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SOME MAXIMUM PROPERTIES FOR A FAMILY OF SINGULAR HYPERBOLIC OPERATORS

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We study some maximum properties of solutions of the equation

$$L_{p,q,c}u \equiv u_{xx} - h^2(x)u_{tt} + ph'(x)u_t + q\frac{h'(x)}{h(x)}u_x + c(x,t)u = 0$$

with real parameters p and q. Some of the results here improve those of L. E. Payne and D. Sather. We also point out that a certain condition given by S. Agmon, L. Nirenberg and M. H. Protter is not only sufficient in order to obtain a kind of maximum property, but also necessary for a special case of $L_{p,q,c}$.

1. Introduction. Since the maximum principles were first established for a class of linear second order hyperbolic operators in two independent variables [**1**], [**3**], many authors have studied various maximum and monotonicity properties of some problems for classes of linear second order hyperbolic operators in two or more independent variables [**5**]–[**10**]. Later Payne and Sather considered a singular hyperbolic operator [**4**]. They obtained some maximum, monotonicity and convexity properties, as well as pointwise bounds, for the solution of some Cauchy and initial-boundary value problems for the Chaplygin operator

$$L \equiv \frac{\partial^2}{\partial x^2} - h^2(x)\frac{\partial^2}{\partial t^2}, \tag{1.1}$$

where h satisfies

$$\text{(a) } h \in C^1(R_+) \cap C^0(\overline{R}_+), \quad \text{(b) } h(0) = 0, \quad \text{(c) } h'(x) > 0, x > 0. \tag{1.2}$$

For example, Theorem 1 in [**4**] states that if h satisfies (1.2) and

$$\lim_{x\to 0} \frac{h(x)}{h'(x)} = 0, \tag{1.3}$$

and if u satisfies the conditions

$$\text{(a) } u \in C^2(E \cup AB) \cap C^1(\overline{E}), \quad \text{(b) } u_x \le 0 \quad \text{on } AB, \quad \text{(c) } Lu \le 0 \quad \text{in } E, \tag{1.4}$$

then

$$u \le \max_{AB} u \quad \text{in } E, \tag{1.5}$$

where AB is a segment of the t-axis and E is the domain bounded by AB and the two characteristics of operator L, through A, B, respectively, which have positive x-coordinate.

In this paper we deal with a family of operators with two real parameters p and q:

$$(1.6)\quad L_{p,q,c} = \frac{\partial^2}{\partial x^2} - h^2(x)\frac{\partial^2}{\partial t^2} + ph'(x)\frac{\partial}{\partial t} + q\frac{h'(x)}{h(x)}\frac{\partial}{\partial x} + c(x,t).$$

The domain E in which the operators $L_{p,q,c}$ are defined is the same as above, and we denote the closed segment AB by C. In §2, we give two lemmas which are used in §3. The main results of this paper are stated and proved in §3. Theorems 1 and 1′ show that condition A in **[1]** is not only sufficient, but necessary for the maximum property for the family of operators under consideration in this paper. Theorems 2 and 3 improve Theorem 1 and Lemma 1 in **[4]**; in fact, some superfluous conditions in the latter are eliminated. In addition we obtain pointwise bounds as given in Corollaries 1 and 2.

2. Two lemmas. It is always supposed in this paper that

$$(2.1)\qquad \text{(a) } h \in C^2(0,M] \cap C^1[0,M] \quad \text{or} \quad \text{(b) } h \in C^1[0,M],$$

$$(2.2)\qquad h(0) = 0; \qquad h'(x) > 0, \quad x > 0,$$

$$(2.3)\qquad c \in C^0(\overline{E} \setminus C), \qquad c \le 0,$$

where $M = \max\{x\colon (x,t) \in \overline{E}\}$. We denote the x- and t-coordinates of any point P in R^2 by x_p, t_p, respectively. Assume $t_A < t_B$, and denote by Γ_1 (Γ_2) the characteristic curve of $L_{p,q,c}$ in (1.6) that passes through A (B) with positive (negative) slope and with positive x-coordinate.

LEMMA 1. *Suppose h satisfies* (2.1)(b) *and* (2.2). *Then for any p and q, there exists a function $g(x,t) \in C^2(\overline{E} \setminus C)$ such that*

(2.4) $L_{p,q,0}g > 0$ *in* $\overline{E} \setminus C$,

(2.5) *g as a function of t decreases strictly on* Γ_1.

