PACIFIC JOURNAL OF MATHEMATICS
Vol 115, No. 2, 1984

NONCOINCIDENCE INDEX OF MANIFOLDS

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For a connected topological manifold M we define the noncoincidence index of M, a topological invariant reflecting the abundance of fixed-point-free self-maps of M. We give some theorems on noncoincidence index and compute the noncoincidence index of the homogeneous manifold $U(n)/H$, where H is conjugate to $U(1)^k \times U(n-k)$.

1. Introduction. Let M be a manifold (connected locally Euclidean Hausdorff space). We define the *noncoincidence index* of M, $\#M$, as follows. If M admits k fixed-point-free self-maps, no pair of which has a coincidence, set $\#M \geq k+1$. If $\#M \geq i$ for all i, put $\#M = \infty$; otherwise, $\#M$ is the greatest number i with $\#M \geq i$. (This definition is inspired by **[5]**.)

Evidently a manifold has noncoincidence index 1 if and only if it has the fixed-point property. On the other hand, if a group G acts freely on M, then $\#M \geq \operatorname{card} G$ if G is finite and $\#M = \infty$ if G is infinite. In particular, any connected nontrivial Lie group has noncoincidence index ∞.

As we see in §2, many manifolds besides Lie groups have noncoincidence index ∞. In §3 we show how the Lefschetz coincidence theorem can be used to put a finite upper bound on $\#M$ for certain compact oriented manifolds M. These results are used in §4 to compute the noncoincidence index of the homogeneous space $U(n)/H$ for H conjugate to $U(1)^k \times U(n-k)$. Section 5 is devoted to proving a classification theorem for endomorphisms of $H^*(U(n)/H; \mathbf{Q})$ which is needed in §4.

I thank my colleague W. Homer for greatly improving Lemma 5.3, and I thank A. Dold for some helpful observations.

2. Sufficient conditions for $\#M = \infty$. In this section we give some sufficient conditions for a manifold M to have $\#M = \infty$. The following result gives some easily checked homological conditions.

THEOREM 2.1. *Let M be a compact manifold. Then $\#M = \infty$ if either of the following is true*:

1. *M has nonzero first Betti number, or*
2. $\chi(M) = 0$.

Proof. For (1), see Corollary 5.1 of [**5**]. Now suppose $\chi(M) = 0$. By [**4**], there is a map s: $[0,1] \times M \to M$ with $s(0,\cdot) = \mathrm{id}_M$ and $s(t,\cdot)$: $M \to M$ fixed-point-free for $t > 0$. Let d be a metric for M, and set

$$N(t) = \inf_{x\in M} d(s(t,x),x), \qquad F(t) = \sup_{x\in M} d(s(t,x),x).$$

Then $F(t) \geq N(t) > 0$ for $t > 0$, and $F(t)$, $N(t) \to 0$ as $t \to 0$. Choose $0 < t_k < t_{k-1} < \cdots < t_1 \leq 1$ so that $F(t_i) < N(t_{i-1})$: then

$$x \to s(t_i,x), \qquad 1 \leq i \leq k,$$

is a set of k fixed-point-free, noncoincident maps. Since we can do this for any k, $\#M = \infty$.

From the preceding result, we see that any odd-dimensional compact manifold has noncoincidence index ∞. It also follows that $\#M = \infty$ for any compact surface M, except $M = S^2$ and $M = \mathbf{R}P^2$ (of course $\#\mathbf{R}P^2 = 1$, and we see in the next section that $\#S^2 = 2$).

The next result gives another useful sufficient condition for $\#M = \infty$.

