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NECESSARY AND SUFFICIENT CONDITIONS FOR CERTAIN HOMOLOGY 3-SPHERES TO HAVE SMOOTH $\mathbf{Z}_p$-ACTIONS

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We derive necessary and sufficient conditions for a broad class of homology 3-spheres, obtained as the gluing of two knot complements, to have $\mathbf{Z}_p$-actions.

We explore when a homology sphere, obtained as the gluing of two knot complements has any smooth periodic diffeomorphisms.

Introduction. Myers **[6]** has given examples of homology spheres with no P.L. involutions. He constructed these examples by gluing together particular types of knot complements. Initially, we construct different types of periodic homology spheres out of different types of periodic knots. One of our constructions yields homology spheres with orientation reversing involutions in a simpler manner than that of Siebenmann and Van Buskirk **[8]**. Next, we prove the necessity of our conditions and are then able to construct infinitely many non-periodic homology spheres. In addition, we establish conditions for such a homology sphere to be the branched cyclic cover of a knot.

We shall use the following notation. Let K_0 and K_1 be distinct knots in S^3. Let $N(K_i)$ be a tubular neighborhood of K_i. Let

$$Q_i = S^3 - \operatorname{Int} N(K_i).$$

Let $l_i \subseteq \partial Q_i$ be an oriented longitude for Q_i in the sense that l_i bounds a surface in Q_i. Let $m_i \subseteq \partial Q_i$ be an oriented meridian for Q_i in the sense that m_i bounds a meridional disk in $N(K_i)$. Let $M(K_0, K_1)$ be the irreducible homology 3-sphere obtained by gluing each l_i on Q_i to a m_j on Q_j, $i \neq j$. Then $Q_0 \cap Q_1 = T$ a torus. We work throughout in the smooth category.

Our main result will be:

THEOREM 3. *Let K_0 and K_1 be distinct prime knots having Property P and neither being a companion of the other. In addition suppose K_0 is not a torus knot or a cable knot. Let p be a prime number, and N_i a tubular neighborhood of K_i.*

1. *$M(K_0, K_i)$ has an orientation reversing $\mathbf{Z}_p$-action h iff K_i is strongly positive amphericheiral and K_j is strongly negative amphicheiral.*

2. *$M(K_0, K_1)$ has an orientation preserving $\mathbf{Z}_p$-action h with* $\text{fix}(h) \cong S^1$ *iff K_i and K_j are both strongly invertible.*

3. *$M(K_0, K_1)$ has a free $\mathbf{Z}_p$-action iff K_0 has a free $\mathbf{Z}_p$-action leaving a $(1, s)$ curve on ∂N_0 invariant and K_1 has a free $\mathbf{Z}_p$-action leaving an $(s, 1)$ curve on ∂N_1 invariant, for some s which is knot a multiple of p.*

We begin with some definitions.

DEFINITION 1. A knot is *strongly negative amphicheiral* if there is a smooth involution g of S^3 which is orientation reversing and g takes K to itself with opposite orientation.

DEFINITION 2. A knot is *strongly positive amphicheiral* if there is a smooth involution g of S^3 which is orientation reversing and g takes K to itself with the same orientation.

DEFINITION 3. A knot is *strongly invertible* if there is an orientation preserving involution g of S^3 and g takes K to itself with opposite orientation.

REMARK. If h is an involution such that $h(K) = -K$ then h fixes two points on K.

DEFINITION 4. A knot K has a *free $\mathbf{Z}_p$-action* if there is an order p diffeomorphism of S^3, leaving K invariant yet which is fixed point free.

DEFINITION 5. A knot K has a *symmetry* if there is a periodic diffeomorphism leaving K invariant yet fixing a simple closed curve disjoint from K.

DEFINITION 6. A (p, q) curve on a torus is a curve wrapping around p times longitudinally and q times meridionally.

DEFINITION 7. $\text{fix}(h)$ shall denote the fixed point set of h.

We begin to construct periodic homology spheres by understanding the behavior of an action in a tubular neighborhood of the knot.

