, p_{i_1}\cdots p_{i_l}$, $b_{i_1\cdots i_l}$, being a polynomial function in the variables $q_j, j > \inf(i_1\cdots i_{l'})$. $\tilde{X}/2\lambda * A$ is thus in A.

(b) Let r be the supremum of the degrees of α_i for all X, $X \in \mathfrak{g}$. We compute the coefficient of $(1/2\lambda)^l$ in our expression. We find a linear combination of terms of the form

$$D_1(\alpha_{j_1 1})\cdot D_2(\alpha_{j_2 2})\cdots D_n(\alpha_{j_n n})p_{i_1}\cdots p_{i_l},$$

where $\tilde{X}_i = \sum_j \alpha_{ji}(q)p_j$ and the D_i are differential operators with constant coefficients. Let i be the inf of $j_1\cdots j_n$. There does not exist a derivation in the variable q_i in $D_1,\ldots,D_n$; hence we do not derive by p_i (see the definition of P^r). Thus:

$$\text{number of } s \text{ such that } i_s = i \leq l.$$

Considering successively the variables $q_{i+1}, q_{i+2},\ldots,q_k$ we easily find:

$$\text{number of } s \text{ such that } i_s = i + t \leq l(r+1)^t,$$

where r is the supremum of $d^0\tilde{X}$ for all X in $\mathfrak{g}$. It means that

$$n \leq l\left[1 + (r+1) + \cdots + (r+1)^k\right],$$

which proves (b). (c) is an easy consequence of (b). In fact, our $*$ exponential is a group morphism.

THEOREM 7.2. (a) *Let G^* be the subset of B defined by*

$$G^* = \{x \in B\colon \exists X \text{ in } \mathfrak{g} \text{ such that } x = e^{*\tilde{X}/2\lambda}\}.$$

Then we can define the product $x * y$ of two elements of G^*, and G^* with that product is a group.

(b) *The map $\Phi\colon G \to G^*$ defined by*

$$\Phi(\exp X) = e^{*\tilde{X}/2\lambda}$$

is a morphism from G onto G^*.

Proof. Let $X_1, X_2, \ldots, X_n \in \mathfrak{g}$. In B the series

$$\sum_{m_1 \geq 0} \left(\frac{1}{2\lambda}\right)^{m_1} \frac{1}{m_1!} \tilde{X}_1^{*m_1} * \sum_{m_2 \geq 0} \left(\frac{1}{2\lambda}\right)^{m_2} \frac{1}{m_2!} \tilde{X}_2^{*m_2}$$
$$* \cdots * \sum_{m_n \geq 0} \left(\frac{1}{2\lambda}\right)^{m_n} \frac{1}{m_n!} \tilde{X}_n^{*n}$$

converges (as a formal power series). Indeed the coefficient of $(1/2\lambda)^l$ depends only on the beginning of each series $e^{*\tilde{X}_i/2\lambda}$. The subset C of B of all these expressions is thus stable for the $*$ product.

Let us now consider the map Φ. We shall prove that Φ is a morphism from G to C by induction on $\dim \mathfrak{g}$. Let $X_1 \cdots X_l$ be a Jordan Hölder basis of $\mathfrak{g}$. We define

$$\Psi(\exp t_1 X_1 \cdots \exp t_l X_l) = e^{*t_1\tilde{X}_1/2\lambda} * \cdots * e^{*t_l\tilde{X}_l/2\lambda}.$$

We suppose that Ψ is a morphism and $\Psi = \Phi$ if $\dim \mathfrak{g} < l$. We thus consider the subgroup G_1 of G with Lie algebra $\mathfrak{g}_1$ generated by $X_1 \cdots X_{l-1}$.

We verify directly the relation

$$e^{*\tilde{X}/2\lambda} * \tilde{Y} * e^{*-\tilde{X}/2\lambda} = \widetilde{\operatorname{Ad}\exp(X)(Y)} \qquad \forall X, Y \in \mathfrak{g}$$

with these hypotheses and compute

$$\begin{aligned}
&\Psi(\exp X \exp t_l X_l \exp X' \exp t_l' X_l) \\
&\quad = \Psi(\exp X \exp(\operatorname{Ad}\exp t_l X_l(X')) \exp(t_l + t_l') X_l) \\
&\quad = \Phi(\exp X) * \Phi(\exp(\operatorname{Ad}\exp t_l X_l X')) * e^{*(t_l+t_l')\tilde{X}_l} \\
&\quad = \Phi(\exp X) * e^{*t_l\tilde{X}_l} * \Phi(\exp X') * e^{*t_l' X_l} \\
&\quad = \Psi(\exp X \exp t_l X_l) * \Psi(\exp X' \exp t_l' X_l), \qquad X, X' \in \mathfrak{g}_1.
\end{aligned}$$

