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ISOMORPHISMS OF SPACES OF NORM-CONTINUOUS FUNCTIONS

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If X and Y are compact Hausdorff spaces and E a uniformly convex Banach space, then the existence of an isomorphism T of $C(X, E)$ onto $C(Y, E)$ with $\|T\|\,\|T^{-1}\|$ small implies that X and Y are homeomorphic.

1. Introduction. Throughout this article, the letters X, Y, Z, and W will denote compact Hausdorff spaces, and E a Banach space. $C(X, E)$ denotes the space of continuous functions on X to E provided with the supremum norm. If E is a dual space then $C(X, E_{\sigma^*})$ stands for the Banach space of continuous functions F on X to E when this latter space is provided with its weak* topology, again normed by $\|F\|_\infty = \sup_{x\in X}\|F(x)\|$. If E is the one-dimensional field of scalars then we write $C(X)$ for $C(X, E)$. The interaction between elements of a Banach space and those of its dual is denoted by $\langle\ \cdot\ ,\cdot\rangle$. We write $E_1 \cong E_2$ to indicate that the Banach spaces E_1 and E_2 are isometric.

The well known Banach-Stone theorem states that if $C(X)$ and $C(Y)$ are isometric then X and Y are homeomorphic. Various authors, beginning with M. Jerison [**13**], have considered the problem of determining geometric properties of E which allow generalizations of this theorem to spaces of norm-continuous vector functions $C(X, E)$. The most exhaustive compilation of results of this nature can be found in the monograph by E. Behrends [**2**]. Another type of generalization of the theorem was obtained independently in [**1**] and [**3**], and, while still dealing with scalar functions, replaces isometries by isomorphisms T with $\|T\|\,\|T^{-1}\|$ small.

The first attempt to combine these two directions of generalization is found in [**4**], where it is shown that if E is a finite-dimensional Hilbert space, then the existence of an isomorphism T of $C(X, E)$ onto $C(Y, E)$ with $\|T\|\,\|T^{-1}\| < \sqrt{2}$ implies that X and Y are homeomorphic. More recently, K. Jarosz [**12**] has obtained a similar generalization for Banach spaces E whose dual space satisfies a geometric condition involving both $\|T\|\,\|T^{-1}\|$ and the number $4/3$. Here we obtain such a theorem for all uniformly convex spaces E. Moreover, given such a space E, the bound on the isomorphisms for which our theorem works depends on the modulus of convexity associated with E.

Our method of proof depends on a characterization of the second dual space of $C(X, E)$, and is analogous to the method used by H. B. Cohen in the scalar case to obtain a new proof of the results of [**1**] and [**3**]. The first dual of $C(X)$ is, of course, given by the Riesz representatioin theorem which states that $C(X)^*$ consists of all finite, regular, scalar-valued Borel measures μ on X. The vector analogue of this result was obtained by I. Singer in [**15**], where it is shown that $C(X, E)^*$ is the Banach space of all regular Borel measures m on X to E^*, with finite variation $|m|$, and norm given by $\|m\| = |m|(X)$. An English version of the proof of this theorem can be found in [**16**, p. 192].

In [**7**] Cohen exploited the fact, first established by Kakutani [**14**], that $C(X)^{**}$ is isometric to a space $C(Z)$ for a particular compact Hausdorff space Z dependent on X. And in [**5**] it is shown that if X is dispersed or if E^* has the Radon-Nikodym property, then $C(X, E)^{**} \cong C(Z, E^{**}_{\sigma^*})$ where Z is that compact Hausdorff space such that $C(X)^{**} \cong C(Z)$. The interaction between the elements of the first dual of $C(X, E)$ (that is, vector measures on X), and functions in $C(Z, E^{**}_{\sigma^*})$ is given explicitly in [**6**]. It is the result of [**5**] on which we base most of our arguments.

We shall assume henceforth, that E is a uniformly convex Banach space. Let U denote the unit ball in E and let

$$\delta(\varepsilon) = \inf_{e_1, e_2 \in U} \{1 - \|(e_1 + e_2)/2\| : \|e_1 - e_2\| \geq \varepsilon\}.$$

Recall that E is uniformly convex means that $\delta(\varepsilon) > 0$ when $0 < \varepsilon \leq 2$. We will frequently use the fact that we always have $\delta(1) \leq \frac{1}{2}$.