LEMMA 1. *Let K be a knot in S^3 with a tubular neighborhood N, and let p be a prime number. Suppose K has an orientation preserving, order p,* *diffeomorphism h such that $h(N) = N$ and h fixes some point of N. Then h fixes 4 points of ∂N and h is an involution with $h(K) = -K$.*

Proof. Since $\text{fix}(h) \neq \varnothing$, by Smith Theory $\text{fix}(h)$ is a circle. Assume $\text{fix}(h) \subseteq \text{Int } N$. Now $\text{fix}(h)$ is not knotted by the Smith Conjecture [**13**]. But K is knotted, so by Schubert [7] $\text{fix}(h)$ must have order zero in N. Let D be a meridional disk of N which misses $\text{fix}(h)$. Let X be the orbit space of N induced by h. Now apply Dehn's Lemma in X to modify D so that $h^i(D) \cap D = \varnothing$ for all $i < p$. Let B be the component of $N - \bigcup_{i=0}^{p-1} h^i(D)$ which contains $\text{fix}(h)$. Let S be the 2-sphere bounding B. Now $h(B) = B$ so $h(S) = S$. Now $\text{fix}(h) \cap S = \varnothing$ so by Smith Theory for S^2, h must be orientation reversing. But this is not possible, since h is orientation preserving. Thus $\text{fix}(h) \subseteq \text{Int } N$. So $\text{fix}(h) \cap \partial N \neq \varnothing$. Let r be the number of points in $\text{fix}(h) \cap \partial N$. Let Y be the orbit space of ∂N induced by h. Then ∂N is the p-fold branched cover of y with r branch points. So by the Riemann-Hurwitz formula $p\chi(Y) = r(p-1)$. Now Y is an orientable 2-manifold and $r(p-1) > 0$. So Y is a 2-sphere. Now $2p = r(p-1)$. Thus $(p-1)$ divides 2 or p. In other words $p - 1 = 2$ or $p - 1 = 1$. If $p = 3$ then $r = 3$. But r is the number of times $\text{fix}(h)$ crosses ∂N. Hence r must be even, since ∂N separates. Therefore $p = 2$, so $r = 4$. Now h takes a meridional disk of N to a meridional disk, and a Seifert surface to a Seifert surface. Thus $h(l)$ is isotopic to $\pm l$, where l is a longitude. Let A be an arc in Y connecting two branch points. Let c be the complete lift of A to ∂N. Then c is a simple closed curve in ∂N such that $h(c) = -c$. Now since h is orientation preserving $h(l)$ must in fact be isotopic to $-l$. Finally $h(K) = -K$ since by hypothesis $h(K) = \pm K$. □

LEMMA 2. *Let K be a knot in S^3 with a tubular neighborhood N, and let p be a prime number. Suppose h is a free $\mathbf{Z}_p$-action of K such that $h(N) = N$. Then $h(c) = c$ for some $(r, 1)$ curve c, and $h(c') = c'$ for some $(1, s)$ curve c', and $rs \equiv 1\ (p)$.*

Proof. It follows from Hartley [3, Theorem 1.1 and the subsequent sentence] that h leaves some $(r, 1)$ curve c invariant.

Now by applying Dehn's Lemma in the orbit space of N we can find a meridional disk D such that $h^i(D) \cap D = \varnothing$. Let $m = \partial D$, and let A' be one component of $\partial N - \bigcup_{i=0}^{p-1} h^i(m)$. Let a' be an arc in A' from some point $x' \in m$ to $h^j(x')$ such that a' goes less than once around A'. Define $c' = \bigcup_{i=0}^{p-1} h^i(a')$. Then c' is a $(1, s)$ curve.

The essential intersection of c and c' must consist of $rs - 1$ points. Now h permutes these points since $h(c) = c$ and $h(c') = c'$. Thus p divides $rs - 1$. So $rs \equiv 1\ (p)$. □

THEOREM 1. *Let K_0 and K_1 be knots in S^3, and let p be a prime number.*

1. *If K_0 is strongly positive amphicheiral, K_1 is strongly negative amphicheiral, and both knots are prime, then $M(K_0, K_1)$ has an orientation reversing involution h with* $\text{fix}(h) \cong S^0$.