THEOREM 2.2. *Let M be a compact manifold which admits a fixed-point-free nonsurjective self-map. Then $\#M = \infty$.*

Proof. Let f: $M \to M$ be fixed-point-free and nonsurjective. By Theorem 1.11 of [**3**], there is a path field nonsingular on the image of f, i.e. a map s: $[0,1] \times M \to M$ such that $s(0,\cdot) = \mathrm{id}_M$ and $s(t,\cdot)$ fixes no point of $f(M)$ for $t > 0$. Let d be a metric for M and take $\varepsilon > 0$ so that $d(f(x),x) \geq \varepsilon$ for $x \in M$. Then there is some $t_0 > 0$ so that

$$\sup_{x\in M} d(s(t,x),x) < \varepsilon$$

for $t < t_0$. Now proceed as in the proof of 2.1; set

$$N(t) = \inf_{x\in f(M)} d(s(t,x),x), \qquad F(t) = \sup_{x\in f(M)} d(s(t,x),x)$$

(note $f(M)$ is compact) and choose $0 < t_k < t_{k-1} < \cdots < t_1 < t_0$ such that $F(t_i) < N(t_{i-1})$. Then there are k fixed-point-free noncoincident self-maps of M given by

$$x \to s(t_i, f(x)), \qquad 1 \leq i \leq k.$$

Since k is arbitrary, $\#M = \infty$.

3. The Lefschetz coincidence theorem. In this section we show how the Lefschetz coincidence theorem can be used to put a finite upper bound on the noncoincidence index in some cases. As we see in the next section, such an upper bound combined with constructions of fixed-point-free maps often gives the noncoincidence index exactly.

Throughout this section, M will be a compact oriented n-manifold. We shall use the following version of the Lefschetz coincidence theorem: for a more general statement, see [**9**].

THEOREM 3.1. *For maps f, g: $M \to M$, set*

$$L(f, g) = \sum_{i=0}^{n} (-1)^i \operatorname{Tr}\left(\Phi_i^{-1} g_* \Phi_i f^*\right),$$

where Φ_i: $H^i(M; \mathbf{Q}) \to H_{n-i}(M; \mathbf{Q})$ is the Poincaré duality isomorphism. If $L(f, g) \neq 0$, then f and g have a coincidence.

REMARKS. 1. It is immediate that $L(f, \mathrm{id}) = L(f)$, the ordinary Lefschetz number of f, so this result implies the Lefschetz fixed-point theorem for M.

2. It follows from properties of trace that $L(f, g) = (-1)^n L(g, f)$.

Let g be a self-map of M. We define the degree of g by $g_*[M] = (\deg g)[M]$, where $[M] \in H_n(M; \mathbf{Q})$ is the fundamental class of M. The following result is useful in computing the Lefschetz coincidence number.

PROPOSITION 3.2. *If g is a self-map of M with $\deg g \neq 0$, then g^*: $H^*(M; \mathbf{Q}) \to H^*(M; \mathbf{Q})$ has an inverse $\bar{g}^*$ and*

$$L(f, g) = (\deg g) L(\bar{g}^* f^*)$$

for any other self-map f of M.

Proof. If $\deg g \neq 0$, it follows from consideration of Poincaré duality that g^* is injective. Then g^* is an automorphism, since each vector space $H^i(M; \mathbf{Q})$ is finite-dimensional. For $u \in H^i(M; \mathbf{Q})$,

$$\begin{aligned}\Phi_1^{-1} g_* \Phi_i f^*(u) &= \Phi_i^{-1} g_*\left(g^* \bar{g}^* f^*(u) \cap [M]\right) \\ &= \Phi_i^{-1}\left(\bar{g}^* f^*(u) \cap g_*[M]\right) = (\deg g) \bar{g}^* f^*(u),\end{aligned}$$

and the conclusion follows from the definition of $L(f, g)$.

By Theorem 3.1, any fixed-point-free self-map f of M must have $L(f) = 0$, and any pair f, g of self-maps without a coincidence must have $L(f, g) = 0$. We put

$$LZ(M) = \{f^* | f: M \to M \text{ and } L(f) = 0\}$$

and say f^*, $g^* \in LZ(M)$ are *compatible* if $L(f, g) = 0$. If $\chi(M) \neq 0$ and $LZ(M)$ consists of automorphisms of $H^*(M; \mathbf{Q})$, we call M *L-rigid*. We then have the following result.