The uniform convexity of E enters into our proof in a number of ways. First, we rely upon a geometric property of uniformly convex spaces which we establish in Lemma 1. Also E uniformly convex implies that E is reflexive [**8**, p. 147], and thus E^* has the Radon-Nikodym property [**9**, p. 218] and the result of [**5**] applies. We wish to prove the following:

THEOREM. *Let X and Y be compact Hausdorff spaces and E a uniformly convex Banach space. If T is an isomorphism of $C(X, E)$ onto $C(Y, E)$ satisfying $\|T\|\,\|T^{-1}\| < (1 - \delta(1))^{-1}$, then X and Y are homeomorphic.*

The proof of the theorem will be established via a sequence of lemmas and a proposition. However we first note the following. By replacing T by the isomorphism $(1 + \varepsilon)\|T^{-1}\|T$ for a sufficiently small positive number ε, we may suppose, without loss of generality, that T is *strictly norm-increasing*—i.e., $\|TF\|_\infty \geq (1 + \varepsilon)\|F\|_\infty$, for $F \in C(X, E)$, and that we have $\|T\| < (1 - \delta(1))^{-1}$. Fix such an ε, and then fix a positive number P with

$1 < P < 1 + \varepsilon$. We will thus assume, throughout the remainder of this article, that we are dealing with an isomorphism T of $C(X, E)$ onto $C(Y, E)$ satisfying $\|TF\|_\infty > P\|F\|_\infty$ for $F \in C(X, E)$, $F \neq 0$ and $\|T\| < (1 - \delta(1))^{-1}$.

Since here we have $E^{**} = E$, it follows that $C(X, E)^{**}$ is of the form $C(Z, E_{\sigma^*})$ for a certain compact Hausdorff space Z. Similarly, $C(Y, E)^{**} \cong C(W, E_{\sigma^*})$ for that compact Hausdorff space W with $C(Y)^{**} \cong C(W)$. We can thus regard T^{**} as a strictly norm-increasing isomorphism of $C(Z, E_{\sigma^*})$ onto $C(W, E_{\sigma^*})$ satisfying $\|T^{**}\| < (1 - \delta(1))^{-1}$ and $\|T^{**}F\|_\infty > P\|F\|_\infty$ for $F \in C(Z, E_{\sigma^*})$, $F \neq 0$.

Next note that if $F^* \in C(Z, E_{\sigma^*})^*$, then the restriction of F^* to $C(Z, E)$ is a continuous linear functional of norm less than or equal to $\|F^*\|$. Thus, by Singer's result, this restriction is given by a regular Borel vector measure n on X to E^* with $\|n\| \leq \|F^*\|$. If z is any point of Z, n can then be uniquely decomposed as $n = \psi \cdot \mu_z + m$, where μ_z denotes the scalar unit point mass at z, $\psi \in E^*$, and $m \in C(Z, E)^*$ with $m(\{z\}) = 0$. (Take $\psi = n(\{z\})$ and $m = n - \psi \cdot \mu_z$.) We then let $\overline{m}$ denote any norm-preserving linear extension of m to an element of $C(Z, E_{\sigma^*})^*$ and set $^{-} = F^* - \psi \cdot \mu_z - \overline{m}$. Then Φ is a continuous linear functional on $Z, E_{\sigma^*})$ which vanishes on $C(Z, E)$ and $F^* = \psi \cdot \mu_z + \overline{m} + \Phi$. Whenever we write an element $F^* \in C(Z, E_{\sigma^*})^*$ in this manner, $F^* = \psi \cdot \mu_z + \overline{m} + \Phi$, it will be implicit that $\psi \in E^*$, that $\overline{m}$ is a fixed Hahn-Banach extension of the vector measure m determined as above, and consequently that $\Phi \in C(Z, E)^{\perp}$. A similar convention applies when we write an element $G^* \in C(W, E_{\sigma^*})^*$ as $G^* = \psi \cdot \mu_w + \overline{m} + \Phi$.

Finally, we let X_0 denote the set of isolated points of Z. It is known that each point of X_0 is of the form tx for some $x \in X$, where t is the canonical (nontopological) injection of X into Z, and every such point tx is isolated [**11**, p. 841]. Similarly, we let Y_0 denote the set of isolated points of W so that Y_0 consists of the points sy, $y \in Y$, where s is the corresponding injection of Y into W.

2. Proof of the Theorem.

LEMMA 1. *If E is a uniformly convex normed linear space and r is a positive integer, and if we are given 2^r elements $e_j \in E$ with $\|e_j\| \geq \eta > 0$ for $1 \leq j \leq 2^r$, then*

(i) *there exists scalars λ_j, $1 \leq j \leq 2^r$, with $|\lambda_j| \leq 1$ for all j such that $\|\sum_{j=1}^{2^r} \lambda_j e_j/\|e_j\|\,\| \geq (1 - \delta(1))^{-r}$, and consequently*

(ii) *there exist scalars α_j, $1 \leq j \leq 2^r$, with $|\alpha_j| \leq 1$ for all j such that $\|\sum_{j=1}^{2^r} \alpha_j e_j\| \geq \eta(1 - \delta(1))^{-r}$.*