2. *If K_0 and K_1 are both strongly invertible then $M(K_0, K_1)$ has an orientation preserving involution h with* $\text{fix}(h) \cong S^1$.

3. *If K_0 has a free $\mathbf{Z}_p$-action leaving a $(1, s)$ curve invariant and K_1 has a free $\mathbf{Z}_p$-action leaving an $(s, 1)$ curve invariant, then $M(K_0, K_1)$ has an orientation preserving $\mathbf{Z}_p$-action h which is fixed point free.*

Proof. 1. Let N_i be a tubular neighborhood of K_i such that $h_i(N_i) = N_i$ where h_i is an orientation reversing involution with $h_0(K_0) = +K_0$ and $h_1(K_1) = -K_1$. By Smith Theory, since the h_i are orientation reversing, $\text{fix}(h_i) \cong S^0$ or $\text{fix}(h_i) \cong S^2$. Suppose $\text{fix}(h_i) \cong S^2$. Then by Smith Theory for K_i, h_i fixes either zero or two points of K_i. Now K_i cannot be contained in one component of $S^3 - \text{fix}(h_i)$ since h_i must trade these components. Thus $\text{fix}(h_i) \cap K_i$ consists of two points. Hence K_i is composite, contrary to hypothesis. So $\text{fix}(h_i) \cong S^0$ for both i. Now we can assume we have picked N_i such that $\text{fix}(h_i) \cap \partial N_i = \varnothing$ for both i. Thus $h_i|\partial N_i$ covers its induced orbit space, which must be a Klein bottle. So $h_0|\partial N_0$ and $h_1|\partial N_1$ are equivalent actions. Let l_i be a longitude and let m_i be a meridian for N_i. Now h_i takes a meridional disk to a meridional disk and a Seifert surface to a Seifert surface. So $h_i(m_i)$ is isotopic to $\pm m_i$ and $h_i(l_i)$ is isotopic to $\pm l_i$. Now since $h_0(K_0) = +K_0$ we must have $h_0(l_0) \sim +l_0$, and since h_0 is orientation reversing $h_0(m_0) \sim -m_0$. Similarly $h_1(K_1) = -K_1$ implies that $h_1(l_1) \sim -l_1$, and so $h_1(m_1) \sim +m_1$ since h_1 is orientation reversing. Let $Q_i = S^3 - \text{Int}(N_i)$. Now we can glue Q_0 to Q_1 along their boundaries longitude to meridian to obtain $M(K_0, K_1)$; and $M(K_0, K_1)$ has an orientation reversing involution h where $h|Q_i = h_i$. By Smith Theory for homology 3-spheres, $\text{fix}(h) \cong S^2$ or $\text{fix}(h) \cong S^0$. But $\text{fix}(h) = \text{fix}(h_0|Q_0) \cup \text{fix}(h_1|Q_1)$; thus $\text{fix}(h)$ cannot be S^2. So $\text{fix}(h) \cong S^0$ as desired.

2. Suppose K_0 and K_1 are strongly invertible. Let h_i be an orientation preserving involution of S^3 such that $h_i(K_i) = -K_i$ and $h_i(N_i) = N_i$ for some tubular neighborhood N_i of K_i. Then h_i fixes two points on K_i for each i. Now by Lemma 1 h_i fixes 4 points on ∂N_i, and so ∂N_i is the

two-fold branched cover of its orbit space, which is a 2-sphere. Now the action induced by h_i on ∂N_i is equivalent to reflecting both meridionally and longitudinally. Let $Q_i = S^3 - \operatorname{Int}(N_i)$. Now glue Q_0 to Q_1 longitude to meridian to obtain $M(K_0, K_1)$; and $M(K_0, K_1)$ will have an orientation preserving involution h where $h_i|Q_i = h|Q_i$. Now by Smith Theory for homology spheres $\operatorname{fix}(h) \cong S^1$ or $\operatorname{fix}(h) = \varnothing$. Now $\operatorname{fix}(h)$ includes the 4 points fixed on ∂N_i, thus $\operatorname{fix}(h) \neq \varnothing$. So $\operatorname{fix}(h) \cong S^1$.