PROPOSITION 3.3. *Suppose M is L-rigid. If $\#M \geq k + 1$, then $LZ(M)$ contains a subset of k pairwise compatible elements.*

Proof. By the hypothesis, there is a set S of k pairwise noncoincident fixed-point-free self-maps of M. Let f, $g \in S$. Then f^* and g^* are compatible elements of $LZ(M)$. We have $f^* \neq g^*$, since otherwise

$$L(f, g) = L(f, f) = (\deg f)L(\mathrm{id}) = (\deg f)\chi(M) \neq 0.$$

Thus, $\{f^* | f \in S\}$ is a set of k pairwise compatible elements of $LZ(M)$.

REMARK. Note that if M is L-rigid, then any pair f^*, $g^* \in LZ(M)$ is compatible if and only if $L(\bar{g}^* f^*) = 0$.

It follows immediately from 3.3 that

$$\#M \leq \operatorname{card} LZ(M) + 1 \tag{1}$$

when M is L-rigid and $LZ(M)$ is finite. Thus we have, e.g., $\#S^{2n} \leq 2$ for any even sphere S^{2n} (and in fact $\#S^{2n} = 2$, since the antipodal map is fixed-point-free). As we see in the next section, however, 3.3 sometimes gives a sharper upper bound than (1).

4. Noncoincidence index of some flag manifolds. Let $F(1^k, n)$ denote the homogeneous space $U(n + k)/H$, where H is conjugate to $U(1)^k \times U(n)$. (We can assume $n = 0$ or $n \geq 2$: in the former case we write $F(1^k)$ instead of $F(1^k, 0)$.) It is proved in [7] that $\#F(1^k) = k!$. In this section we compute $\#F(1^k, n)$ for all k and n.

The manifold $F(1^k, n)$ can be thought of as in the space of k-tuples of orthogonal lines in $\mathbf{C}^{n+k}$. Thus, there are maps

$$\pi_i \colon F(1^k, n) \to \mathbf{C}P^{n+k-1}, \qquad 1 \leq i \leq k,$$

given by picking out the ith line. If we let $t \in H^2(\mathbf{C}P^{n+k-1}; \mathbf{Q})$ be the first Chern class of the canonical line bundle over $\mathbf{C}P^{n+k-1}$ and put $t_i = \pi_i^*(t)$, we have the following description of $H^*(F(1^k, n); \mathbf{Q})$ [**1**].

$$H^*(F(1^k, n); \mathbf{Q}) = \mathbf{Q}[t_1, t_2, \ldots, t_k]/\{h_i | n + 1 \leq i \leq n + k\},$$

where h_i is the ith complete symmetric function in $t_1, t_2, \ldots, t_k$, i.e.

$$h_i = \sum_{p_1 + \cdots + p_k = i} t_1^{p_1} t_2^{p_2} \cdots t_k^{p_k}.$$

There is a free action of the symmetric group Σ_k on $F(1^k, n)$ by permutation of lines, and this action evidently permutes the t_i in cohomology.

For any $m \in \mathbf{Q}$ and $\sigma \in \Sigma_k$, let h_m^σ denote the endomorphism of $H^*(F(1^k, n); \mathbf{Q})$ given by

$$h_m^\sigma(t_i) = mt_{\sigma(i)}.$$

The following classification theorem for endomorphisms of $H^*(F(1^k, n); \mathbf{Q})$ is proved in §5.

THEOREM 4.1. *Unless $k = 2$ and n is a positive even number, all endomorphisms of $H^*(F(1^k, n); \mathbf{Q})$ are of the form h_m^σ for some $m \in \mathbf{Q}$ and $\sigma \in \Sigma_k$. If $k = 2$ and $n \geq 2$ is even, the only additional endomorphisms are*

$$t_i \to (-1)^i mt_q, \qquad i = 1, 2,$$

for $q \in \{1, 2\}$ and $m \in \mathbf{Q}$.