Proof. The proof is established by induction on r. First assume that $r = 1$ and that $e_1, e_2 \in E$, with $\|e_j\| \geq \eta, j = 1, 2$. Then

$$e_1/\|e_1\| = \tfrac{1}{2}(e_1/\|e_1\| + e_2/\|e_2\|) + \tfrac{1}{2}(e_1/\|e_1\| - e_2/\|e_2\|),$$

and, since a uniformly convex space is strictly convex, we must thus have either

$$\|e_1/\|e_1\| + e_2/\|e_2\|\,\| > 1 \quad \text{or} \quad \|e_1/\|e_1\| - e_2/\|e_2\|\,\| > 1,$$

and both of these norms are less than or equal to 2. Let M be the maximum of these two norms. Then by taking $\lambda_1 = 1$ and $\lambda_2 = 1$ or -1 we can find scalars λ_j of modulus one such that

$$(*) \qquad \|\lambda_1 e_1/\|e_1\| + \lambda_2 e_2/\|e_2\|\,\| = M > 1.$$

Now

$$a = (1/M)(\lambda_1 e_1/\|e_1\| + \lambda_2 e_2/\|e_2\|)$$

and

$$b = (1/M)(\lambda_1 e_1/\|e_1\| - \lambda_2 e_2/\|e_2\|)$$

are in the closed unit ball U of E and $(1/M)(\lambda_1 e_1/\|e_1\|)$ is the midpoint of the segment joining them. Also, since $\|a - b\| = 2/M$ and M is less than or equal to 2, we have

$$1 - 1/M = 1 - \|(1/M)(\lambda_1 e_1/\|e_1\|)\| \geq \delta(2/M) \geq \delta(1),$$

giving $M \geq (1 - \delta(1))^{-1}$ and establishing (i) for $r = 1$.

Next let $N = \min\{\|e_1\|, \|e_2\|\}$. Then from (*) we have

$$\|(N\lambda_1/\|e_1\|)e_1 + (N\lambda_2/\|e_2\|)e_2\| = N \cdot M \geq \eta(1 - \delta(1))^{-1}.$$

Thus letting $\alpha_j = N\lambda_j/\|e_j\|$ for $j = 1, 2$ we have established (ii) for $r = 1$.

Now assume the lemma is valid for all r with $1 \leq r \leq k$, and that we are given elements $e_j \in E$, $1 \leq j \leq 2^{k+1}$, with $\|e_j\| \geq \eta$ for all j. By the inductive hypothesis there exist scalars $\hat{\lambda}_j$, $1 \leq j \leq 2^{k+1}$, with $|\hat{\lambda}_j| \leq 1$ for all j such that

$$\left\| \sum_{j=1}^{2^k} \hat{\lambda}_j e_j/\|e_j\| \right\| = M_1 \geq (1 - \delta(1))^{-k}$$

and

$$\left\| \sum_{j=2^k+1}^{2^{k+1}} \hat{\lambda}_j e_j/\|e_j\| \right\| = M_2 \geq (1 - \delta(1))^{-k}.$$

Then

$$c = \left(\frac{1}{M_1}\right) \sum_{j=1}^{2^k} \hat{\lambda}_j e_j/\|e_j\| \quad \text{and} \quad d = \left(\frac{1}{M_2}\right) \sum_{j=2^k+1}^{2^{k+1}} \hat{\lambda}_j e_j/\|e_j\|$$

belong to U and $c = (\frac{1}{2})(c + d) + (\frac{1}{2})(c - d)$. Since $\|c\| = 1$, again we must have either $\|c + d\| > 1$ or $\|c - d\| > 1$, and both of these norms are ≤ 2.

Let M be the maximum of these two norms. Thus taking either $\tilde{\lambda}_j = \hat{\lambda}_j$ for all j with $2^k + 1 \leq j \leq 2^{k+1}$, or $\tilde{\lambda}_j = -\hat{\lambda}_j$ for all such j, we can find $\tilde{\lambda}_j$ with $|\tilde{\lambda}_j| \leq 1$ such that

$$(**) \qquad \left\| \left(\frac{1}{M_1}\right) \sum_{j=1}^{2^k} \hat{\lambda}_j e_j/\|e_j\| + \left(\frac{1}{M_2}\right) \sum_{j=2^k+1}^{2^{k+1}} \tilde{\lambda}_j e_j/\|e_j\| \right\| = M > 1.$$

Let $e = (1/M_2)\sum_{j=2^k+1}^{2^{k+1}} \tilde{\lambda}_j e_j/\|e_j\|$. Now $a = (1/M)(c + e)$ and $b = (1/M)(c - e)$ are in U and $(1/M)c$ is the midpoint of the segment joining them. Also $\|a - b\| = 2/M$. Hence

$$1 - 1/M = 1 - \|(1/M)c\| \geq \delta(2/M) \geq \delta(1),$$

giving $M \geq (1 - \delta(1))^{-1}$.

Let $M_0 = \min\{M_1, M_2\}$. Then from (**) we have

$$\left\| \sum_{j=1}^{2^k} \left(\frac{M_0 \hat{\lambda}_j}{M_1}\right) \frac{e_j}{\|e_j\|} + \sum_{j=2^k+1}^{2^{k+1}} \left(\frac{M_0 \tilde{\lambda}_j}{M_2}\right) \frac{e_j}{\|e_j\|} \right\| = M \cdot M_0 \geq (1 - \delta(1))^{-k-1},$$

so that, by letting $\lambda_j = M_0\hat{\lambda}_j/M_1$ for $1 \leq j \leq 2^k$ and $\lambda_j = M_0\tilde{\lambda}_j/M_2$ for $2^k + 1 \leq j \leq 2^{k+1}$, we have established (i) for $r = k + 1$.