Proof. We select the function g in the class $C^2(0,M]$; in other words, g will be a function of the single variable x. Then we have

$$(2.6)\qquad L_{p,q,0}g = g'' + q\frac{h'}{h}g'.$$

(a) The case $q \le 0$. It is sufficient that g satisfy

$$g' < 0, \qquad g'' > 0.$$

In this case, g can be chosen to be in the class $C^\infty(R^1)$, say

$$g(x) = x^2 - 3Mx, \tag{2.7}$$

and it follows that

$$L_{p,q,0}g = 2 - 3Mqh'/h > 0 \quad \text{in } \bar{E} \setminus C. \tag{2.8}$$

(b) The case $q > 0$. Choose

$$g(x) = \int_x^M (h(s))^{-2q}\, ds, \tag{2.9}$$

which satisfies the equation

$$g'' + 2q\frac{h'}{h}g' = 0 \quad \text{in } \bar{E} \setminus C.$$

Therefore we get

$$L_{p,q,0}g = qh'h^{-2q-1} > 0 \quad \text{in } \bar{E} \setminus C. \tag{2.10}$$

Thus, the proof of Lemma 1 is complete.

LEMMA 2. *If* (2.1)(b), (2.2) *and* (2.3) *hold, then, for any* p, q, *there exists a function* $g(x, t) \in C^2(\bar{E} \setminus C)$ *which satisfies*

$$L_{p,q,c}g < 0 \quad \textit{in } \bar{E} \setminus C \tag{2.11}$$

and

$$g_x(x, t) < 0 \quad \textit{in } \bar{E} \setminus C. \tag{2.12}$$

Proof. (a) The case $q \geq 0$. We choose

$$g(x, t) = M^2 - x^2. \tag{2.13}$$

A simple calculation shows that

$$L_{p,q,c}g = -2 - 2q\frac{h'(x)}{h(x)}x + c(M^2 - x^2) < 0 \quad \text{in } \bar{E} \setminus C \tag{2.14}$$

and that

$$g_x(x, t) = -2x < 0 \quad \text{in } \bar{E} \setminus C. \tag{2.15}$$

(b) The case $q < 0$. We choose

$$g(x, t) = \int_x^M \exp\left(\int_s^M 2q\frac{h'(r)}{h(r)}\, dr \right) ds. \tag{2.16}$$

Then we obtain

$$g_x(x, t) = -\exp\left(\int_x^M 2q\frac{h'(s)}{h(s)}\, ds \right) < 0 \quad \text{in } \bar{E} \setminus C$$

and

$$L_{p,q,c}g = q\frac{h'(x)}{h(x)}\exp\left(\int_x^M 2q\frac{h'(s)}{h(s)}ds\right) + cg$$
$$\leq q\frac{h'(x)}{h(x)}\exp\left(\int_x^M 2q\frac{h'(s)}{h(s)}ds\right) < 0 \quad \text{in } \overline{E}\setminus C.$$

3. Main results. First we give two definitions.

DEFINITION 1. Suppose (2.3) holds. The operator $L_{p,q,c}$ is said to have the maximum property (P) if the conditions

(3.1) $$u \in C^2(\overline{E}\setminus C) \cap C^1(\overline{E}),$$

(3.2) $$L_{p,q,c}u \geq 0 \quad \text{in } E,$$

(3.3) $$u \text{ as a function of } t \text{ decreases on } \Gamma_1,$$

(3.4) $$\max_{\overline{E}} u \geq 0 \ \textit{if}\ c \not\equiv 0$$

imply

(3.5) $$\max_C u = \max_{\overline{E}} u.$$

REMARK. (3.4) is not needed if $c \equiv 0$.

DEFINITION 2. Suppose (2.3) holds. The operator $L_{p,q,c}$ has the maximum property $(L)_s$ $[(L)_w]$ if the conditions

$(3.6)_s$ $$u \in C^2(E) \cap C^1(\overline{E}),$$

$[(3.6)_w$ $$u \in C^2(\overline{E}\setminus C) \cap C^1(\overline{E})],$$

$(3.7)_s$ $$L_{p,q,c}u \leq 0 \quad \text{in } E,$$

$[(3.7)_w$ $$L_{p,q,c}u \leq 0 \quad \text{in } \overline{E}\setminus C],$$

$(3.8)_s$ $$u_x \leq 0 \quad \text{in } C,$$

$[(3.8)_w$ $$u_x < 0 \quad \text{on } C]$$

(3.9) $$\max_C u < 0 \quad \text{if } c \not\equiv 0$$

imply (3.5).

REMARK. (3.9) is not needed if $c \equiv 0$.

We now state and prove the main theorems.