Finally if $X = \sum a_i X_i$ belongs to $\mathfrak{g}$, we can write

$$\exp tX = \exp \alpha_1(t) X_1 \cdots \exp \alpha_l(t) X_l,$$

where each α_i is a polynomial function and $\partial\alpha_i/\partial t|_{t=0} = a_i$, $\alpha_i(0) = 0$ ([**15**]). $\Psi(\exp tX)$ is thus a formal power series with polynomial coefficients in t. By differentiation,

$$\frac{\partial}{\partial t} \Psi(\exp X)|_{t=0} = \frac{1}{2\lambda} \tilde{X},$$

and since Ψ is a morphism,

$$\frac{\partial}{\partial t}\left(\Psi(\exp X) * e^{*-\tilde{X}/2\lambda}\right) = 0.$$

That proves that $\Psi = \Phi$ and $C = G^*$. G^* is thus a group for $*$.

REMARKS. (a) In §10 we prove that G^* does not depend on the choice of the chart 6.1.

(b) It is easy to add variables p_0, q_0 to W, and then consider direct products of orbits W_i in such a manner that each $\tilde{X}$ is nonconstant on $\prod_i W_i = W$. Thus on W our computation gives an isomorphism of groups between G and G^* and we are able to find the Campbell-Hausdorff formula.

8. A $*$ polarization and its representation.

DEFINITION 8.1. A subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ is subordinate to $\xi \in W$ if

$$\langle \xi, [\mathfrak{n}, \mathfrak{n}] \rangle = 0.$$

The unitary irreducible representation of G associated to W is constructed from such an algebra. We easily see that:

LEMMA 8.1. *With our notation, the algebra*

$$\mathfrak{n} = \{ X \in \mathfrak{g} \textit{ such that } \tilde{X}(p, 0) = \tilde{X}(0, 0)\ \forall p = (p_1 \cdots p_k)\}$$

is subordinate to $(0, 0)$ *and maximal.*

The reader can easily prove that lemma by induction on $\dim \mathfrak{g}$. We recall that k $(= \frac{1}{2} \dim W)$ is then the codimensionality of $\mathfrak{n}$ in $\mathfrak{g}$. Starting

with this subalgebra, adjusted to our chart, we define a $*$ polarization, following an idea of Fronsdal ([**4**]):

DEFINITION 8.2. We call polarization with respect to $\mathfrak{n}$ the space S of all x in B such that

$$x * \frac{1}{2\lambda}\tilde{X} = \frac{1}{2\lambda}\tilde{X}(0,0)\cdot x \qquad \forall X \in \mathfrak{n}.$$

In fact, we will define (see Theorem 8.1) a representation of G on S, G acting on the left as in induced representations. Let us determine S:

PROPOSITION 8.1. *If* $\mathfrak{n}$ *is the algebra defined in Lemma* 8.1, *then*:
(a) $x \in S$ *if and only if*

$$x * q_i = 0 \qquad \forall i = 1,2,\ldots,k.$$

(b) $x \in S$ *if and only if*

$$x = \sum_{r\geq 0}\left(\frac{1}{\lambda}\right)^{r+1}\sum_{s=0}^{r}\frac{(-1)^s}{s!}\left(\sum_{i=1}^{k}p_iq_i\right)^s a_{r-s}(q)\left(\frac{p_0}{2}\right)^{r-s+1}$$
$$= e^{-pq/\lambda}\sum_{t\geq 0}a_t(q)\left(\frac{p_0}{2\lambda}\right)^{t+1}.$$

Proof. (a) $\tilde{X} = q_k$ is in $\mathfrak{n}$ by construction ([**15**]). Thus $x * q_k$ has to vanish if x is in S. But there exists in $\mathfrak{n}$ an element

$$\tilde{X} = q_{k-1} + \sum_{l>0}(q_k)^l\sum_{i=1}^{k-1}\alpha_{li}(q)p_i,$$

where $\alpha_{li} \in \mathbf{R}[q_{i+1},\ldots,q_{k-1}]$. Then $x * q_{k-1}$ also vanish. Thus, we show inductively that for each x in S the relations

$$x * q_i = 0, \qquad \forall i = 1,\ldots,k,$$

hold. The converse is clear.