Finally let $N = \min\{\|e_j\| : j = 1, \ldots, 2^{k+1}\}$. We then have

$$\left\| \sum_{j=1}^{2^{k+1}} \left(\frac{N\lambda_j}{\|e_j\|}\right) e_j \right\| \geq N(1 - \delta(1))^{-k-1} \geq \eta(1 - \delta(1))^{-k-1}$$

and thus, setting $\alpha_j = N\lambda_j/\|e_j\|$ for $1 \leq j \leq 2^{k+1}$, we have established (ii) for $r = k + 1$. This completes the proof.

LEMMA 2. *If $w \in W$ and $tx \in X_0$ then there exists an element ϕ of E^* with $\|\phi\| = 1$ such that $T^{***}\phi \cdot \mu_w$ is of the form $\psi \cdot \mu_{tx} + \overline{m} + \Phi$ with $\|\psi\| > P$ if, and only if, for some $e \in E$ with $\|e\| = 1$ we have $\|T^{**}(\chi_{\{tx\}} \cdot e)(w)\| > P$.*

Proof. Suppose that for some $e \in E$ with $\|e\| = 1$ we have $\|T^{**}(\chi_{\{tx\}} \cdot e)(w)\| > P$. Choose $\phi \in E^*$ with $\|\phi\| = 1$ such that

$$\langle T^{**}(\chi_{\{tx\}} \cdot e)(w), \phi \rangle = \|T^{**}(\chi_{\{tx\}} \cdot e)(w)\|.$$

Then writing $T^{***}\phi \cdot \mu_w$ as $\psi \cdot \mu_{tx} + \overline{m} + \Phi$ we would have

$$P < \left\| T^{**}(\chi_{\{tx\}} \cdot e)(w) \right\| = \left\langle T^{**}(\chi_{\{tx\}} \cdot e)(w), \phi \right\rangle$$

$$= \int T^{**}(\chi_{\{tx\}} \cdot e)\, d(\phi \cdot \mu_w) = \left\langle \chi_{\{tx\}} \cdot e, T^{***}\phi \cdot \mu_w \right\rangle$$

$$= \int (\chi_{\{tx\}} \cdot e)\, d(\psi \cdot \mu_{tx} + m) + \left\langle \chi_{\{tx\}} \cdot e, \Phi \right\rangle = \langle e, \psi \rangle,$$

and hence $\|\psi\| > P$.

Conversely, suppose there exists a $\phi \in E^*$ with $\|\phi\| = 1$ such that $T^{***}\phi \cdot \mu_w$ has the specified form. Take $e \in E$ with $\|e\| = 1$ such that $\langle e, \psi \rangle > P$. A computation exactly like that above then gives

$$\left\langle T^{**}(\chi_{\{tx\}} \cdot e)(w), \phi \right\rangle = \langle e, \psi \rangle > P$$

and, consequently, $\|T^{**}(\chi_{\{tx\}} \cdot e)(w)\| > P$.

We now let W_1 denote the set of all $w \in W$ such that for some $\phi \in E^*$ with $\|\phi\| = 1$ there exists a $tx \in X_0$ with $T^{***}\phi \cdot \mu_w = \psi \cdot \mu_{tx} + \overline{m} + \Phi$, where $\|\psi\| > P$. Then define $\rho\colon W_1 \to X_0$ by $\rho(w) = tx$ if w and tx are related as in the previous sentence.

We first note that ρ is a well defined map from W_1 to X_0. For by Lemma 2 we have $w \in W_1$ and $\rho(w) = tx$ if, and only if, for some $e \in E$ with $\|e\| = 1$ we have $\|T^{**}(\chi_{\{tx\}} \cdot e)(w)\| > P$. Thus if we assume that there exist $\phi_1, \phi_2 \in E^*$ with $\|\phi_1\| = \|\phi_2\| = 1$ and

$$T^{***}\phi_i \cdot \mu_w = \psi_i \cdot \mu_{tx_i} + \overline{m}_i + \Phi_i$$

for $i = 1, 2$, with $\|\psi_i\| > P$ and $tx_1 \neq tx_2$, then for all choices of scalars α_i with $|\alpha_i| \leq 1$ and all $e_i \in E$ with $\|e_i\| = 1$, $i = 1, 2$, we would have $\|\alpha_1\chi_{\{tx_1\}} \cdot e_1 + \alpha_2\chi_{\{tx_2\}} \cdot e_2\|_\infty \leq 1$. However, it follows from Lemmas 1 and 2 that for appropriate choices of such α_i and e_i we would have

$$\left\| T^{**}\left(\alpha_1\chi_{\{tx_1\}} \cdot e_1 + \alpha_2\chi_{\{tx_2\}} \cdot e_2\right) \right\|_\infty$$

$$\geq \left\| \alpha_1 T^{**}\left(\chi_{\{tx_1\}} \cdot e_1\right)(w) + \alpha_2 T^{**}\left(\chi_{\{tx_2\}} \cdot e_2\right)(w) \right\|$$

$$\geq P(1 - \delta(1))^{-1} > (1 - \delta(1))^{-1},$$

contradicting the fact that $\|T^{**}\| < (1 - \delta(1))^{-1}$. Consequently ρ is well defined as claimed.