THEOREM 1. *Suppose* (2.1)(a), (2.2), (2.3) *hold. Then the operator* $L_{p,q,c}$ *has the maximum property* (P) *if*

$$p - q - 1 \le 0, \tag{3.10}$$

$$4h^2c + (p - q - 1)\left[2hh'' + (p + q - 3)(h')^2\right] \ge 0 \quad \text{in } (0, M]. \tag{3.11}$$

Moreover, (3.5) *holds without the requirement* (3.4) *if* $p - q - 1 = 0$.

Conversely, (3.5) *is violated if* (3.10) *doesn't hold even though all the remaining conditions are satisfied.*

Proof. (a) First we consider the case $p - q - 1 < 0$ and $c \not\equiv 0$. Suppose u satisfies (3.1)–(3.4). If the result (3.5) were false, there would be a point $Q \in E \cup \Gamma_2$ such that

$$u(Q) = \max_{\bar{E}} u \ge 0$$

because of the condition (3.3). We have the identity

$$\begin{aligned} D_-\left[(h(x))^\alpha D_+ v\right] &= (h(x))^\alpha L_{p,q,c} v - D_-(Av) \\ &\quad + \left[A' - (h(x))^\alpha c\right] v, \\ &\qquad \forall v \in C^2(\bar{E} \setminus C), \text{ in } \bar{E} \setminus C, \end{aligned} \tag{3.12}$$

where

$$D_\pm = \frac{\partial}{\partial x} \pm h(x)\frac{\partial}{\partial t}, \qquad \alpha = \frac{p + q - 1}{2},$$

$$A = \frac{q - p + 1}{2} h'(x)(h(x))^{\alpha - 1}.$$

Draw the characteristic Γ from Q to a point P which is on Γ_1, and integrate (3.12) with respect to x in which v is replaced by u. We find

$$\begin{aligned} (h(x))^\alpha D_+ u\big|_Q^P &\ge (-Au)\big|_Q^P + \int_\Gamma (A' - h^\alpha c) u \, dx \\ &= (-Au)\big|_Q^P + \int_\Gamma (h^\alpha c - A')(u(Q) - u) \, dx \\ &\quad - u(Q) \int_\Gamma h^\alpha c \, dx + u(Q) A\big|_Q^P \\ &= A(P)(u(Q) - u(P)) + \int_\Gamma (h^\alpha c - A')(u(Q) - u) \, dx \\ &\quad - u(Q) \int_\Gamma h^\alpha c \, dx > 0, \end{aligned}$$

since $u(Q) > u(P)$, $A(P) > 0$, $h^\alpha c \le 0$, $u(Q) \ge 0$, $u(Q) - u \ge 0$, $h^\alpha c - A' \ge 0$, where the integral along Γ is from Q to P. Hence we have

$$D_+u(Q) < \left(\frac{h(P)}{h(Q)}\right)^\alpha D_+u(P) \le 0 \tag{3.13}$$

since (3.3); this contradicts the fact that $u(Q) = \max_{\bar{E}} u$.

(b) Suppose $p - q - 1 = 0$. It follows immediately from (2.3), (3.11) that $c \equiv 0$. Let g be a function which has the properties mentioned in Lemma 1. Let

$$v_\varepsilon = u + \varepsilon g, \qquad \varepsilon > 0. \tag{3.14}$$

It is easy to see that v_ε has the properties (3.2) with strict inequality and (3.3) for every $\varepsilon > 0$. We claim that for every $\delta > 0$ ($\delta < M$) and $\varepsilon > 0$,

(3.15) the maximum of v_ε on $\bar{E}_\delta$ is only achieved on C_δ,

where $E_\delta = E \cap \{(x, t): x > \delta\}$ and $C_\delta = \partial E_\delta \cap \{(x, t): x = \delta\}$. In fact, identity (3.12) in this case is

$$D_-\left[(h(x))^\alpha D_+v_\varepsilon\right] = (h(x))^\alpha L_{p,q,0}v_\varepsilon \quad \text{in } \bar{E}\setminus C. \tag{3.12$'$}$$

With reasoning similar to the case (a) we get (3.15) (notice that we haven't used condition (3.4)). Hence we obtain

$$\max_{C_\delta} u = \max_{\bar{E}_\delta} u \quad \text{for every } \delta \in (0, M),$$

and (3.5) follows.

(c) In the case $c \equiv 0$, it is obvious that we can obtain (3.5) without condition (3.4) because we can add any constant to u.