(b) Of course, the element

$$e^{-pq/\lambda}\sum_{t\geq 0}a_t(q)\left(\frac{p_0}{2\lambda}\right)^{t+1}$$

is in S. Conversely, the relations

$$x * q_j = \left(\sum_{n \geq 0} \lambda^{-n} x_n \right) * q_j = 0, \qquad \forall j = 1, \ldots, k,$$

are equivalent to

$$x_n q_j = -\frac{\partial x_{n+1}}{\partial p_j}, \quad n \geq 0, \qquad \forall j = 1, \ldots, k.$$

But they impose

$$x_0 = 0, \qquad x_1 = a_0(q) p_0,$$

and if we suppose the existence of $a_0, \ldots, a_{r-1}$ such that

$$x_r = \sum_{s=0}^{r-1} \frac{(-1)^s}{s!} \left(\sum p_i q_i \right)^s a_{r-s-1}(q) p_0^{r-s},$$

let us put

$$y_{r+1} = \sum_{s=1}^{r} \frac{(-1)^s}{s!} \left(\sum p_i q_i \right)^s a_{r-s}(q) p_0^{r-s+1}.$$

We obtain

$$\frac{\partial}{\partial p_j} (y_{r+1} - x_{r+1}) = 0 \quad \text{for all } j,$$

or

$$x_{r+1} = y_{r+1} + a_n(q) p_0^{r+1}. \qquad \square$$

THEOREM 8.1. *If x belongs to S and g to G, then $\Phi(g) * x$ is well defined and belongs to S. We can define on S a representation π of G by*

$$\pi(g) x = \Phi(g) * x.$$

Proof. Keeping all our notations, we compute $1/2\lambda \tilde{X} * x$ for x in S and $\tilde{X} = \sum_{t \geq 0} \alpha_i(q) p_i$:

$$\frac{1}{2\lambda} \tilde{X} * x = e^{-pq/\lambda} \sum_{t \geq 0} \left(\frac{p_0}{2\lambda} \right)^{t+1} \left(\frac{1}{2} \sum_{i > 0} \alpha_t(2q) \frac{\partial a_t}{\partial q_i} + \frac{p_0}{2\lambda} \alpha_0(2q) a_t \right).$$

From that, we deduce that $\tilde{X}/2\lambda * x$ is in S and the coefficient of $(1/2\lambda)^l$ in the series $\tilde{X}_1/2\lambda * \tilde{X}_2/2\lambda * \cdots * \tilde{X}_n/2\lambda * x$ vanishes if n is sufficiently large. $\Phi(\exp X) * x$ converges in the topology of formal power series and belongs to S by construction. π is of course a representation. □

Our next step is to intertwine π with the unitary irreducible representation associated to W. For that, we have to determine π more precisely.

LEMMA 8.2. *Let X be an element of $\mathfrak{g}$ such that*

$$\tilde{X} = \sum_{i \geq 0} \alpha_i(q) p_i.$$

Let us define on $\mathbf{R}^k$ the vector field

$$X^- = \sum_{i > 0} \alpha_i(2q) \frac{1}{2} \frac{\partial}{\partial q_i}.$$

Then X^- is a complete vector field and its flow $\exp -tX^-$ *has the form*

$$(\exp -tX^- \cdot q)_j = q_j - B_j(t; q_{j+1}, \ldots, q_k)$$

where B_j is polynomial.

Proof. Computing successively $(\exp -tX^- \cdot q)_j$ for $j = k, k-1, \ldots, 1$, we find

$$(\exp -tX^- \cdot q)_j = q_j - B_j(t; q_{j+1}, \ldots, q_k),$$

where

$$B_j(t; q_{j+1}, \ldots, q_k) = \int_0^t \frac{1}{2} \alpha_j\big(2(q_{j+1} - B_{j+1}(s; q)), \ldots, 2(q_k - B_k(s))\big)\, ds.$$

PROPOSITION 8.2. *Let*

$$x = e^{-pq/\lambda} \sum_{t \geq 0} a_t(q) \left(\frac{p_0}{2\lambda}\right)^{t+1}$$

be an element of S and X an element of $\mathfrak{g}$ such that

$$\tilde{X} = \sum_{i \geq 0} \alpha_i(q) p_i.$$

Then

$$\pi(\exp X)x = e^{-pq/\lambda}\exp\left\{\frac{p_0}{2\lambda}\int_0^1 \alpha_0(2\exp -sX^-\cdot q)\,ds\right\}$$
$$\times \sum_{t\geq 0} a_t(\exp -X^-\cdot q)\left(\frac{p_0}{2\lambda}\right)^{t+1}.$$

Let us remark that this last expression is well defined and in S, thanks to Lemma 8.2.