3. Suppose each K_i has a smooth free $\mathbf{Z}_p$-action h_i. Let N_i be a tubular neighborhood of K_i such that $h_i(N_i) = N_i$ and there is a $(1, s)$ curve c_0, and an $(s, 1)$ curve c_1 on ∂N_i such that $h_i(c_i) = c_i$. Pick numbers r_0 and r_1 such that $(h_i^{r_i})$ is just a $2\pi/p$ rotation along c_i. Now glue the

$$Q_i = S^3 - \operatorname{Int}(N_i)$$

together longitude to meridian attaching c_0 to c_1. Now $M(K_0, K_1)$ has an orientation preserving $\mathbf{Z}_p$-action h where $h|Q_i = h_i^{r_i}|Q_i$. Also, $h_i^{r_i}$ is fixed point free since p is a prime, so h is fixed point free as well. □

REMARKS. 1. The construction in the first case of Theorem 1 provides a simpler method of obtaining an irreducible homology 3-sphere with an orientation reversing involution than that of Siebenmann and Van Buskirk [**8**].

2. We can find knots satisfying the conditions of case 3 of Theorem 1 by lifting knots in the appropriate lens spaces.

Now that we are able to construct homology spheres with $\mathbf{Z}_p$-actions with different types of fixed point sets we want to know whether the conditions on knot pairs in Theorem 1 are, in fact, necessary. That is, if $M(K_0, K_1)$ has a $\mathbf{Z}_p$-action in what sense do K_0 and K_1 inherit the action? We show that with a few added hypotheses our conditions are necessary. The first hypothesis we will add is that one knot complement contains no essential annulus. According to Simon [**9**, Lemmas 2.1 and 2.2] if the knot is not a composite, torus or cable knot then its complement contains no essential annulus.

LEMMA 3. *Suppose K_0 is a prime knot which is not a torus knot or a cable knot, and suppose K_1 is any knot. Let Q_i be the closed complement of K_i. Then every incompressible torus in $M(K_0, K_1)$ is isotopic to one disjoint from $T = \partial Q_i$.*

Proof. Let T' be an incompressible torus in M. Isotop T' so that T and T' meet transversely in a minimal number of components. Suppose some component J of $T \cap T'$ bounds a disk D' in T'. Pick J to be innermost, i.e. $D' \cap T = \partial D'$. Since T is incompressible, J also bounds a disk D in T. Now since $M(K_0, K_1)$ is irreducible, $D \cup D'$ bounds a 3-ball B. By pushing D' across B we can remove J from $T \cap T'$, thus contradicting minimality. So there is no component of $T \cap T'$ which bounds a disk in T'. Hence $T' \cap Q_0$ consists of properly embedded incompressible annuli. But by Simon [**9**, Lemmas 2.1 and 2.2] any such annulus must be boundary parallel in Q_0. So again we could remove the boundaries of this annulus by an isotopy of T', and so contradict minimality. Thus $T \cap T' = \varnothing$. □

REMARK 3. Let h be a periodic diffeomorphism of a 3-manifold M, then, as is well known, we can choose a Riemannian metric for M which makes h an isometry. (Take the average of the h-transforms of any Riemannian metric.)

LEMMA 4. *Let K_0 be a prime knot other than a torus knot or a cable knot, and let K_1 be any knot. Let Q_i be the closed complement of K_i. Let h be a periodic diffeomorphism of $M(K_0, K_1)$. Then $T = \partial Q_i$ is isotopic to a surface S such that either $h(S) = S$ or $h(S) \cap S = \varnothing$.*

Proof. By Theorem 1.1 of Freedman, Hass and Scott [**2**] since $M(K_0, K_1)$ is P^2-irreducible there is a least area immersion $f\colon T \to M$ which is homotopic to the inclusion $i\colon T \to M$. Since i is incompressible f must also be incompressible. Now by Alexander duality, since $H_1(M(K_0, K_1)) = 0$ every closed surface in $M(K_0, K_1)$ is two-sided. By [**2**, Theorem 5.1] f is an embedding. Hence $f(T)$ is a two-sided least area incompressible embedded torus. Since h is an isometry, $h(f(T))$ is also a two-sided least area incompressible embedded torus.