(d) We give an example to show that the last conclusion is true. For the sake of convenience, let

$$\Gamma_1 = \{(x, t) : t - H(x) = 0, 0 < x \le M\},$$

$$\Gamma_2 = \{(x, t) : t + H(x) = 2H(M), 0 < x \le M\},$$

where $H(x) = \int_0^x h(s)\, ds$.

(i) The case $c \equiv 0$. The function we desire is

$$u_{p,q}(x, t) = g_{p,q}(x) f(t - H(x)), \tag{3.16}$$

where f satisfies

$$\begin{aligned} &\text{(a) } f \in C^2[0, 2H(M)], \quad \text{(b) } f(0) = 0, \quad \text{(c) } f' \ge 0, \\ &\text{(d) } f(s) = f(2H(M)) > 0, \; 2H(M) - 2H\left(\frac{M}{2}\right) \le s \le 2H(M), \end{aligned} \tag{3.17}$$

and $g_{p,q}$ is defined as follows:

(3.18)

$$g_{p,q}(x)=\begin{cases} G_{p,q}\equiv 2(n+1)\left(\frac{3M}{4}\right)^n \Big/ (p-q-1)\displaystyle\min_{M/4\le x\le M} h'(x), \\ \hfill 0\le x\le \frac{M}{4}, \\ G_{p,q}+\left(x-\frac{M}{4}\right)^{n+1}, \quad \frac{M}{4}<x\le M, \end{cases}$$

where n satisfies

$$\text{(3.19)} \quad \text{(a) } n>1, \qquad \text{(b) } n\ge \max_{M/4\le x\le M}\left(-q\left(x-\frac{M}{4}\right)\frac{h'(x)}{h(x)}\right).$$

It is not difficult to verify that

$$\text{(3.20)} \quad \begin{aligned} L_{p,q,0}u_{p,q} &= \left[(p-q-1)g_{p,q}h' - 2hg'_{p,q}\right]f'(t-H(x)) \\ &\quad + \left(g''_{p,q} + qg'_{p,q}\frac{h'}{h}\right)f(t-H(x)) \\ &\ge 0 \quad \text{in } \overline{E}\setminus C \end{aligned}$$

if $p-q-1>0$, and that (3.3) holds for $u_{p,q}$. But

$$\text{(3.21)} \quad \begin{aligned} &u_{p,q}\left(\frac{M}{2}, 2H(M)-H\left(\frac{M}{2}\right)\right) \\ &= g_{p,q}\left(\frac{M}{2}\right)f\left(2H(M)-2H\left(\frac{M}{2}\right)\right) \\ &= \left(G_{p,q}+\left(\frac{M}{4}\right)^{n+1}\right)f(2H(M)) \\ &= g_{p,q}(0)f(2H(M)) + \left(\frac{M}{4}\right)^{n+1} f(2H(M)) \\ &= \max_C u_{p,q} + \left(\frac{M}{4}\right)^{n+1} f(2H(M)) > \max_C u_{p,q}, \end{aligned}$$

that is to say, (3.5) doesn't hold.

(ii) The case $c\le 0$, $c\not\equiv 0$. Define the function

$$\text{(3.22)} \qquad v_{p,q}(x,t) = u_{p,q}(x,t) + A,$$

where $u_{p,q}(x, t)$ is the function which appears in case (i), and $A = -\max_{\bar{E}} u_{p,q}$. It is obvious that

$$v_{p,q} \leq 0 \quad \text{in } \bar{E} \qquad \text{and} \qquad \max_{\bar{E}} v_{p,q} = 0. \tag{3.23}$$

And we have (3.3) (for $v_{p,q}$) and

$$\begin{aligned} L_{p,q,c} v_{p,q} &= L_{p,q,c}(u_{p,q} + A) \\ &= L_{p,q,0} u_{p,q} + c v_{p,q} \geq 0 \quad \text{in } \bar{E} \setminus C \end{aligned} \tag{3.24}$$

if $p - q - 1 > 0$, because of (2.3), (3.20) and (3.23). Thus, the function $v_{p,q}$ satisfies conditions (3.1)–(3.4). However, we have

$$\begin{aligned} v_{p,q}\left(\frac{M}{2}, 2H(M) - H\left(\frac{M}{2}\right)\right) &= u_{p,q}\left(\frac{M}{2}, 2H(M) - H\left(\frac{M}{2}\right)\right) + A \\ &> \max_C u_{p,q} + A = \max_C v_{p,q}, \end{aligned}$$

because of (3.21); i.e., (3.5) doesn't hold. The proof of Theorem 1 is complete.

