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A PROBLEM ON CONTINUOUS AND PERIODIC FUNCTIONS

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Let $f(x)$ be continuous and of period one on the real line. If d_j, $j = 1, 2, \ldots, n$, are n numbers such that each $d_j - d_1$ is rational, then there are two rational numbers r and r' for which

$$f(r) \leq f(r + d_j) \quad \text{and} \quad f(r') \geq f(r' + d_j), \qquad j = 1, 2, \ldots, n.$$

This problem was communicated to the author by K. L. Chung and P. Erdös.

1. Introduction. Let $f(x)$ be a real valued function. We say that $f(x)$ is of period one if

$$f(x + 1) = f(x) \quad \text{for} -\infty < x < \infty.$$

A problem (communicated by Chung and Erdös) asks that if $f(x)$ is continuous and of period one, and if $d_j, j = 1, 2, \ldots, n$, are n numbers, can one find a rational number r such that

$$f(r) \leq f(r + d_j), \qquad j = 1, 2, \ldots, n.$$

In this note, we present the following partial solution.

THEOREM 1. *Let $f(x)$ be continuous and of period one. If d_j, $j = 1, 2, \ldots, n$, are n numbers such that each $d_j - d_1$ is rational, then there are two rational numbers r and r' for which*

$$(1) \quad f(r) \leq f(r + d_j) \quad \textit{and} \quad f(r') \geq f(r' + d_j), \qquad j = 1, 2, \ldots, n.$$

2. Uniform distribution. Let x be a positive number and let $[x]$ be the largest integer less or equal to x. By a theorem of Hardy and Wright [**1**, Theorem 445], we know that if θ is irrational then the points $(n\theta) = n\theta - [n\theta]$ are uniformly distributed in $(0, 1)$. In particular, the points $(n\theta)$ are dense in $(0, 1)$. Based on this theorem, we shall prove the following result.

LEMMA 1. *Let d_1 be irrational and let $d_j - d_1$ be rational, $j = 2, 3, \dots, n$. If $I_j(k), j = 1, 2, \dots, n$, are non-negative integral valued functions such that*

$$\sum_{j=1}^{n} I_j(k) = k, \quad \text{for each } k = 1, 2, \dots, \tag{2}$$

then the points (a_k) are dense in $(0, 1)$, where

$$a_k = \sum_{j=1}^{n} I_j(k) d_j, \quad \text{for } k = 1, 2, \dots.$$

Proof. According to the first hypothesis, we may write

$$d_j = d_1 + p_j/q_j, \quad \text{for some integers } p_j \text{ and } q_j.$$

This together with the second hypothesis (2) yields that

$$\sum_{j=1}^{n} I_j(k) d_j = k d_1 + \sum_{j=1}^{n} I_j(k) p_j/q_j.$$

Multiplying the product $\prod_1^n q_j$ on both sides, we obtain the assertion from Hardy and Wright's theorem.

3. Proof of Theorem 1. Let $f(x)$ be continuous and of period one. We shall prove the first set of inequalities in (1). For this, we let m be the minimum of $f(x)$ and let S_m be the set of all minimum points in $(0, 1)$, i.e.

$$S_m = \{x : f(x) = m, 0 < x < 1\}.$$

We then have two cases to be considered: either there is a point $y \in S_m$ such that

$$f(y) < f(y + d_j), \quad \text{for all } j = 1, 2, \dots, n; \tag{3}$$

or for each $x \in S_m$, there is a $j = j(x)$ such that

$$f(x) = f(x + d_j), \qquad 1 \le j \le n. \tag{4}$$

If the first case occurs, then there is a $\delta > 0$ such that (3) holds for each z in $|z - y| < \delta$. By choosing a rational number r in $|r - y| < \delta$ we obtain the desired result. Therefore only the second case needs to be settled in the sequel. In this case, by applying (4) successively, we obtain a sequence of points in S_m as follows:

$$x, x + d_{j_1}, x + d_{j_1} + d_{j_2}, \dots. \tag{5}$$

Clearly, this sequence (5) can be represented by

$$x_k = x + \sum_{j=1}^{n} I_j(k) d_j = x + a_k, \qquad k = 1, 2, \dots, \tag{6}$$

where $I_j(k)$ and a_k are defined in Lemma 1. If d_1 is irrational, then by Lemma 1 the points (x_k) are dense in $(0, 1)$. It follows from the continuity of $f(x)$ that the function $f(x) = m$ for each $x \in (0,1)$. This clearly yields the assertion.

It remains to consider the case that d_1 is rational. Let x_k be a number of the form defined in (6) and let S_x be the set of all points (x_k) in $(0,1)$, $k = 1, 2, \ldots$. Clearly, this set S_x is a subset of S_m. As before, we write

$$d_j = p_j/q_j, \quad j = 1, 2, \ldots, n, \qquad \text{and} \qquad Q = \prod_1^n q_j.$$

Then the set S_x contains at most Q points.

We now begin with the first minimum

$$f(d_{j_1}) = \min_{1 \le j \le n} f(d_j), \quad \text{for some } 1 \le j_1 \le n.$$

If this minimum point $r = d_{j_1}$ satisfies (1) we are done, otherwise, we consider the second minimum

$$f(d_{j_1} + d_{j_2}) = \min_{1 \le j \le n} f(d_{j_1} + d_j) < f(d_{j_1}).$$

Again, if $r = d_{j_1} + d_{j_2}$ satisfies (1), we are done, otherwise, we consider the third minimum

$$f(d_{j_1} + d_{j_2} + d_{j_3}) = \min_{1 \le j \le n} f(d_{j_1} + d_{j_2} + d_j) < f(d_{j_1} + d_{j_2}) < f(d_{j_1}).$$

Since the set S_x contains at most Q points, there are two positive integers $M < N \le Q$ such that

$$f(d_{j_1} + \cdots + d_{j_M}) = f(d_{j_1} + \cdots + d_{j_N}) < \cdots < f(d_{j_1} + \cdots + d_{j_M}),$$

which is absurd. This proves the first set of inequalities in (1).