Since T and $f(T)$ are homotopic incompressible surfaces in an irreducible 3-manifold, by Waldhausen [**12**, Corollary 5.5] T and $f(T)$ are ambient isotopic. Thus we can apply Lemma 3 to $f(T)$ to conclude that $h(f(T))$ can be isotoped disjoint from $f(T)$. Now apply Theorem 6.2 of [**2**] to conclude that either $h(f(T)) = f(T)$ or $h(f(T)) \cap f(T) = \varnothing$; and let $S = f(T)$. □

DEFINITION 8. If h is a periodic diffeomorphism of $M(K_0, K_1)$ and $h(Q_i) = Q_i$, then we say h is "good".

THEOREM 2. *Suppose K_0 and K_1 are distinct prime knots having property P and neither is a companion of the other. In addition suppose K_0 is* *not a torus knot or a cable knot. Then if $M(K_0, K_1)$ has a periodic diffeomorphism h then $M(K_0, K_1)$ has a good diffeomorphism h' which is conjugate to h.*

Proof. By the conditions on K_0 and K_1 we can apply Lemma 4 to get an isotopy f_t of $M(K_0, K_1)$ such that f_0 is the identity and either $h(f_1(T)) = f_1(T)$ or $h(f_1(T)) \cap f_1(T) = \varnothing$. Let $h' = f_1^{-1} \circ h \circ f_1$. Then h' has the same order as h and either $h'(T) = T$ or $h'(T) \cap T = \varnothing$. Suppose $h'(T) \cap T = \varnothing$. Recall, Q_i is the closed complement of K_i in S^3. Assume $h'(T) \subseteq \mathrm{Int}(Q_0)$. If $h'(Q_0) \subseteq \mathrm{Int}(Q_0)$ then

$$Q_0 = (h')^p(Q_0) \subseteq \mathrm{Int}(Q_0).$$

This contradiction implies that $h'(Q_0) \not\subseteq \mathrm{Int}(Q_0)$ and hence

$$h'(Q_1) \subseteq \mathrm{Int}(Q_0) \subseteq S^3,$$

since $h'(Q_1) \neq Q_0$ by Property P. Now $h'(T)$ is essential in $M(K_0, K_1)$ so it is essential in Q_0 by Waldhausen [**11**, satz 1.9]. Let V be the component of $S^3 - h'(T)$ containing K_0. Thus

$$V \cup h'(Q_1) = S^3 \quad \text{and} \quad V \cap h'(Q_1) = h'(T).$$

Hence V must be a solid torus. Let J be the core of V. Then J is isotopic to K_1, since K_1 has Property P. Now since $h'(T)$ is essential in Q_0, we must have either K_1 is a companion of K_0 or K_1 is K_0 itself. Either case contradicts our hypotheses. We now use the same argument to show that $h'(T) \not\subseteq \mathrm{Int}(Q_1)$. Thus $h'(T) = T$. Now since both knots have Property P, $h'(Q_i) = Q_i$. □

REMARKS.

4. Since l_i bounds a Seifert surface in Q_i, $h(l_i)$ must also. So on T, $h(l_i)$ is isotopic to $\pm l_i$. But since the l_i are identified with the m_j, for $i \neq j$, $h(m_j)$ is isotopic to $\pm m_i$.

5. By the above remark we can interpret the conclusion of Theorem 2 as saying that if $M(K_0, K_1)$ has a period p diffeomorphism h, then K_0 and K_1 each have a period p diffeomorphism h_i.