Moreover, ρ maps W_1 onto X_0. For given $tx \in X_0$ then for any $e \in E$ with $\|e\| = 1$ there exists some $w \in W$ such that $\|T^{**}(\chi_{\{tx\}} \cdot e)(w)\| > P$. Thus, as noted in the second sentence of the previous paragraph, we have $w \in W_1$ and $\rho(w) = tx$.

By arguments exactly analogous to those given above, one obtains the companion result:

LEMMA 2′. *If* $z \in Z$ *and* $sy \in Y_0$ *then there exists an element* ϕ *of* E^* *with* $\|\phi\| = 1$ *such that* $T^{***-1}\phi \cdot \mu_z$ *is of the form* $\psi \cdot \mu_{sy} + \overline{m} + \Phi$ *with* $\|\psi\| > 1 - \delta(1)$ *if, and only if, for some* $e \in E$ *with* $\|e\| = 1$ *we have* $\|T^{**-1}(\chi_{\{sy\}} \cdot e)(z)\| > 1 - \delta(1)$.

We then let Z_1 denote the set of all $z \in Z$ such that for some $\phi \in E^*$ with $\|\phi\| = 1$ there exists an $sy \in Y_0$ with $T^{***-1}\phi \cdot \mu_z = \psi \cdot \mu_{sy} + \overline{m} + \Phi$, where $\|\psi\| > 1 - \delta(1)$. And we define $\tau\colon Z_1 \to Y_0$ by $\tau(z) = sy$ if z and sy are related as in the previous sentence. Just as before one establishes that τ is a well defined map carrying Z_1 onto Y_0. Moreover, by Lemma 2′, we have $z \in Z_1$ and $\tau(z) = sy$ if and only if for some $e \in E$ with $\|e\| = 1$ we have $\|T^{**-1}(\chi_{\{sy\}} \cdot e)(z)\| > 1 - \delta(1)$.

LEMMA 3. (i) *For each* $tx \in X_0$, $\rho^{-1}(\{tx\})$ *is a finite open set of points, and consequently* $W_1 \subset Y_0$.

(ii) *For each* $sy \in Y_0$, $\tau^{-1}(\{sy\})$ *is a finite open set of points, and consequently* $Z_1 \subseteq X_0$.

Proof. Suppose $tx \in X_0$ and $w \in \rho^{-1}(\{tx\})$. Then there exists an $e_w \in E$ with $\|e_w\| = 1$ such that $\|T^{**}(\chi_{\{tx\}} \cdot e_w)(w)\| > P$. Let

$$\hat{e}_w = T^{**}(\chi_{\{tx\}} \cdot e_w)(w) / \left\| T^{**}(\chi_{\{tx\}} \cdot e_w)(w) \right\|$$

and take any continuous $g\colon W \to [0,1]$ such that $g(w) = 1$. Then define $G \in C(W, E) \subseteq C(W, E_{\sigma^*})$ by $G(w') = g(w') \cdot \hat{e}_w$, $w' \in W$. Now

$$\left\| G + T^{**}(\chi_{\{tx\}} \cdot e_w) \right\|_\infty \geq \left\| G(w) + T^{**}(\chi_{\{tx\}} \cdot e_w)(w) \right\| > 1 + P,$$

so that

$$\left\| T^{**-1}(G) + \chi_{\{tx\}} \cdot e_w \right\|_\infty > (1 + P)(1 - \delta(1)) \geq (1 + P)/2.$$

Thus as $\|T^{**-1}(G)\|_\infty < 1$ we must have $\|T^{**-1}(G)(tx)\| > (P - 1)/2$.

Now pick any element $\phi_w \in E^*$ with $\|\phi_w\| = 1$ such that $\langle \hat{e}_w, \phi_w \rangle = 1$. Then $w \in \{ w' \in W\colon |\langle T^{**}(\chi_{\{tx\}} \cdot e_w)(w'), \phi_w \rangle| > P \}$, and this set is open. Moreover, for any w' in this set, we have $\|T^{**}(\chi_{\{tx\}} \cdot e_w)(w')\| > P$ and thus w' must belong to $\rho^{-1}(\{tx\})$. Hence fixing such elements e_w and ϕ_w for each $w \in \rho^{-1}(\{tx\})$ we have

$$\rho^{-1}(\{tx\}) = \bigcup_{w \in \rho^{-1}(\{tx\})} \left\{ w' \in W\colon \left| \left\langle T^{**}(\chi_{\{tx\}} \cdot e_w)(w'), \phi_w \right\rangle \right| > P \right\},$$

an open set.