Proof. Put

$$\varphi(u,x) = e^{-pq/\lambda}\exp\left\{\frac{p_0}{2\lambda}\int_0^1 u\alpha_0(2\exp -suX^-\cdot q)\,ds\right\}$$
$$\times \sum_{t\geq 0} a_t(\exp -uX^-\cdot q)\left(\frac{p_0}{2\lambda}\right)^{t+1}, \qquad u\in \mathbf{R}.$$

We define a formal power series element of S. We directly see

$$\frac{\partial}{\partial u}\varphi(u,x) = \varphi\left(u, \frac{1}{2\lambda}\tilde{X} * x\right).$$

From that, we deduce

$$\frac{\partial}{\partial u}\varphi(u, \pi(\exp -uX)\cdot x) = 0 \qquad \text{for all } u.$$

Then

$$\varphi(1,x) = \varphi(1, \pi(\exp -X)\pi(\exp X)\cdot x)$$
$$= \varphi(0, \pi(\exp X)x) = \pi(\exp X)\cdot x. \qquad \square$$

9. The UIR U and the intertwining operator. First we recall the description of the unitary irreducible representation U associated to W.

PROPOSITION 9.1. *Let U be the unitary irreducible representation associated to W. Then*:

(a) *U is induced by the character χ on $N = \exp \mathfrak{n}$*:

$$\chi(\exp X) = e^{\sqrt{-1}\langle \xi, x\rangle} \quad \textit{where } \xi = (0,0).$$

(b) *The space H where U is defined is $L^2(\mathbf{R}^k)$ with variables q_i, $i = 1,\ldots,k$, and Lebesgue measure.*

(c) *The space H^∞ of C^∞ vectors for U is the Schwartz space $\mathscr{S}$ of C^∞ functions on $\mathbf{R}^k$, rapidly decreasing.*

(d) *For a good choice of variables q, if $X \in \mathfrak{g}$ is such that $\tilde{X} = \sum_{i \geq 0} \alpha_i(q) p_i$, we can write*

$$(U(\exp X)f)(q) = \exp\left\{\sqrt{-1}\int_0^1 \alpha_0(2\exp -sX^- \cdot q)\, ds\right\} f(\exp -X^- \cdot q)$$

(X^- is defined in Lemma 8.2).

Proof. (a)–(c) are well known (see [7] and [**15**] for instance). As usual, we prove (d) by induction on $\dim \mathfrak{g}$. The only nontrivial case is when $\operatorname{Ker} \mathfrak{w} = 0$ (see proof of Proposition 6.1), then U is induced by the representation U_1 of G_1 associated to W_1 ([**15**]). We introduce the variables $\tilde{q} = q_1, \ldots, q_{k-1}$ in $\mathbf{R}^{k-1}$. A function f in the space H of U is a function from $\mathbf{R}$ with variables q_k to $L^2(\mathbf{R}^{k-1})$ with variables $\tilde{q}$. We identify H to $L^2(\mathbf{R}^k)$. With this identification and the notations of the proof of Proposition 6.1, we have

$$\begin{aligned}[U(\exp X_1 \exp tX_0)f](q_k) &= U_1(\exp \operatorname{Ad} \exp -q_k X_0(X_1)) \cdot f(q_k - t) \\ &= (\text{multiplier})\, f([\exp -\operatorname{Ad} \exp -q_k X_0(X_1)]^- \cdot \tilde{q}, q_k - t).\end{aligned}$$

But we remark that

$$\begin{aligned}&[(\exp -tX_0^- \circ \exp -X_1^-)q]_j \\ &\quad = [[\exp -\operatorname{Ad} \exp -q_k X_0(X_1)]^- \tilde{q}]_j \quad \text{if } j < k \\ &\quad = q_k - t \qquad\qquad\qquad\qquad\qquad\qquad \text{if } j = k.\end{aligned}$$

These relations are proved for $j = k$, then for $j = k-1, k-2, \ldots, 1$.