REMARK 1. In a special case of operators $L_{p,q,c}$ with

$$h(x) \equiv x, \quad q = 0, \qquad c \equiv 0,$$

(we denote $L_{p,q,c}$ by L' in this case), condition A in [**1**] is sufficient and necessary for L' to have the maximum property (P). It is stated as follows:

THEOREM 1′. *The operator L' has the maximum property (P) if and only if*

$$p \leq 1. \tag{3.25}$$

In fact, we note that conditions (3.10), (3.11) in this case become $p - 1 \leq 0, (p-1)(p-3) \geq 0$, i.e., (3.25).

REMARK 2. The first part of Theorem 1 can be stated in an equivalent way.

THEOREM 1″. *Suppose* (2.1)(a), (2.2), (2.3), (3.1)–(3.3), (3.10), (3.11) *hold. Then we have*

$$(3.26) \qquad \max_{\bar{E}} u < 0,$$

if

$$(3.27) \qquad \max_{C} u < 0.$$

Proof. The reasoning from Theorem 1 to Theorem 1″ is obvious. On the other hand, if (3.4) holds, we define

$$(3.28) \qquad v = u - \max_{\bar{E}} u.$$

Then $\max_{\bar{E}} v = 0$. According to Theorem 1″, we must have $\max_C v = 0$, i.e., (3.5) holds.

We now deal with the operators $L_{p,0,c}$.

THEOREM 2. *Suppose* (2.1)(b), (2.2), (2.3) *hold. Then the operator* $L_{p,0,c}$ *has the maximum property* $(L)_s$ *if*

$$(3.29) \qquad |p| \leq 1.$$

If c *satisfies* (2.3) *and, in addition,*

(2.3)′ c *is bounded if* $p > 1$,

(2.3)″ c *is bounded by a certain constant depending on* M *and* p *if* $p < -1$,

the operator $L_{p,0,c}$ *doesn't have property* $(L)_s$ *when* $|p| > 1$. *When* $c \equiv 0$, *the result holds without condition* (3.9).

Proof. (a) The case $|p| \leq 1$.

(i) Suppose all of the conditions in the theorem are satisfied and $c \not\equiv 0$. We will show that

$$(3.30) \qquad u < 0 \quad \text{in } \bar{E}.$$

If it were not true, then there would exist a point P' which belongs to the union of Γ_1, Γ_2 and E, and is such that

$$(3.31) \qquad u(P') = 0,$$

$$(3.32) \qquad u(Q) < 0 \quad \text{for any } Q \in \bar{E} \text{ with } 0 \leq x_Q < x_{P'},$$

because of (3.9). Draw two characteristics Γ_1', Γ_2' through P', with positive and negative slope respectively. Let A' (B') denote the unique point of intersection of the t-axis and the characteristic Γ_1' (Γ_2'), and let E' be the domain bounded by Γ_1', Γ_2' and the t-axis. Then, by Green's formula, we have

$$\iint_{E'} L_{p,0,c}u\,dx\,dt = \oint_{\partial E'} \left(h^2u_t - ph'u\right) dx + u_x\,dt + \iint_{E'} cu\,dx\,dt$$

$$= -\int_{A'}^{B'} u_x\,dt + \int_{\Gamma_1'} h^2u_t\,dx + u_x\,dt - \int_{\Gamma_2'} h^2u_t\,dx + u_x\,dt$$

$$-\int_{\Gamma_1'} ph'u\,dx + \int_{\Gamma_2'} ph'u\,dx + \iint_{E'} cu\,dx\,dt$$

$$= -\int_{A'}^{B'} u_x\,dt + \int_{\Gamma_1'} h\,du + \int_{\Gamma_2'} h\,du - \int_{\Gamma_1'} ph'u\,dx$$

$$+\int_{\Gamma_2'} ph'u\,dx + \iint_{E'} cu\,dx\,dt$$

$$= -\int_{A'}^{B'} u_x\,dt + hu|_{A'}^{P'} + hu|_{B'}^{P'} - (p+1)\int_{\Gamma_1'} h'u\,dx$$

$$+(p-1)\int_{\Gamma_2'} h'u\,dx + \iint_{E'} cu\,dx\,dt$$

$$= -\int_{A'}^{B'} u_x\,dt + 2h(P')u(P') - h(A')u(A') - h(B')u(B')$$

$$-(p+1)\int_{\Gamma_1'} h'u\,dx + (p-1)\int_{\Gamma_2'} h'u\,dx + \iint_{E'} cu\,dx\,dt,$$

where the integral along Γ_1' (Γ_2') is from A' (B') to P'. Therefore we find that

$$(3.33)\qquad 2h(P')u(P') = h(A')u(A') + h(B')u(B') + \iint_{E'} L_{p,0,c}u\,dx\,dt - \iint_{E'} cu\,dx\,dt + \int_{A'}^{B'} u_x\,dt + (p+1)\int_{\Gamma_1'} h'u\,dx + (1-p)\int_{\Gamma_2'} h'u\,dx.$$

According to assumptions (2.2), (2.3), (3.6), (3.7), (3.8), (3.29), (3.31) and (3.32), we have

$$0 = 2h(P')u(P') = \text{the right-hand side of } (3.33) < 0.$$

This is a contradiction and (3.30) follows.