The next result gives a formula for $L(h_m^\sigma)$.

THEOREM 4.2. *Let $\lambda_1 \geq \lambda_2 \geq \cdots$ be the cycle-type of $\sigma \in \Sigma_k$ (so λ_1 is the length of the longest cycle in σ, λ_2 is the length of the next longest cycle, etc.). Then h_m^σ: $H^*(F(1^k, n); \mathbf{Q}) \to H^*(F(1^k, n); \mathbf{Q})$ has Lefschetz number*

$$L(h_m^\sigma) = \frac{(1 - m^{n+1})(1 - m^{n+2}) \cdots (1 - m^{n+k})}{(1 - m^{\lambda_1})(1 - m^{\lambda_2}) \cdots}. \tag{1}$$

Proof. For h_m^σ: $H^*(F(1^k, n); \mathbf{Q}) \to H^*(F(1^k, n); \mathbf{Q})$, let $P_{k,n}(\sigma, m)$ denote $L(h_m^\sigma)$. From [7] we have the formula

$$P_{k,0}(\sigma, m) = \frac{(1 - m)(1 - m^2) \cdots (1 - m^k)}{(1 - m^{\lambda_1})(1 - m^{\lambda_2}) \cdots}. \tag{2}$$

Now the spectral sequence of the fibration

$$F(1^k) \to F(1^k, n) \to G_k(\mathbf{C}^{n+k}),$$

where $G_k(\mathbf{C}^{n+k})$ is the Grassmannian of k-planes in $\mathbf{C}^{n+k}$, collapses for degree reasons. Thus

$$H^*(F(1^k, n); \mathbf{Q}) \cong H^*(F(1^k); \mathbf{Q}) \otimes H^*(G_k(\mathbf{C}^{n+k}); \mathbf{Q})$$

additively. Now $H^*(G_k(\mathbf{C}^{n+k}); \mathbf{Q})$ can be regarded as the invariant subring of $H^*(F(1^k, n); \mathbf{Q})$ under the Σ_k-action, and $H^*(F(1^k); \mathbf{Q})$ is a quotient of $H^*(F(1^k, n); \mathbf{Q})$ (the projection is the obvious map sending $t_i \in H^2(F(1^k, n); \mathbf{Q})$ to $t_i \in H^2(F(1^k); \mathbf{Q})$). Any endomorphism h_m^σ of

$H^*(F(1^k, n); \mathbf{Q})$ restricts to the endomorphism of $H^*(G_k(\mathbf{C}^{n+k}); \mathbf{Q})$ which multiplies dimension $2i$ by m^i, and gives rise to the corresponding h_m^σ on $H^*(F(1^k); \mathbf{Q})$. Since trace is multiplicative on tensor products,

$$(3) \qquad P_{k,n}(\sigma, m) = P_{k,0}(\sigma, m) \sum_{i \geq 0} m^i \dim H^{2i}\big(G_k(\mathbf{C}^{n+k}); \mathbf{Q}\big).$$

It is well known (see e.g. [1]) that

$$\sum_{i \geq 0} m^i \dim H^{2i}\big(G_k(\mathbf{C}^{n+k}); \mathbf{Q}\big) = \frac{(1 - m^{n+1})(1 - m^{n+2}) \cdots (1 - m^{n+k})}{(1 - m)(1 - m^2) \cdots (1 - m^k)},$$

and this together with (2) and (3) implies the conclusion.

We can use Theorems 4.1 and 4.2 to show that many of the manifolds $F(1^k, n)$ are L-rigid.