Up to the multiplier, (d) is proved. But the multiplier comes from U_1. Its form is

$$\exp\left[\sqrt{-1}\int_0^1 \alpha_{\circ q_k}(2[\exp -s \operatorname{Ad} \exp -q_k X_0(X_1)]^- \tilde{q}]\, ds\right),$$

where, by definition, $\alpha_{\circ q_k}$ is the function of q:

$$\alpha_{\circ q_k}(q_1, \ldots, q_{k-1}) = [\operatorname{Ad} \exp -q_k X_0(X_1)](0, \ldots, 0, q_1, \ldots, q_{k-1}, 0),$$

i.e. $\alpha_{\circ q_k}(q_1, \ldots, q_{k-1}) = \alpha_0(q_1, \ldots, q_k)$. □

Now we define our intertwining operator T. Intuitively T "fixes $p_0/2\lambda = \sqrt{-1}$ and multiplies x by $e^{pq/\lambda}$".

Let us introduce the spaces D and V of S and $L^2(\mathbf{R}^k)$ which are, respectively, the domain of T and its range.

LEMMA 9.1. *Let V be the subspace of $L^2(\mathbf{R}^k)$ defined by*

$$V = \{ f \in L^2(\mathbf{R}^k) : f \text{ is the restriction to } \mathbf{R}^k \text{ of an entire function } F \text{ on } \mathbf{C}^k \}.$$

Let V_0 be $V \cap \mathscr{S}$. Then

(a) *V and V_0 are dense in $L^2(\mathbf{R}^k)$.*

(b) *V and V_0 are stable under the action of $U(\exp X)$, $\forall X \in \mathfrak{g}$.*

(c) *V_0 is stable under the action of $dU(X)$, $X \in \mathfrak{g}$, and dense in $\mathscr{S}$ for the topology of $\mathscr{S}$.*

Proof. (a) is obvious. Now we extend the operators $U(\exp X)$ to complex functions F by

$$(\overline{U}(\exp X)F)(z) = \exp\left\{\sqrt{-1}\int_0^1 \alpha_0(2\exp -sX^- \cdot z)\, ds\right\} F(\exp -X^- \cdot z),$$

where $\exp -X^- \cdot z$ is the polynomial function extending $q \hookrightarrow \exp -X^- \cdot q$. Then $\overline{U}(\exp X)F$ is entire if and only if F is entire and

$$\overline{U}(\exp X)F|_{\mathbf{R}^k} = U(\exp X)(F|_{\mathbf{R}^k}).$$

This proves (b) for V; for V_0, $\mathscr{S}$ is stable since it is H^∞.

(c) By construction the generator $dU(X)$ of the one parameter group $U(\exp tx)$ is a differential operator with polynomial coefficients. Then $dU(X)$ leaves V_0 invariant, and a classical result of Poulsen ([**14**]) proves (c).

LEMMA 9.2. *Let D (resp. D_0) be the subspace of S defined by the relations $x \in D$ (resp. D_0) if and only if there exists an integer n, $x_1, \ldots, x_n$ in S and $X_1, \ldots, X_n$ in $\mathfrak{g}$, such that:*

(1) $x = \sum_{i=1}^n x_i$.

(2)

$$\pi(\exp X_i)x_i = e^{-pq/\lambda} \sum_{t \geq 0} a_{ti}(q)\left(\frac{p_0}{2\lambda}\right)^{t+1},$$

where the polynomial functions a_{ti} are homogeneous with degree t.

(3) $\sum_{t \geq 0}(\sqrt{-1})^{t+1} a_{ti}(z)$ *is the development on $\mathbf{C}^k$ of an entire function F_i $\forall i = 1, \ldots, n$.*

(4) $f_i = F_{i|\mathbf{R}^k}$ *belongs to V (resp. V_0) $\forall i = 1, \ldots, n$. Then the vector of $L^2(\mathbf{R}^k)$,*

$$\sum_i U(\exp X_i) f_i,$$

depends on x only. In particular it does not depend on the chosen decomposition (1).