LEMMA 5. *Let K be a knot in S^3 with a tubular neighborhood N and let p be a prime number. Suppose h is a symmetry of K such that $h(N) = N$; and that for some (r, s) curve c on ∂N, $r \neq 0$, $h(c) = c$. Then s is a multiple of p.*

Proof. By Edmonds and Livingston [**1**, Corollary 2.2] K bounds a Seifert surface which is invariant under h. By intersecting this surface with

∂N we can find a longitude l such that $h(l) = l$. Now c and l must intersect essentially in s points; and h permutes these essential intersections. Thus p must divide s. □

THEOREM 3. *Let K_0 and K_1 be distinct prime knots having Property P and neither being a companion of the other. In addition suppose K_0 is not a torus knot or a cable knot. Let p be a prime number. Then*:

1. *$M(K_0, K_1)$ has an orientation reversing $\mathbf{Z}_p$-action h iff K_i is strongly positive amphicheiral and K_j is strongly negative amphicheiral.*

2. *$M(K_0, K_1)$ has an orientation preserving $\mathbf{Z}_p$-action h with* $\operatorname{fix}(h) \cong S^1$ *iff K_i and K_j are both strongly invertible.*

3. *$M(K_0, K_1)$ has a free $\mathbf{Z}_p$-action iff K_0 has a free $\mathbf{Z}_p$-action leaving a $(1, s)$ curve, on the boundary of a tubular neighborhood, invariant; and K_1 has a free $\mathbf{Z}_p$-action leaving an $(s, 1)$ curve, on the boundary of a tubular neighborhood, invariant, and $s \not\equiv 0\ (p)$.*

Proof. We have already established the sufficiency of the conditions in Theorem 1. We now establish necessity. Let h be an order p diffeomorphism of $M(K_0, K_1)$. By Theorem 2 we can assume that $h_i(Q_i) = Q_i$, where Q_i is the closed complement of K_i. By Remarks 4 and 5 each K_i has an order p diffeomorphism h_i such that $h_i|Q_i = h|Q_i$.

Case 1. If h is orientation reversing. Then since p is a prime, in fact $p = 2$. By Remark 4 $h(l_i)$ is isotopic to $\pm l_i$ on T. Now l_i is identified with m_j, where m_j is a meridian for Q_j. Since h is orientation reversing if $h(l_i)$ is isotopic to $+l_i$ then $h(m_i)$ is isotopic to $-m_i$. Thus $h(l_j)$ is isotopic to $-l_j$. Hence $h(K_i) = +K_i$ but $h(K_j) = -K_j$. So K_i is strongly positive amphicheiral, whereas K_j is strongly negative amphicheiral.

Case 2. Now suppose h is orientation preserving and $\operatorname{fix}(h) \cong S^1$. First, assume $\operatorname{fix}(h) \cap \partial Q_i = \varnothing$. Then for some i, say $i = 0$, $\operatorname{fix}(h) \subseteq Q_i$. Thus $\operatorname{fix}(h_1|Q_1) = \varnothing$. Now if h_1 fixed any point of $N_1 = S^3 - \operatorname{Int}(Q_1)$ then by Lemma 1, h would fix points on ∂Q_1. Hence $\operatorname{fix}(h_1) = \varnothing$. So h_1 is a free $\mathbf{Z}_p$-action on K_1. Now by Lemma 2 there is an $(s, 1)$ curve c on ∂Q_1 such that $h_1(c) = c$. Now c is also a $(1, s)$ curve on ∂Q_0 and $h_0(c) = c$. But $\operatorname{fix}(h) \subseteq Q_0$ so h_0 is a symmetry of K_0. Thus by Lemma 5 s is a multiple of p. This is impossible, so $\operatorname{fix}(h) \cap \partial Q_i \neq \varnothing$. Now by Lemma 1 h is an involution and both K_i are strongly invertible.

Case 3. Here h is a free $\mathbf{Z}_p$-action of $M(K_0, K_1)$. So as in Case 2 we can apply Lemma 1 to conclude that h_i is a free $\mathbf{Z}_p$-action for both i. Now apply Lemma 2 to h_0, to get a $(1, s)$ curve c on ∂Q_0 which is invariant under h_0. Now c is also a $(s, 1)$ curve on ∂Q_1 and is thus invariant under $h_1|\partial Q_1 = h_0|\partial Q_0$. Also by Lemma 1, s has a multiplicative inverse in $\mathbf{Z}/p\mathbf{Z}$ so $s \not\equiv 0\,(p)$. □

REMARK 6. Note, in case 1, that fix$(h) \not\cong S^2$ since the knots are prime. Hence by Smith Theory fix$(h) \cong S^0$.