PROPOSITION 4.3. *Suppose $k \neq 2$, n is odd, or $n = 0$. Then $F(1^k, n)$ is L-rigid. Further, if Γ is the set of products of $[k/2]$ disjoint transpositions in Σ_k, then*

$$LZ\big(F(1^k, n)\big) = \begin{cases} \{h_1^\sigma | \sigma \neq \mathrm{id}\} \cup \{h_{-1}^\sigma | \sigma \in \Sigma_k\}, & kn \text{ odd}, \\ \{h_1^\sigma | \sigma \neq \mathrm{id}\} \cup \{h_{-1}^\sigma | \sigma \notin \Gamma\}, & kn \text{ even}. \end{cases}$$

Proof. By 4.1, an element of $LZ(F(1^k, n))$ must be an endomorphism h_m^σ of $H^*(F(1^k, n); \mathbf{Q})$ with $L(h_m^\sigma) = 0$. The above list follows from consideration of (1) (it is easy to see that any h_m^σ, $m = \pm 1$, is induced by a self-map of $F(1^k, n)$). Now

$$\chi\big(F(1^k, n)\big) = k!\binom{n+k}{k}$$

and clearly h_m^σ is an automorphism for $m \neq 0$ (in fact $\deg h_m^\sigma = m^d \operatorname{sgn} \sigma$, where $d = \dim_{\mathbf{C}} F(1^k, n)$), so $F(1^k, n)$ is L-rigid.

Now we can give an upper bound for $\#F(1^k, n)$ when n and k satisfy the hypothesis of the preceding result.

PROPOSITION 4.4. *Suppose $k \neq 2$, n is odd, or $n = 0$. Then $\#F(1^k, n) \leq k!$ if kn is even, and $\#F(1^k, n) \leq 2k!$ if kn is odd.*

Proof. The statement about $\#F(1^k, n)$ for kn odd follows immediately from 3.3, since $LZ(F(1^k, n))$ has $2k! - 1$ elements by 4.3. Again by 3.3, to prove the statement about $\#F(1^k, n)$ for kn even it suffices to show that any set of pairwise compatible elements of $LZ(F(1^k, n))$ has at most $k! - 1$ elements in this case. Suppose kn even and let $S \subset LZ(F(1^k, n))$ be a set of compatible elements. Let

$$H = \{\sigma \in \Sigma_k | h_1^\sigma \in S\} \cup \{\mathrm{id}\},$$

$$K = \{\sigma \in \Sigma_k | h_{-1}^\sigma \in S\}.$$

Then card S = card H + card $K - 1$. Since

$$L(h_{-1}^{\tau}, h_1^{\sigma}) = (\deg h_1^{\sigma}) L\left(h_1^{\sigma^{-1}} h_{-1}^{\tau}\right) = \pm L\left(h_{-1}^{\sigma^{-1}\tau}\right),$$

we must have $\sigma^{-1}\tau \notin \Gamma$ for $\tau \in K$ and $\sigma \in H$ (here Γ is as in 4.3). But then, if we take $\rho \in \Gamma$, we have $\sigma\rho \notin K$ for every $\sigma \in H$: hence card H + card $K \le k!$, and the conclusion follows.

Next we show that the inequalities of 4.4 are equalities. To do this, we construct fixed-point-free, noncoincident maps. For any even number $2r$, define J: $\mathbf{C}^{2r} \to \mathbf{C}^{2r}$ by

$$J(z_1, z_2, \ldots, z_{2r-1}, z_{2r}) = (-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{2r}, \bar{z}_{2r-1}).$$

Then J is a conjugate-linear map of $\mathbf{C}^{2r}$ with $J^2 = -\mathrm{id}$. Under the evident identification $\mathbf{C}^{2r} \cong \mathbf{H}^r$, we can regard J as multiplication by the quaternion j. Any subspace of $\mathbf{C}^{2r}$ invariant under J can be given the structure of a quaternionic vector space, and thus must be even-dimensional (cf. the proof of Theorem 1 of [**6**]). Further, if $\langle\ ,\ \rangle$ denotes inner product,

$$\langle Jv, Jw \rangle = \langle w, v \rangle \quad \text{for } v, w \in \mathbf{C}^{2r}. \tag{4}$$

Thus J preserves orthogonality.