Proof. Let us extend U to the representation $\overline{U}$ on $\mathscr{S}'$ by putting

$$\langle \overline{U}(\exp X)T, f\rangle = \langle T, U(\exp -X)f\rangle \qquad \forall f \in \mathscr{S}, T \in \mathscr{S}'.$$

For each function φ on $\mathbf{R}^k$, C^∞ with compact support, we have

$$\left\langle \sum_i U(\exp -X_i)f_i, \varphi\right\rangle = \sum_i \langle f_i, U(\exp X_i)\varphi\rangle.$$

Now $U(\exp X_i)\varphi$ has compact support, the development of f_i converges uniformly on it, and we have

$$\begin{aligned}\left\langle \sum_i U(\exp -X_i)f_i, \varphi\right\rangle &= \sum_i \sum_{t\geq 0} \left\langle (\sqrt{-1})^{t+1} a_{ti}, U(\exp X_i)\varphi\right\rangle \\ &= \sum_{t\geq 0} \left\langle \sum_i (\sqrt{-1})^{t+1} U'(\exp -X_i) a_{ti}, \varphi\right\rangle.\end{aligned}$$

But we know U and π (see Propositions 9.1 and 8.2). Thus

$$(\sqrt{-1})^{t+1} \sum_i \overline{U}(\exp -X_i) a_{ti}(q) = (\sqrt{-1})^{t+1} b_t(q) \quad \text{for all } t,$$

where $b_t(q)$ is given by the development of x:

$$x = e^{pq/\lambda} \sum_{t\geq 0} \left(\frac{p_0}{2\lambda}\right)^{t+1} b_t(q).$$

Thus $\langle \Sigma_i U(exp -X_i)f_i, \varphi\rangle$ depends only on x. □

Let us preserve our notations. We are now able to prove

THEOREM 9.1.

(a) *D and D_0 are π stable subspaces of S.*

(b) *The operator T: $D \to V$, defined by*

$$Tx = \sum_i U(\exp -X_i)f_i,$$

is a linear intertwining of π and U:

$$T\pi(g) = U(g)T.$$

(c) *The range of T is V.*

(d) *The conclusions of* (b) *and* (c) *hold with $T_0 = T|_{T_0}$, D_0 and V_0, respectively.*

Proof. The restriction to D_0 is obvious.

(a) D is of course π stable since if x is in D,

$$\pi(\exp X)x = \sum_i \pi(\exp X)x_i,$$

and the $\pi(\exp X_i \exp -X)(\pi(\exp X)x_i)$ satisfy conditions (2)–(4) of Lemma 9.2.

(b) Lemma 9.2 defines T and, with that definition,

$$\begin{aligned} T\pi(\exp X)x &= \sum_i U(\exp X \exp -X_i)f_i \\ &= U(\exp X)Tx \qquad \forall x \in D, \forall X \in \mathfrak{g}. \end{aligned}$$

T is linear by construction.

(c) For each f in V, f is the restriction to $\mathbf{R}^k$ of $\sum_{t\geq 0} a_t(z)(\sqrt{-1})^{t+1}$ and thus

$$f = Tx,$$

where x is the element of D:

$$x = e^{-pq/\lambda} \sum_{t\geq 0} \left(\frac{p_0}{2\lambda}\right)^{t+1} a_t(q).$$

10. Unicity. Until now we only considered a given chart for W. In this section we study the dependence of our construction on the choice of the chart in Proposition 6.1. First, we prove that G^* is, up to isomorphism, unique. More precisely:

PROPOSITION 10.1. *Let $\xi \hookrightarrow (p, q)$ and $\xi \hookrightarrow (p', q')$ be two charts of W constructed as in Proposition* 6.1, *and let G^* and $G^{*\prime}$ be the two groups defined from these charts* (*see Theorem* 7.2). *Then the correspondence*

$$\Psi\colon G^* \to G^{*\prime}, \qquad \Psi(e^{*\tilde{X}/2\lambda}) = e^{*\prime\tilde{X}/2\lambda},$$

is a group isomorphism.

Proof. Let us denote by Φ, π, U and T the same objects as in previous sections for the first chart. In particular, Φ is the group morphism Φ: $G \to G^*$ defined by $\Phi(\exp X) = e^{*\tilde{X}/2\lambda}$.