(ii) The reasoning from the fact that "$\max_C u < 0 \Rightarrow \max_{\overline{E}} u < 0$" to the fact that "$\max_C u < 0 \Rightarrow \max_C u = \max_{\overline{E}} u$" is as follows: Let $v_\varepsilon = u - \max_C u - \varepsilon$, where $0 < \varepsilon < -\max_C u$; then we see that v_ε satisfies all the conditions of the theorem. So we obtain $\max_{\overline{E}} v_\varepsilon < 0$ in $\overline{E}$. Let ε tend to zero; we get $u \leq \max_C u$ in $\overline{E}$, i.e., $\max_C u = \max_{\overline{E}} u$.

(iii) $c \equiv 0$. The result in this case is obvious because we can add any constant to u and insure a negative maximum of u on C without violating any conditions of the theorem.

(b) The case $|p| > 1$ and $c \equiv 0$. Let Γ_1, Γ_2 and E be as in the proof of Theorem 1, (d). We have a counterexample as follows:

$$u_p(x, t) = H(x) - \frac{t}{p}. \tag{3.34}$$

It is easy to check that

$$L_{p,0,0} u_p = 0, \quad (u_p)_x(0, t) = H'(x)|_{x=0} = h(x)|_{x=0} = 0. \tag{3.35}$$

(i) $p > 1$. We have

$$\max_C u_p = \max_{0 \leq t \leq 2H(M)} \left(-\frac{t}{p} \right) = 0. \tag{3.36}$$

However, when $(x, t) \in \Gamma_1$, $t > 0$, we have

$$u_p(x, t) = H(x) - \frac{t}{p} = H(x) - t + \left(1 - \frac{1}{p}\right)t = \left(1 - \frac{1}{p}\right)t > 0.$$

(ii) $p < -1$. We have now, instead of (3.36),

$$\max_C u_p = \max_{0 \leq t \leq 2H(M)} \left(-\frac{t}{p} \right) = -\frac{2H(M)}{p}. \tag{3.37}$$

But an easy calculation shows that

$$\begin{aligned} u_p(M, H(M)) &= H(M) - \frac{H(M)}{p} \\ &= -\frac{2H(M)}{p} + \left(1 + \frac{1}{p}\right) H(M) > -\frac{2H(M)}{p}. \end{aligned}$$

(c) The case $|p| > 1$ and $c \not\equiv 0$. Define the function

$$v_p(x, t) = u_p(x, t) - (G_p + \varepsilon_p)\exp\left(\sqrt{C_0}\, x\right), \tag{3.38}$$

where $u_p(x, t)$ is the function given in (3.34), and the constants G_p, ε_p, C_0 satisfy the following conditions:

$$G_p = \begin{cases} 0, & p > 1, \\ -\dfrac{2H(M)}{p}, & p < -1, \end{cases} \tag{3.39}$$

$$\text{(3.40)}\qquad \begin{aligned} &\max_{\bar{E}} |c| < C_0, && \text{if } p > 1;\\ &\max_{\bar{E}} |c| < C_0 < \left(\ln\left(\frac{1-p}{2}\right)\right)^2 \Big/ M^2 && \text{if } p < -1, \end{aligned}$$

(the number $(\ln((1-p)/2))^2/M^2$ is the constant mentioned in condition (2.3)″) and

$$\text{(3.41)}\qquad \begin{cases} 0 < \varepsilon_p < \dfrac{p-1}{p\left[\exp\left(\sqrt{C_0}M\right)-1\right]} H(M) & \text{if } p > 1,\\ 0 < \varepsilon_p < \left[\dfrac{p+1}{p\left[\exp\left(\sqrt{C_0}M\right)-1\right]} + \dfrac{2}{p}\right] H(M) & \text{if } p < -1. \end{cases}$$

A not too complicated calculation shows that

$$\text{(3.42)}\qquad \begin{cases} L_{p,0,c}v_p = -(C_0 + c)(G_p + \varepsilon_p)\exp\left(\sqrt{C_0}\,x\right) < 0 \quad \text{in } \bar{E},\\ (v_p)_x\big|_{x=0} = -(G_p + \varepsilon_p)\sqrt{C_0} < 0,\\ \max_C v_p = -\varepsilon_p < 0,\\ v_p(M, H(M)) > -\varepsilon_p. \end{cases}$$

The proof is complete.