We now show that $\rho^{-1}(\{tx\})$ is a finite set. Suppose that w_k, $1 \leq k \leq 2^r$, are elements of $\rho^{-1}(\{tx\})$. We have seen that for each k we can find $G_k \in C(W, E_{\sigma^*})$ with $\|G_k\|_\infty = 1$ and $\|T^{**-1}(G_k)(tx)\| > (P-1)/2$. If we choose the G_k to have pairwise disjoint supports, then for all scalars α_k, $1 \leq k \leq 2^r$, with $|\alpha_k| \leq 1$, we have $\|\Sigma_{k=1}^{2^r} \alpha_k G_k\|_\infty \leq 1$. But by Lemma 1(ii), we can choose the α_k such that

$$\left\| \sum_{k=1}^{2^r} \alpha_k T^{**-1}(G_k)(tx) \right\| \geq \frac{(P-1)(1-\delta(1))^{-r}}{2}.$$

Hence $\rho^{-1}(\{tx\})$ must be finite as claimed.

Thus for each $tx \in X_0$, $\rho^{-1}(\{tx\})$ is a finite open set of points, and thus consists entirely of isolated points. Hence $W_1 = \bigcup_{tx \in X_0} \rho^{-1}(\{tx\})$ consists of isolated points and so $W_1 \subseteq Y_0$, proving (i). The proof of (ii) is analogous.

LEMMA 4. *Given an element of $C(Z, E_{\sigma^*})^*$ of the form $\psi \cdot \mu_{tx} + \overline{m} + \Phi$, where $tx \in X_0$ is an isolated point of Z, then*

$$\|\psi \cdot \mu_{tx} + \overline{m} + \Phi\| = \|\psi\| + \|\overline{m} + \Phi\|.$$

Proof. Suppose $\varepsilon > 0$ is given. Choose $F \in C(Z, E_{\sigma^*})$ with $\|F\|_\infty \leq 1$ such that $\langle F, \overline{m} + \Phi \rangle$ is real and greater than $\|\overline{m} + \Phi\| - \varepsilon$. Let $e_1 = F(tx)$. Then both $\overline{m}$ and Φ annihilate $e_1 \cdot \chi_{\{tx\}}$ so that $\langle F - e_1\chi_{\{tx\}}, \overline{m} + \Phi \rangle > \|\overline{m} + \Phi\| - \varepsilon$. Choose an element $e_2 \in E$ with $\|e_2\| = 1$ and $\langle e_2, \psi \rangle = \|\psi\|$. Then $\|F + (e_2 - e_1) \cdot \chi_{\{tx\}}\|_\infty \leq 1$ and thus

$$\begin{aligned} &\|\psi \cdot \mu_{tx} + \overline{m} + \Phi\| \\ &\quad \geq \left|\left\langle F + (e_2 - e_1) \cdot \chi_{\{tx\}}, \psi \cdot \mu_{tx} + \overline{m} + \Phi \right\rangle\right| \\ &\quad = \int e_2 \cdot \chi_{\{tx\}}\, d(\psi \cdot \mu_{tx}) + \left\langle F - e_1 \cdot \chi_{\{tx\}}, \overline{m} + \Phi \right\rangle \\ &\quad > \|\psi\| + \|\overline{m} + \Phi\| - \varepsilon. \end{aligned}$$

LEMMA 5. *If $sy \in W_1 \subseteq Y_0$ and $\rho(sy) = tx$, then $tx \in Z_1$ and $\tau(tx) = sy$.*

Proof. Let sy belong to W_1 and let $\rho(sy) = tx$. Suppose that either tx is not an element of Z_1, or that $tx \in Z_1$, but $\tau(tx) \neq sy$. Either supposition leads to the conclusion that for all $e \in E$ with $\|e\| = 1$ we have $\|T^{**-1}(\chi_{\{sy\}} \cdot e)(tx)\| \leq 1 - \delta(1)$.

Fix an $e \in E$ with $\|e\| = 1$ and let $Q = \sup_{z \in Z}\|T^{**-1}(\chi_{\{sy\}} \cdot e)(z)\|$. Then by Lemma 3(ii), and the paragraph preceding the statement of

Lemma 3, we have

$$\left\{z \in Z: \left\|T^{**-1}(\chi_{\{sy\}} \cdot e)(z)\right\| > 1 - \delta(1)\right\}$$
$$= \left\{tx' \in X_0: \left\|T^{**-1}(\chi_{\{sy\}} \cdot e)(tx')\right\| > 1 - \delta(1)\right\} \subseteq \tau^{-1}(\{sy\}),$$

a finite set, and thus we can find a $tx' \in X_0$ such that

$$\|T^{**-1}(\chi_{\{sy\}} \cdot e)(tx')\| = Q.$$

Now $tx' \neq tx$ since $\tau(tx) \neq sy$.