Let us determine the kernel of Φ. If $\Phi(\exp X) = 1$ we can write

$$\Phi(\exp tX) = \sum_{l\geq 0} \left(\frac{1}{2\lambda}\right)^l x_l(t),$$

where $x_l(t)$ is a polynomial function, since x_l depends only on the first terms of the expansion of $\Phi(\exp X)$. For each integer m, $x_l(m)$ vanishes, since

$$\Phi(\exp mX) = \Phi(\exp X) * \Phi(\exp X) * \cdots * \Phi(\exp X).$$

Then $\Phi(\exp tX)$ is identically equal to 1. Now for each x in D,

$$T\pi(\exp tX)x = T(\Phi(\exp tX) * x) = Tx = U(\exp tX)Tx.$$

Thus Tx belongs to the domain of $dU(X)$ and $dU(X)Tx$ vanishes. That means, since V is dense in $L^2(\mathbf{R}^k)$ and $dU(X)$ is a closed operator, that

$$dU(X) \equiv 0.$$

Conversely, if $dU(X)$ is the null operator, it vanishes identically on $\mathscr{S}$ and on $\mathscr{S}'$ by duality. Concerning the function $1 \in \mathscr{S}'$, if $\tilde{X} = \sum \alpha_i(q)p_i$, we have

$$dU(X) \cdot 1 = \alpha_0(2q) \equiv 0.$$

In the same way, concerning the function $q_i \in \mathscr{S}'$, $i = 1,\ldots,k$, we find

$$dU(X)q_i = \tfrac{1}{2}\alpha_i(2q) \equiv 0.$$

$\tilde{X}$ is the null function on W, and $\Phi(\exp X)$ is the function 1. Let us now consider the kernel of our representation U:

$$\operatorname{Ker} U = \{g \in G\colon U(g) = \mathrm{Id}\}.$$

We saw that $\operatorname{Ker}\Phi \subset \operatorname{Ker} U$. But the converse is also true since if g belongs to $\operatorname{Ker} U$, $\overline{U}(g)$ defined on $\mathscr{S}'$ is the identity operator and

$$\overline{U}(g) \cdot 1 = 1$$

means that our multiplier is 1.

$$\overline{U}(g) \cdot q_i = q_i$$

means that $g = \exp X$, where the vector field X^- vanishes identically. Thus

$$g = \exp X \quad \text{where } \tilde{X} = 0.$$

We are now able to define an isomorphism Φ^* from $G/\operatorname{Ker} U$ to G^*. But for our second chart, the same holds: there exists an isomorphism $\Phi^{*\prime}$ from $G/\operatorname{Ker} U$ to $G^{*\prime}$. The proof is now complete since, by construction,

$$\Psi = \Phi^{*\prime} \circ \Phi^{*-1}. \qquad \square$$

The following example shows that π is not uniquely defined. Of course the choice of a polarization in order to define S and π implies the nonuniqueness.

LEMMA 10.1. *Let $\mathfrak{g}$ be the three-dimensional Heisenberg algebra: $\mathfrak{g}$ is generated by X, Y and Z with commutation relations $[X, Y] = Z$. Let W be the two dimensional orbit in $\mathfrak{g}^*$ such that $\tilde{Z} = 1$. Then we choose the first chart*

$$\tilde{X} = p, \qquad \tilde{Y} = q,$$

and the second chart

$$\tilde{X} = q', \qquad \tilde{Y} = -p'.$$

We keep our above notations with "primes" for the second chart. Then if $x_0 = e^{-pq/\lambda} p_0/2\lambda \in S$ we have

$$\pi(\exp tX)x_0 = x_0 \quad \text{for all } t,$$

and in S' there does not exist an element x_0' such that

$$\pi'(\exp tX)x_0' = x_0' \qquad (t \neq 0).$$

Thus there does not exist an isomorphism between S and S' intertwining π and π'.

Proof. We recall that

$$S = \left\{ x \in B \colon x = e^{-pq/\lambda} \sum_{t \geq 0} \left(\frac{p_0}{2\lambda} \right)^{t+1} a_t(q) \right\}$$

and

$$S' = \left\{ x \in B \colon x' = e^{-p'q'/\lambda} \sum \left(\frac{p_0}{2\lambda} \right)^{t+1} a_t(q') \right\}.$$

Thus

$$\pi(\exp \tau X)x = e^{-pq/\lambda} \sum_{t \geq 0} \left(\frac{p_0}{2\lambda} \right)^{t+1} a_t(q - \tau),$$

$$\pi'(\exp \tau X)x = e^{-p'q'/\lambda} e^{(-p_0/2\lambda)2q'\tau} \sum_{t \geq 0} \left(\frac{p_0}{2\lambda} \right) a_t(q').$$