REMARK 3. The operator to be considered in Theorem 1 of [4] is a special case of operators $L_{p,0,c}$, i.e., the case that $p = 0$, $c \equiv 0$. Moreover, we eliminate the superfluoud condition that $\lim_{x\to 0}[h^2(x)/h'(x)] = 0$.

REMARK 4. Of course, we have the following (compare also [4]).

COROLLARY 1. *Suppose h satisfies* (2.1)(b) *and* (2.2) *and* $|p| \le 1$. *Then, in E,*

$$\text{(3.43)}\qquad u(x,t) \le \max_C u + x \max_C u + \frac{x^2}{2} \max_{\bar{E}} L_{p,0,0}u, \qquad u \in C^2(E) \cap C^1(\bar{E}).$$

Finally, we deal with the family of operators $L_{p,q,c}$ again.

THEOREM 3. *Suppose* (2.1)(b), (2.2), (2.3) *hold. If*

$$\text{(3.44)}\qquad p - q - 1 \ge 0, \qquad p + q + 1 \le 0,$$

then the operator $L_{p,q,c}$ *has the maximum property* $(L)_w$. *Actually, we have*

$$u < \max_C u \quad \text{in } \bar{E} \setminus C, \tag{3.45}$$

$$D_+ u \le \max_C u_x, \qquad D_- u \le \max_C u_x \quad \text{in } \bar{E}, \tag{3.46}$$

under conditions (2.1)(b), (2.2), (2.3), (3.44), $(3.6)_w$, $(3.7)_w$, $(3.8)_w$ *and* (3.9). *When* $c \equiv 0$, (3.45) *and* (3.46) *hold without* (3.9).

Proof. (a) First of all, we suppose that strict inequality holds in $(3.7)_w$. Suppose (3.45) didn't hold. Then there would exist a point $P_1 \in \bar{E} \setminus C$ such that

$$u(P_1) = 0; \qquad u(Q) < 0, \quad \forall Q \in \bar{E}, 0 \le x_Q < x_{P_1}. \tag{3.47}$$

Therefore we would have

$$D_+ u(P_1) \ge 0, \qquad D_- u(P_1) \ge 0. \tag{3.48}$$

We could get a point P_2 with $0 < x_{P_2} \le x_{P_1}$ such that

$$D_+ u(P_2) \cdot D_- u(P_2) = 0, \tag{3.49}$$

$$D_+ u(Q) < 0, \qquad D_- u(Q) < 0, \quad \forall Q \in \bar{E}, 0 \le x_Q < x_{P_2}, \tag{3.50}$$

since $(3.8)_w$. Suppose

$$D_+ u(P_2) = 0. \tag{3.51}$$

Then the maximum of $h^\lambda D_+ u$ in the set $((\bar{E} \setminus C) \cap \{(x, t)\colon x < x_{P_2}\}) \cup \{P_2\}$ is achieved at P_2 because of (3.50), (3.51) and (2.2), where the real number λ is arbitrary. Hence it follows that

$$\big(D_-(h^\lambda D_+ u)\big)(P_2) \ge 0, \quad \text{for any } \lambda. \tag{3.52}$$

But according to the identity

$$D_-(h^\alpha D_+ u) = h^\alpha L_{p,q,c} u + \frac{p - q - 1}{2} h' h^{\alpha - 1} D_- u - c h^\alpha u, \tag{3.53}$$

$$\alpha = \frac{p + q - 1}{2},$$

and conditions (2.3), (3.44), (3.47), (3.50) and $(3.7)_w$ with strict inequality, we have

$$D_-(h^\alpha D_+ u)(P_2) < 0. \tag{3.54}$$

This is inconsistent with (3.52) with $\lambda = \alpha$. It follows that

$$u < 0 \quad \text{in } \bar{E}. \tag{3.55}$$

If $D_{-}u(P_2) = 0$, then we use another identity, namely,

$$(3.56)\quad D_{+}\left(h^{\beta}D_{-}u\right) = h^{\beta}L_{p,q,c}u - \frac{p+q+1}{2}h'h^{\beta-1}D_{+}u - ch^{\beta}u,$$

$$\beta = \frac{q-p-1}{2}.$$

We now show that

$$(3.57)\qquad D_{+}u < 0,\quad D_{-}u < 0 \quad \text{in } \overline{E}.$$

In fact, suppose there were a point $P \in \overline{E} \setminus C$ such that

$$(3.58)\quad D_{-}u(P) = 0, \qquad D_{-}u(Q) < 0, \quad \text{for any } Q \in \overline{E}, 0 \le x_Q < x_P.$$