Let $\hat{e} = T^{**-1}(\chi_{\{sy\}} \cdot e)(tx')$ and $\tilde{e} = \hat{e}/\|\hat{e}\|$. Then consider the element $\chi_{\{tx'\}} \cdot \tilde{e}$ of $C(Z, E) \subseteq C(Z, E_{\sigma^*})$. There exists a $w \in W$ such that $\|T^{**}(\chi_{\{tx'\}} \cdot \tilde{e})(w)\| > P$. Hence this w belongs to $W_1 \subseteq Y_0$ so $w = sy'$ for some $sy' \in Y_0$. Moreover $sy' \neq sy$ since $\rho(sy') = tx' \neq tx = \rho(sy)$.

From the proof of Lemma 2, we know that if $\phi \in E^*$ with $\|\phi\| = 1$ is such that

$$\left\langle T^{**}(\chi_{\{tx'\}} \cdot \tilde{e})(sy'), \phi\right\rangle = \left\|T^{**}(\chi_{\{tx'\}} \cdot \tilde{e})(sy')\right\|$$

then

$$T^{***}\phi \cdot \mu_{sy'} = \psi \cdot \mu_{tx'} + \overline{m} + \Phi \quad \text{where } \langle \tilde{e}, \psi\rangle > P.$$

Hence $\langle \hat{e}, \psi\rangle = \|\hat{e}\|\langle \tilde{e}, \psi\rangle > QP > Q$. We have

$$0 = \int \chi_{\{sy\}} \cdot e \, d(\phi \cdot \mu_{sy'}) = \left\langle \chi_{\{sy\}} \cdot e, \phi \cdot \mu_{sy'}\right\rangle$$
$$= \left\langle T^{**-1}(\chi_{\{sy\}} \cdot e), T^{***}\phi \cdot \mu_{sy'}\right\rangle$$
$$= \int T^{**-1}(\chi_{\{sy\}} \cdot e)\, d(\psi \cdot \mu_{tx'}) + \left\langle T^{**-1}(\chi_{\{sy\}} \cdot e), \overline{m} + \Phi\right\rangle$$
$$= \langle \hat{e}, \psi\rangle + \left\langle T^{**-1}(\chi_{\{sy\}} \cdot e), \overline{m} + \Phi\right\rangle.$$

But the modulus of the first term on the right is greater than Q while, by Lemma 4, the modulus of the second term on the right is less than or equal to $(\|T\| - \|\psi\|)Q < Q$. This contradiction completes the proof of the lemma.

Note that Lemma 5 implies that $X_0 = \rho(W_1) \subseteq Z_1$, so that $X_0 = Z_1$. It also shows that $Y_0 = \tau(Z_1) \subseteq W_1$. For ρ maps W_1 onto X_0; hence, given $tx \in Z_1 = X_0$ there exists an $sy \in W_1$ with $\rho(sy) = tx$. And by Lemma 5 $\tau(tx) = sy \in W_1$. Thus ρ maps Y_0 onto X_0, ρ is injective since τ is a function and $\tau = \rho^{-1}$. It follows that $\hat{\rho} = t^{-1} \circ \rho \circ s$ is a one-one map of Y onto X. We would like to show that $\hat{\rho}$ is a homeomorphism.

To this end again recall that we have $sy \in W_1 = Y_0$ and $\rho(sy) = tx$ if, and only if, for some $e \in E$ with $\|e\| = 1$ we have $\|T^{**}(\chi_{\{tx\}} \cdot e)(sy)\| > P$. Since for any $e \in E$ with $\|e\| = 1$ we must have $\|T^{**}(\chi_{\{tx\}} \cdot e)(w)\| > P$

for some $w \in W$, it now follows that for *all* $e \in E$ with $\|e\| = 1$ the only candidate for this w is sy. That is, given $tx \in X_0$ let $sy = \tau(tx)$. Then for each $e \in E$ with $\|e\| = 1$ we must have $\|T^{**}(\chi_{\{tx\}} \cdot e)(sy)\| > P$ and sy is the only point of W for which such an inequality holds.

Next note that for $e \in E$, $\phi \in E^*$, $tx \in X_0$ and $sy \in Y_0$ we have

$$\langle T^{**}(\chi_{\{tx\}} \cdot e), \phi \cdot \mu_{sy} \rangle = \langle \phi \cdot \mu_y, T^{**}(\chi_{\{tx\}} \cdot e) \rangle,$$

the equality holding by the proof of Theorem 2 in **[6]**. We next have

$$\langle \phi \cdot \mu_y, T^{**}(\chi_{\{tx\}} \cdot e) \rangle = \langle T^*(\phi \cdot \mu_y), \chi_{\{tx\}} \cdot e \rangle$$

by definition of the adjoint map, and then

$$\langle T^*(\phi \cdot \mu_y), \chi_{\{tx\}} \cdot e \rangle = \langle e, (T^*\phi \cdot \mu_y)(\{x\}) \rangle,$$

again by the proof of Theorem 2 in **[6]**. Thus

$$\langle T^{**}(\chi_{\{tx\}} \cdot e), \phi \cdot \mu_{sy} \rangle = \langle e, (T^*\phi \cdot \mu_y)(\{x\}) \rangle.$$