The stabilizer of x_0 contains $\exp \tau X$ for each τ, but the equation

$$\pi'(\exp \tau X)x = x \qquad (\tau \neq 0)$$

implies

$$\pi'(\exp \tau X)x = x \quad \forall \tau \qquad \text{and} \qquad \frac{p_0}{2\lambda} 2q' \cdot x = 0. \qquad \square$$

In fact the intertwining operator between U and U' is the Fourier transform and constants in $\mathscr{S}'$ are transformed in Dirac measures.

Nevertheless, we can define an intertwining operator between the two $\mathfrak{g}$ modules S and S'. Indeed in V_0 there exist some very particular functions (see [**1**]). As usual, we denote by π, U, S, V_0 the representations and space defined with a chart (p, q) and by π', U', S', V_0' those associated to another chart (p', q'). The representations U, π, U', π' give rise by differentiation to representations $dU, d\pi, \ldots$ of $\mathfrak{g}$ and of its enveloping algebra $\mathscr{U}(\mathfrak{g})$. We define

$$f_0(q) = \exp\left\{-\sum q_j^2 + e^{-\Sigma q_j^2}\right\} \in V_0.$$

f_0' is defined similarly in V_0' with variables q'. x_0 is the element of S defined by the development of the entire function f_0: $f_0 = Tx_0$. x_0' is in S' with a similar definition.

PROPOSITION 10.2.

(a) *The* $\mathfrak{g}$ *modules* $dU(\mathscr{U}(\mathfrak{g}))\, f_0$ *and* $\Delta_0 = d\pi(\mathscr{U}(\mathfrak{g}))\, x_0$ *are isomorphic to*

$$\mathscr{U}(\mathfrak{g})/\operatorname{Ker} dU.$$

(b) *The operator*

$$\theta\colon \Delta_0 \to \Delta_0', \qquad d\pi(u)x_0 \hookrightarrow d\pi'(u)x_0',$$

is an intertwining operator of $\mathfrak{g}$ *modules.*

Proof. The proof is very easy. In fact in [**1**] it is proved that

$$dU(u)f_0 = 0 \quad \text{implies} \quad dU(u) \equiv 0.$$

Thus with T, the same property holds for x_0 and $d\pi$, and of course for x_0' and $d\pi'$. This proves that θ is a well-defined isomorphism between Δ_0 and Δ_0'. □

Proposition 10.2 is somewhat unsatisfactory because as a $\mathfrak{g}$-module, S is too rich. We can find in it many other $\mathfrak{g}$-submodules equivalent to the corresponding submodule in S'.

Finally, let us remark that the notion of $*$ product is a geometrical one and depends essentially on W itself, not on the choice of the duality which allows us to consider W as an orbit in $\mathfrak{g}^*$. Thus the notion of the group G^* is the canonical one. Polarizations are introduced in this context only in order to find (more or less) irreducible representations of G. As usual, see [**4**] for instance, we are able from the orbit W to construct all

unitary irreducible representations in $L^2(\mathbf{R}^k)$ with the same kernel. More precisely, let us choose such a representation U' induced by a character χ' of the subgroup N'. Let W' be the associated orbit, $\tilde{X}'$ the function $\tilde{X}$ computed on W'. We have

$$e^{\sqrt{-1}\,\tilde{X}'(0,0)} = \chi'(\exp X) \qquad \forall X \in \mathfrak{n}',$$

the Lie algebra of N'. Since $\operatorname{Ker} U' = \operatorname{Ker} U$, we see, as in the proof of Proposition 10.1, that

$$\{X \in \mathfrak{g} : \tilde{X} \equiv 0 \text{ on } W\} = \{X \in \mathfrak{g} : X' \equiv 0 \text{ on } W'\}.$$

Thus, the groups G^* and $G^{*\prime}$ are isomorphic, and we define on W the polarization

$$S'' = \{x \in B : \tilde{X} * x = \tilde{X}'(0,0) \cdot x \; \forall X \in \mathfrak{n}'\}.$$

We define a representation π'' on W and an intertwining operator between π'' and U' as in §9.

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UNIVERSITÉ DE DIJON
BP 138
21004-DIJON, FRANCE