THEOREM 4.5. *Unless $k = 2$ and n is a positive even number,*

$$\#F(1^k, n) = \begin{cases} 2k!, & kn \text{ odd}, \\ k!, & kn \text{ even}. \end{cases}$$

Proof. Since there is a free Σ_k-action on $F(1^k, n)$ (i.e., permutation of lines), we have $\#F(1^k, n) \ge k!$; together with 4.4, this disposes of the case kn even. Now suppose kn is odd. Then $n + k$ is even, and we have the map J: $\mathbf{C}^{n+k} \to \mathbf{C}^{n+k}$ defined above. Consider the $2k! - 1$ self-maps of $F(1^k, n)$ defined by

$$(l_1, l_2, \ldots, l_k) \to (l_{\pi(1)}, l_{\pi(2)}, \ldots, l_{\pi(k)}), \qquad \pi \in \Sigma_k - \{\mathrm{id}\} \tag{5}$$

and

$$(l_1, l_2, \ldots, l_k) \to (Jl_{\pi(1)}, Jl_{\pi(2)}, \ldots, Jl_{\pi(k)}), \qquad \pi \in \Sigma_k. \tag{6}$$

We claim these maps are fixed-point-free and pairwise noncoincident. Clearly the maps in (5) are fixed-point-free and pairwise noncoincident, and the maps in (6) are pairwise noncoincident. Suppose now we have a fixed point of a map in (6) or a coincidence between a map in (5) and one in (6), i.e., an element $(l_1, l_2, \ldots, l_k)$ of $F(1^k, n)$ with

$$l_{\pi(i)} = Jl_{\sigma(i)}, \qquad 1 \le i \le k,$$

for some $\pi, \sigma \in \Sigma_k$. Then J fixes $l_1 \oplus l_2 \oplus \cdots \oplus l_k$. But this is impossible, since J cannot fix an odd-dimensional subspace of $\mathbf{C}^{n+k}$.

Finally, we dispose of the case $k = 2$ and $n \geq 2$ even.

THEOREM 4.6. *If $n \geq 2$ is even, then $\#F(1^2, n) = \infty$.*

Proof. Let $J\colon \mathbf{C}^{n+2} \to \mathbf{C}^{n+2}$ be as defined above. Note that for $v \in \mathbf{C}^{n+2}$,

$$\langle Jv, v \rangle = \langle Jv, J^2 v \rangle = -\langle Jv, v \rangle$$

by (4) above; thus $\langle Jv, v \rangle = 0$, and Jl is orthogonal to l for any line l. Define $\psi\colon F(1^2, n) \to F(1^2, n)$ by $\psi(l_1, l_2) = (Jl_1, l_1)$: then ψ is fixed-point-free and nonsurjective, and the conclusion follows by 2.2.

5. Proof of the endomorphism theorem. This section is devoted to a proof of Theorem 4.1. We use the notation of the previous section.

Since $t_i \in H^2(F(1^k, n); \mathbf{Q})$ is pulled back from $H^2(\mathbf{C}P^{n+k-1}; \mathbf{Q})$, we have $t_i^{n+k} = 0$. The next result gives a converse: it is proved in **[8]** for $k \leq n$, and in **[2]** without restriction.

THEOREM 5.1. *If $u \in H^2(F(1^k, n); \mathbf{Q})$ and $u^{n+k} = 0$, then u is of the form at_i for some $a \in \mathbf{Q}$ and $1 \leq i \leq k$.*

Now suppose f is an endomorphism of $H^*(F(1^k, n); \mathbf{Q})$. Then $f(t_i)^{n+k} = f(t_i^{n+k}) = 0$ for $1 \leq i \leq k$, and it follows from 5.1 that

$$f(t_i) = m_i t_{p(i)}, \qquad 1 \leq i \leq k,$$

for some function $p\colon \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$ and rational numbers m_i. We shall prove that p is a permutation and all the m_i are equal unless $k = 2$ and n is even.