PROPOSITION. *$\hat{\rho}$ is a homeomorphism of Y onto X.*

Proof. As noted above we have $\hat{\rho}(y) = x$ if, and only if, for all $e \in E$ with $\|e\| = 1$ we have $\|T^{**}(\chi_{\{tx\}})(sy)\| > P$, which will be true if, and only if, for every e there exists a $\phi \in E^*$ (depending on e and y) with $\|\phi\| = 1$ such that $\langle T^{**}(\chi_{\{tx\}} \cdot e), \phi \cdot \mu_{sy} \rangle = \langle e, (T^*\phi \cdot \mu_y)(\{x\}) \rangle$ is real and greater than P.

Now suppose that $\{y_\beta : \beta \in B\}$ is a net in Y, $y_\beta \to y_0$ but $x_\beta = \hat{\rho}(y_\beta) \nrightarrow \hat{\rho}(y_0) = x_0$. Then there exists a compact neighborhood V of x_0 such that for all $\beta_0 \in B$ there is a $\beta \geq \beta_0$ with x_β outside V.

Fix an $e \in E$ with $\|e\| = 1$. By the paragraph before last there is a $\phi_0 \in E^*$ with $\|\phi_0\| = 1$ and $\langle e, (T^*\phi_0 \cdot \mu_{y_0})(\{x\}) \rangle > P$. Write $T^*\phi_0 \cdot \mu_{y_0}$ as $\psi_0 \cdot \mu_{x_0} + m$, where $\psi_0 \in E^*$ and m is a regular Borel vector measure on X to E^* with $m(\{x_0\}) = 0$. Then $\langle e, \psi_0 \rangle > P$. Choose a neighborhood V_1 of x_0, $V_1 \subseteq V$, such that $|m|(V_1) < P - 1$. Next choose a continuous function $f_1 : X \to [0, 1]$ such that the support of f_1 is contained in V_1 and $f_1(x_0) = 1$. Then define $F_1 \in C(X, E)$ by $F_1(x) = f_1(x) \cdot e$, $x \in X$. We have

$$\begin{aligned} |\langle (TF_1)(y_0), \phi_0 \rangle| &= |\langle (TF_1), \phi_0 \cdot \mu_{y_0} \rangle| = |\langle F_1, T^*(\phi_0 \cdot \mu_{y_0}) \rangle| \\ &= |\langle F_1, \psi_0 \cdot \mu_{x_0} + m \rangle| = \left| \langle F_1(x_0), \psi_0 \rangle + \int F_1 \, dm \right| \\ &\geq \langle e, \psi_0 \rangle - \int \|F_1\| \, d|m| > 1. \end{aligned}$$

Thus $\|(TF_1)(y_0)\| > 1$.

Since $y_\beta \to y_0$ and TF_1 is continuous in the norm topology, there is a $\beta_0 \in B$ such that $\beta \geq \beta_0$ implies $\|(TF_1)(y_\beta)\| > 1$. Thus fix a β such that $\|(TF_1)(y_\beta)\| > 1$ and $x_\beta = \hat{\rho}(y_\beta)$ lies outside V. Then for some $\phi_\beta \in E^*$ with $\|\phi_\beta\| = 1$ we have $\left\langle e, \left(T^*\phi_\beta \cdot \mu_{y_\beta}\right)\left(\{x_\beta\}\right)\right\rangle > P$. Write $T^*\phi_\beta \cdot \mu_{y_\beta}$ as $\psi_\beta \cdot \mu_{x_\beta} + n$ where $\psi_\beta \in E^*$ and $n(\{x_\beta\}) = 0$. Then $\langle e, \psi_\beta \rangle > P$. Take a neighborhood V_2 of x_β disjoint from V with $|n|(V_2) < P - 1$ and choose continuous f_2: $X \to [0,1]$ such that the support of f_2 is contained in V_2 and $f_2(x_\beta) = 1$. If we then define $F_2 \in C(X, E)$ by $F_2(x) = f_2(x) \cdot e$, $x \in X$, it follows as above that $\|(TF_2)(y_\beta)\| > 1$.

Now since F_1 and F_2 have disjoint supports, for every choice of scalars α_i with $|\alpha_i| \leq 1$, $i = 1, 2$, we have $\|\alpha_1 F_1 + \alpha_2 F_2\|_\infty \leq 1$. However, by Lemma 1, there exist such scalars α_i with

$$\left\|T(\alpha_1 F_1 + \alpha_2 F_2)\right\|_\infty \geq \left\|\alpha_1 (TF_1)(y_\beta) + \alpha_2 (TF_2)(y_\beta)\right\| > (1 - \delta(1))^{-1},$$

which contradicts our assumptions about the norm of T. Thus $\hat{\rho}$ is a continuous, one-one map of Y onto X, and is hence a homeomorphism.

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