COROLLARY 1. *Let K_0 and K_1 be distinct prime knots having Property P and neither being a companion of the other. In addition suppose K_0 is not a torus knot or a cable knot. If $M(K_0, K_1)$ is any finite cyclic branched cover with branch set S^1, then both K_i are strongly invertible.*

Proof. If $M(K_0, K_1)$ were a finite cyclic branched cover of a knot then $M(K_0, K_1)$ has a covering translation f and fix$(f) \cong S^1$. Let h be f raised to a power, if necessary, so that the order of h is a prime. Now f was orientation preserving so h is orientation preserving. Now by Smith Theory since fix$(f) \subseteq$ fix(h) in fact fix$(h) \cong S^1$. Now apply Case 2 of Theorem 3. □

DEFINITION 9. A knot is *simple* if its complement in S^3 contains no essential torus.

REMARK 7. A simple knot has no companions. So if the K_i are simple non-torus knots, it is enough to assume $Q_0 \not\cong Q_1$ in the proof of Theorem 2, we do not actually need Property P. Hence also in Theorem 3.

COROLLARY 2. *Let K_0 and K_1 be simple knots with non-homeomorphic exteriors. Suppose at least one of the knots is non-amphicheiral, one is non-invertible, and one has no free $\mathbf{Z}_p$-action. Then $M(K_0, K_1)$ has no periodic diffeomorphisms.*

Proof. Observe that a torus knot has free $\mathbf{Z}_p$-actions. So if K_0 is the knot with no free $\mathbf{Z}_p$-actions then K_0 is not a torus knot. Since the knots are simple they are prime, neither is a companion of the other, and neither is a cable knot. Hence we can apply Theorem 3. □

This corollary easily provides examples of irreducible homology 3-spheres with no smooth $\mathbf{Z}_p$-actions. We show how to get an infinite collection of such homology spheres. Let K_0 be the 5_2 knot. Then K_0 is

2-bridge so it is simple by Kawauchi [5] and is non-amphicheiral. The Alexander polynomial of K_0 is $\Delta_{K_0}(t) = -2t^2 + 3t - 2$ so by Hartley [3] K_0 has no free $\mathbf{Z}_p$-actions. Now, if q, r and s are distinct odd numbers bigger than one then the pretzel knot $K(q, r, s)$ is non-invertible by Trotter [10]. Since it is prime and 3-bridge, by Kawauchi it is simple. So we let $K_1 = K(q, r, s)$. Now let $n = qs + qr + rs$, then $K(q, r, s)$ has Alexander Polynomial

$$\Delta_{K_1}(t) = t^2\left(\frac{n+1}{4}\right) - t\left(\frac{n+1}{2}\right) + \left(\frac{n+1}{4}\right).$$

By our conditions on q, r, s we have $n \geq 71$ so $\Delta_{K_0} \neq \Delta_{K_1}$ and thus $Q_0 \not\cong Q_1$. Now for distinct n the knots $K(q, r, s)$ will have distinct Alexander polynomials and hence their complements are non-homeomorphic. Let $M(K_0, K_1)$ be one such homology sphere and let $M(K_0, K_1')$ be another. Then since 5_2 and $K(q, r, s)$ are simple $M(K_0, K_1)$ and $M(K_0, K_1')$ each have precisely one incompressible torus. Suppose $M(K_0, K_1') \cong M(K_0, K_1)$, then there is a homeomorphism

$$h\colon M(K_0, K_1') \to M(K_0, K_1)$$

taking the incompressible torus T' to the incompressible torus T. Now since $Q_0 \not\cong Q_1$ we must have $h(Q_1') = Q_1$ hence the n for K_1' must be the same as the n for K_1. So for the infinite collection of distinct n we will get infinitely many different non-periodic homology 3-spheres.

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