First we prove a technical lemma. The expression $h_i(x_1, x_2, \ldots, x_r)$ denotes the ith complete symmetric function in $x_1, x_2, \ldots, x_r$.

LEMMA 5.2. *Let $n \geq 0$ be an integer, $a_1, a_2, \ldots, a_r$ real numbers, and suppose*

$$h_{n+i}(a_1, a_2, \ldots, a_r) = 0, \qquad 1 \leq i \leq r - 1.$$

Then unless $r = 2$ and n is even, $a_1 = a_2 = \cdots = a_r = 0$. If $r = 2$ and n is even, $a_2 = -a_1$.

Proof. Suppose first that $r = 2$. Then we have

$$(1) \qquad h_{n+1}(a_1, a_2) = a_1^{n+1} + a_1^n a_2 + \cdots + a_1 a_2^n + a_2^{n+1} = 0.$$

If $a_1 = a_2$, this evidently implies $a_1 = a_2 = 0$. If $a_1 \neq a_2$, then (1) is

$$\frac{a_1^{n+2} - a_2^{n+2}}{a_1 - a_2} = 0,$$

from which it follows that $a_2 = -a_1$ and n is even.

Now suppose $r = 3$. We have

$$h_{n+1}(a_1, a_2, a_3) = h_{n+2}(a_1, a_2, a_3) = 0.$$

Then from the relations

$$h_{n+2}(a_1, a_2, a_3) = a_1^{n+2} + a_1^{n+1}h_1(a_2, a_3) + \cdots + a_1 h_{n+1}(a_2, a_3) + h_{n+2}(a_2, a_3)$$

and

$$a_1 h_{n+1}(a_1, a_2, a_3) = a_1^{n+2} + a_1^{n+1}h_1(a_2, a_3) + \cdots + a_1 h_{n+1}(a_2, a_3)$$

we get

$$h_{n+2}(a_2, a_3) = 0. \tag{2}$$

Similarly,

$$h_{n+2}(a_1, a_3) = 0 \tag{3}$$

and

$$h_{n+2}(a_1, a_2) = 0. \tag{4}$$

By the argument of the preceding paragraph, these equations imply $a_1 = a_2 = a_3 = 0$ unless n is odd. In this case, (2) gives $a_3 = -a_2$, (4) gives $a_2 = -a_1$, and (3) gives $a_1 = -a_3$: but then $a_1 = a_2 = a_3 = 0$. It is now clear how to prove the result by induction for any $r > 3$.

As noted in the previous section, $H^*(F(1^k, n); \mathbf{Q})$ is the quotient of $\mathbf{Q}[t_1, \ldots, t_k]$ by the ideal generated by $R_1, R_2, \ldots, R_k$, where

$$R_i = h_{n+i}(t_1, t_2, \ldots, t_k), \qquad 1 \leq i \leq k.$$

For f to be a well-defined endomorphism of $H^*(F(1^k, n); \mathbf{Q})$ there must be relations

$$f(R_i) = N_i R_i + \sum_{1 \leq |\alpha| < i} N_i^\alpha t^\alpha R_{i-|\alpha|}, \qquad 1 \leq i \leq k, \tag{5}$$

in $\mathbf{Q}[t_1, \ldots, t_k]$, where the sum is over multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ and

$$t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k}.$$

Then we have the following result.