Similarly, by replacing minimum by maximum, we obtain the second set of inequalities in (1). This completes the proof.

4. Finite set of extreme points. In view of Theorem 1, we may ask the question as to whether the condition that each $d_j - d_1$ be rational can be replaced by some suitable conditions on the set of minimum or maximum points. For this, we prove the following

THEOREM 2. *Let $f(x)$ be continuous and of period one, and let d_j, $j = 1, 2, \ldots, n$ be n numbers. If the set of minimum and maximum points of $f(x)$ in $(0,1)$ is finite, then there are two rational numbers r and r' satisfying* (1).

Proof. As before, we let m be the minimum of $f(x)$ and let S_m be the set of all minimum points in $(0,1)$. Then by the hypothesis the cardinality $|S_m|$ of S_m is finite. As before, we have two cases described in (3) and (4). The first case gives the assertion immediately while the second case implies a sequence of points in S_m defined in (5). It follows that there are two positive integers $M < N \le |S_m|$ such that

$$(7) \qquad f(x + d_{j_1} + \cdots + d_{j_M}) = f(x + d_{j_1} + \cdots + d_{j_N}).$$

We now let δ be the minimum distance between any two points in S_m. We then consider a point $y \in S_m$ and a rational number r_1 with $y < r_1 < y + \delta$. If this number r_1 satisfies (1), we are done, otherwise, we have

$$f(r_1) > f(r_1 + d_{j_1}), \quad \text{for some } 1 \le j_1 \le n.$$

By the continuity of $f(x)$, there is a $0 < \delta_1 \le \delta/2$ such that

$$(8) \qquad f(x) > f(x + d_{j_1}), \quad \text{for each } x \text{ in } |x - r_1| < \delta_1.$$

Choose a rational number r_2 with $|r_2 - r_1 - d_{j_1}| < \delta_1/2$. If r_2 satisfies (1), we are done again, otherwise, we have

$$f(r_2) > f(r + d_{j_2}), \quad \text{for some } 1 \le j_2 \le n.$$

Again, there is a $0 < \delta_2 < \delta_1/2$ such that

$$(9) \qquad f(z) > f(z + d_{j_2}), \quad \text{for each } z \text{ in } |z - r_2| < \delta_2.$$

We now write

$$z = (z - d_{j_1}) + d_{j_1} = x + d_{j_1}.$$

Then we have

$$|x - r_1| = |z - r_2 + r_2 - r_1 - d_{j_1}| < \delta_1.$$

It follows from (8) and (9) that

$$f(x) > f(x + d_{j_1}) > f(x + d_{j_1} + d_{j_2}).$$

Continuing this process, we finally obtain

$$\begin{aligned} f(x) > f(x + d_{j_1}) > \cdots > f(x + d_{j_1} + \cdots + d_{j_M}) \\ > \cdots > f(x + d_{j_1} + \cdots + d_{j_M} + \cdots + d_{j_N}), \end{aligned}$$

which contradicts (7). This concludes the existence of a rational number r satisfying the first set of inequalities in (1). By the same argument, we obtain another rational number r' satisfying the second set of inequalities in (1). This completes the proof.

5. Problem. In closing this note, let us pose the following

Problem. If d_1 and d_2 are two independent numbers, and if $I_1(k)$ and $I_2(k)$ are non-decreasing, is it true that the points

$$(a_k), \quad \text{where } a_k = I_1(k)d_1 + I_2(k)d_2,$$

are dense in a subinterval of $(0, 1)$.

Note that the monotonicity of $I_1(k)$ and $I_2(k)$ in the above problem is simply meant that we take either d_1 or d_2 after each term in the sequence a_k. However, if this hypothesis is omitted then there does exist a sequence (a_k) which tends to any prescribed number $0 \leq \delta \leq 1$. The following construction is due to Komlos:

For each positive integer k, we choose a positive integer $i_k \leq k$ such that

$$\varepsilon_k = |(i_k d + kd_2 - \delta)| = \min_{1 \leq i \leq k} |(id + kd_2 - \delta)|$$

where $d = d_1 - d_2$. Since d_1 and d_2 are independent, the difference d is an irrational number and hence the sequence (id) forms an ε-net in the sense that for any x there is a positive integer i for which $|(id - x)| < \varepsilon$. In particular, when $x = \delta - kd_2$ there is an integer $I(\varepsilon)$ such that for each $k \geq I(\varepsilon)$ there exists an $i_k \leq k$ satisfying $|(i_k d - x)| < \varepsilon$. This in turn implies that $\varepsilon_k < \varepsilon$ and therefore $\varepsilon_k \to 0$ as $k \to \infty$. We now let $I_1(k) = i_k$ and $I_2(k) = k - i_k$. Then we obtain

$$a_k = I_1(k)d_1 + I_2(k)d_2 = i_k d + kd_2 = \delta \pm \varepsilon_k \to \delta,$$

which is the desired result.

Also note that this example shows that the condition that $d_j - d_1$ be rational is essential in Lemma 1.

References

[1] G. H. Hardy and E. M. Wright, *The Theory of Numbers*, Oxford Univ. Press, 1945.

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