We could, without loss of generality, suppose

$$(3.59)\qquad D_{+}u(Q) < 0, \quad \text{for any } Q \in \overline{E}, 0 \le x_Q < x_P.$$

Then we get a contradiction by using the identity (3.56). So (3.57) follows.

It is easy to obtain (3.46) from (3.57) if the above result is applied to the function

$$v_\varepsilon = u - \left(\max_C u_x + \varepsilon\right)x,$$

where $0 < \varepsilon < -\max_C u_x$, and if we let ε tend to zero. (Notice, we have used the fact that $q \le -1$, which is a consequence of (3.44)). Then we obtain

$$u_x \le \max_C u_x \quad \text{in } \overline{E} \quad \text{and} \quad u_x < 0 \quad \text{in } \overline{E},$$

because $u_x = (D_{+}u + D_{-}u)/2$. Therefore (3.45) follows. (b) We now consider the general case; in other words, we do not assume that $(3.7)_w$ with strict inequality holds. If u is the function given in Theorem 3, we define a family of functions

$$v_\varepsilon = u + \varepsilon g, \qquad \varepsilon > 0,$$

where g is the function mentioned in Lemma 2. If we concentrate on the domain E_δ and C_δ is a part of its boundary, where $\delta > 0$ is sufficiently small, it is easily seen that all of the conditions, including strict inequality in $(3.7)_w$, in Theorem 3 are satisfied if $\varepsilon > 0$ is sufficiently small. It follows then that

$$(3.60)\qquad D_{+}v_\varepsilon \le \max_{C_\delta}\,(v_\varepsilon)_x,\quad D_{-}v_\varepsilon \le \max_{C_\delta}\,(v_\varepsilon)_x \quad \text{in } \overline{E}_\delta,$$

and we therefore have

$$(3.61)\qquad D_{+}u \le \max_C u_x,\quad D_{-}u \le \max_C u_x \quad \text{in } \overline{E},$$

if first we let ε tend to zero and then δ tend to zero. It is an immediate consequence of (3.61) and (3.8) that (3.45) holds.

The result in the case $c \equiv 0$ is obvious because we can add any constant to the function u without violating any condition of Theorem 3.

REMARK 5. We can obtain an estimate which is more explicit than (3.45).

COROLLARY 2. *Under all conditions of Theorem* 3, *i.e., if* (2.1)(b), (2.2), (2.3), (3.44) *hold and if u satisfies* $(3.6)_w$, $(3.7)_w$, $(3.8)_w$ *and* (3.9), *then*

$$u \le \max_C u + x \max_C u_x + \frac{x^2}{2} \max_{\bar{E}} L_{p,q,c} u \quad \text{in } \bar{E}. \tag{3.62}$$

When $c \equiv 0$, we have

$$u \le \max_C u + x \max_C u_x + \frac{x^2}{2} \max_{\bar{E}} L_{p,q,0} u \quad \text{in } \bar{E} \tag{3.63}$$

without the requirement (3.9).

Proof. For every ε, $0 < \varepsilon < -\max_C u_x$, define a family of functions

$$v_\varepsilon = u - x\left(\max_C u_x + \varepsilon\right) - \frac{x^2}{2} \max_{\bar{E}} L_{p,q,c} u. \tag{3.64}$$

It is easy to verify that $(3.6)_w$, $(3.7)_w$, $(3.8)_w$ and (3.9) hold for every v_ε, $0 < \varepsilon < -\max_C u_x$. (Notice that we have used here the fact $q \le -1$, a consequence of (3.44)). Therefore we have

$$v_\varepsilon < \max_C v_\varepsilon, \quad 0 < \varepsilon < -\max_C u_x, \quad \text{in } \bar{E} \setminus C. \tag{3.65}$$

The reasoning from (3.65) to (3.62) is obvious. The proof in the case $c \equiv 0$ is similar to that in the case $c \not\equiv 0$. The proof is complete.

REMARK 6. The operator M in [4] is the special case of $L_{p,q,c}$ with $p = 0$, $q = -2$, $c \equiv 0$.

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