LEMMA 5.3. *If r of the elements $t_1, t_2, \ldots, t_k$ are missing from the image of f, then $f(R_i) = 0$ in $\mathbf{Q}[t_1, \ldots, t_k]$ for $1 \leq i \leq r$.*

Proof. Permuting the t_i if necessary, we can assume that $t_1, t_2, \ldots, t_r$ are missing from the image of f. Define π: $F(1^k, n) \to F(1^{k-r}, n + r)$ by

$$\pi(l_1, l_2, \ldots, l_k) = (l_{r+1}, \ldots, l_k).$$

Then π^* sends $t_i \in H^2(F(1^{k-r}, n + r); \mathbf{Q})$ to $t_{r+i} \in H^2(F(1^k, n); \mathbf{Q})$ for $1 \leq i \leq k - r$, and is injective since the spectral sequence of the fibration

$$F(1^r, n) \to F(1^k, n) \xrightarrow{\pi} F(1^{k-r}, n + r)$$

collapses for degree reasons. Now $t_1, \ldots, t_r$ are missing from $\operatorname{im} f$, so $\operatorname{im} f \subset \operatorname{im} \pi^*$ in $H^*(F(1^k, n); \mathbf{Q})$. Hence $f = \pi^* g$ for

$$g = (\pi^*)^{-1} f: H^*(F(1^k, n); \mathbf{Q}) \to H^*(F(1^{k-r}, n + r); \mathbf{Q}).$$

But the first nontrivial relation in $H^*(F(1^{k-r}, n + r); \mathbf{Q})$ is in dimension $2(n + r + 1)$, so $g(R_i) = 0$ in $\mathbf{Q}[t_1, \ldots, t_{k-r}]$ for $1 \leq i \leq r$ and the conclusion follows.

Suppose $n \geq 2$. For each i from 1 to k, we define the *weight* of i to be the cardinality of $\{r | p(r) = i \text{ and } m_r \neq 0\}$, i.e., the number of t_r that f maps to t_i with nonzero coefficient. The following result is the key to the proof of Theorem 4.1.

PROPOSITION 5.4. *Let $n \geq 2$. Then i has weight at most 1 for $1 \leq i \leq k$ unless $k = 2$ and n is even. If $k = 2$, n is even, and $q \in \{1, 2\}$ has weight 2, then f has the form*

$$f(t_i) = (-1)^i m t_q, \qquad i = 1, 2.$$

Proof. Suppose q has weight $w > 1$. Then at least $w - 1$ of the t_i are missing from the image of f, and by 5.3

$$f(R_i) = 0, \qquad 1 \leq i \leq w - 1, \tag{6}$$

in $\mathbf{Q}[t_1, \ldots, t_k]$. We can assume $t_1, t_2, \ldots, t_w$ map to t_q with nonzero coefficient. Examine the coefficient of t_q^{n+i} in (6) to get

$$h_i(m_1, m_2, \ldots, m_w) = 0, \qquad 1 \leq i \leq w - 1.$$

Then unless $w = 2$ and n is even, $m_1 = m_2 = \cdots = m_w = 0$ by 5.2, a contradiction. If $w = 2$ and n is even, 5.2 gives $m_2 = -m_1 \neq 0$. In this case, $f(R_1) = 0$ and $f(R_2) \neq 0$ in $\mathbf{Q}[t_1, \ldots, t_k]$: but then no i can have weight 1 and no more than one t_i can be missing from the image of f, from which follows $k = 2$.

REMARK. The case $n = 0$ is disposed of in [7], where it is proved that all endomorphisms of $H^*(F(1^k); \mathbf{Q})$ have the form h_m^σ.

By the preceding result, the function p of $\{1, 2, \ldots, k\}$ can be assumed a permutation if $k \neq 2$ or n is odd. To finish the proof of 4.1, we need only show all the m_i are equal in this case. Now in $f(R_1)$ the coefficient of t_r^{n+1} is m_s^{n+1}, where $p(s) = r$. The coefficient of t_r^{n+1} on the right-hand side of (5) (with $i = 1$) is N_1. Thus $m_s^{n+1} = N_1$ for $1 \leq s \leq k$. For n even, this shows all the m_s are equal. For n odd, it is also necessary to inspect the coefficients of terms $t_r^n t_s$, $r \neq s$, in equation (5) with $i = 1$.

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Received March 4, 1983 and in revised form April 29, 1983.

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