

ON THE LOCATION OF ZEROES OF OSCILLATORY SOLUTIONS OF $y^{(n)} = c(x)y$

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We locate the zeroes of oscillatory solutions of wide classes of differential equations, $y^{(n)} = c(x)y$. Asymptotic techniques are used. The asymptotic behaviour of solutions and their derivatives up to the n th order are also provided.

New results are obtained in addition to old results becoming more transparent.

1. Introduction. The main purpose of this paper is to demonstrate a method for *locating* the zeroes of oscillatory solutions of the differential equation

$$(1.1) \quad y^{(n)} = c(x)y.$$

As shown by the references cited, the differential equation (1.1) attracted a considerable amount of attention. However, the location of zeroes of oscillatory solutions of (1.1) does not seem to be available in the literature. It is the purpose of this paper to fill this gap for a wide class of differential equations (1.1).

The method to be used exploits concepts of classical asymptotics which seem to us the most appropriate ones to handle problems of singular differential equations. The singularity of the differential equation (1.1) stems from the fact that the independent variable x ranges on an infinite interval and also from the fact that $c(x)$ may be unbounded.

We do assume an amount of smoothness on the coefficient $c(x)$ which is more restrictive than a continuity assumption made e.g. by Kim [12]. However, this is a reasonable price to be paid for obtaining the fine structure of $y^{(\nu)}(x)$, $\nu = 0, 1, \dots, n - 1$ as $x \rightarrow \infty$.

In particular, most of the asymptotic properties known so far on oscillatory and nonoscillatory solutions of (1.1) can be better understood by the techniques employed in this work.

An additional reward of this paper is that we produce Prufer type representatives for solutions of (1.1) which belong to certain subspaces of the linear space of solutions of (1.1). See e.g. Hille [9] p. 394.

The course of events in this paper will be as follows: After this introduction we proceed to §2, which contains preparations for an

asymptotic decomposition theorem. In §3 we prove an asymptotic decomposition theorem, and in §4 we find the location of zeroes of oscillatory solutions of (1.1). This paper is an attempt to generalize results of Gingold [5] for the case $n = 2$. Results of §4, in this paper, depend heavily on §5 in Gingold [5].

2. Preliminaries. We first convert the scalar differential equation (1.1) into a vectorial differential system for the vector

$$(2.1) \quad \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix}.$$

The resulting *matrix* differential system is

$$(2.2) \quad Y' = AY$$

with

$$(2.3) \quad A = \begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ c(x) & 0 & \cdot & \cdots & 0 \end{bmatrix}, \quad A = (a_{jk}), \quad j, k = 1, \dots, n.$$

In this matrix

$$(2.4) \quad a_{n1} = c(x),$$

$$(2.5) \quad a_{j,j+1} \equiv 1, \quad j = 1, \dots, n-1$$

and all other entries in this matrix are zero.

Notation 2.1. We will denote

$$(2.6) \quad r := r(x) = |c(x)|^{1/n}.$$

It is a simple exercise to verify that the eigenvalues of the matrix A denoted by $\lambda_k(x)$, $k = 1, \dots, n$, are easily found to be

$$(2.7) \quad \lambda_k(x) = r(x)(e^{i\theta_k}) = r(x)(\cos \theta_k + i \sin \theta_k),$$

where

$$(2.8) \quad \theta_k = \frac{\arg c(x)}{n} + \frac{2\pi}{n}(k-1), \quad k = 1, \dots, n, \quad c(x) \neq 0,$$

and $\arg z$ is defined for z complex

$$(2.9) \quad -\pi < \arg z = \operatorname{Im} \ln z \leq \pi.$$

We use the following asertion throughout this paper and we will not restate it in the sequel unless it will be replaced.

Assumption 2.2. Let J be the interval $[a, \infty)$. Then

(i)

$$(2.10) \quad |c(x)| \neq 0, \quad c(x) \text{ is real for } x \in J.$$

(ii) $c(x) \in C^2(J)$.

It is a simple consequence of Assumption 2.2 that $r(x) \in C^2(J)$. It is also a simple exercise to verify that the matrix A possesses n linearly independent eigenvectors which form the columns of the following Vandermonde matrix V ,

$$(2.11) \quad V = (v_{lk}), \quad l, k = 1, \dots, n,$$

$$(2.12) \quad v_{lk} = \lambda_k^{-1}(x).$$

Therefore, the transformation

$$(2.13) \quad Y = VZ$$

takes the differential equation (2.2) into

$$(2.14) \quad Z' = A_1 Z,$$

with

$$(2.15) \quad A_1 = [V^{-1}AV - V^{-1}V'],$$

and with

$$(2.16) \quad V^{-1}AV = \text{diag}\{\lambda_1(x), \dots, \lambda_n(x)\}.$$

We will need more properties of the matrix $V^{-1}V'$ in (2.15). Therefore we will need the following lemma:

LEMMA 2.3. *Denote by K a constant matrix,*

$$(2.17) \quad K = (k_{sd}), \quad s, d = 1, \dots, n,$$

to be introduced later. Denote by b also a constant to be introduced later. Then if V is the Vandermonde matrix given by (2.11), V is differentiable and

$$(2.18) \quad V^{-1}V' \equiv (\ln r)'bK.$$

Proof. In order to find explicitly $V^{-1}V'$ we let

$$(2.19) \quad V = \begin{bmatrix} 1 & 1 & \dots & 1 \\ (e^{i\theta_1})r & (e^{i\theta_2})r & \dots & (e^{i\theta_n})r \\ (e^{2i\theta_1})r^2 & (e^{2i\theta_2})r^2 & \dots & (e^{2i\theta_n})r^2 \\ \vdots & \vdots & \dots & \vdots \\ (e^{(n-1)i\theta_1})r^{n-1} & (e^{(n-1)i\theta_2})r^{n-1} & \dots & (e^{(n-1)i\theta_n})r^{n-1} \end{bmatrix}.$$

Then

$$(2.20) \quad \text{Det } V = r^{[1+2+\dots+(n-1)]} (\text{Det } V_1)$$

with,

$$(2.21) \quad V_1 = \begin{bmatrix} 1 & 1 & \dots & 1 \\ e^{i\theta_1} & e^{i\theta_2} & & e^{i\theta_n} \\ e^{2i\theta_1} & e^{2i\theta_2} & & e^{2i\theta_n} \\ \vdots & \vdots & & \vdots \\ e^{(n-1)i\theta_1} & e^{(n-1)i\theta_2} & & e^{(n-1)i\theta_n} \end{bmatrix}.$$

Thus, one can verify that

$$(2.22) \quad \text{Det } V = r^{[n(n-1)]/2} \left[\prod_{k < l} (e^{i\theta_l} - e^{i\theta_k}) \right].$$

Next we turn to the adjugant of V in order to find V^{-1} . To this end we evaluate c_{lk} the cofactors of V as follows: Let \tilde{V}_{lk} be the following $(n - 1) \times (n - 1)$ matrix which is obtained from V by eliminating the l th row and the k th column in an obvious manner. (The arrows point respectively to the eliminated row and column in (2.23)).

$$(2.23) \quad \begin{array}{c} \text{column } k \\ \downarrow \\ \begin{array}{l} V_{lk} = \\ \text{row } l \rightarrow \end{array} \left[\begin{array}{cccc} 1 & & & 1 \\ (e^{i\theta_1})r & & (e^{i\theta_k})r & (e^{i\theta_n})r \\ (e^{i(l-1)\theta_1})r^{l-1} & (e^{i(l-1)\theta_2})r^{l-1} & \dots & (e^{i(l-1)\theta_k})r^{l-1} \dots (e^{i(l-1)\theta_n})r^{l-1} \\ (e^{i(n-1)\theta_1})r^{n-1} & \dots & & (e^{i(n-1)\theta_n})r^{n-1} \end{array} \right]. \end{array}$$

Then,

$$(2.24) \quad c_{lk} = (-1)^{l+k} \frac{r^{[1+2+\dots+(n-1)]}}{r^{(l-1)}} (\text{Det } \tilde{V}_k)$$

with

$$(2.25) \quad \begin{array}{c} \text{column } k \\ \downarrow \\ \begin{array}{l} \tilde{V}_{lk} = \\ \text{row } l \rightarrow \end{array} \left[\begin{array}{cccc} 1 & & & 1 \\ e^{i\theta_1} & & e^{i\theta_k} & e^{i\theta_n} \\ \vdots & & \vdots & \vdots \\ e^{i(l-1)\theta_1} & \dots & e^{i(l-1)\theta_k} & \dots & e^{i(l-1)\theta_n} \\ \vdots & & & & \vdots \\ e^{i(n-1)\theta_1} & & & \dots & e^{i(n-1)\theta_n} \end{array} \right]. \end{array}$$

Obviously $(\text{Det } \tilde{V}_{lk})$ is a constant. Therefore

$$(2.26) \quad c_{lk} = (-1)^{l+k} r^{[n(n-1)/2-(l-1)]} (\text{Det } \tilde{V}_{lk}).$$

The entries of V^{-1} are given by

$$(2.27) \quad (\text{Det } V)V^{-1} = (\hat{v}_{sj}), \quad s, j = 1, \dots, n,$$

with

$$(2.28) \quad \hat{v}_{sj} = c_{js}.$$

By (2.19) one has

$$(2.29) \quad V' = (v'_{lk}), \quad v'_{lk} = (l-1)e^{i(l-1)\theta_k} r^{(l-2)} r'.$$

Denote by u_{sd} , $s, d = 1, \dots, n$ the entries of

$$(2.30) \quad (\text{Det } V)V^{-1}V' = (u_{sd}).$$

Then

$$(2.31) \quad \begin{aligned} u_{sd} &= \sum_{j=1}^n \hat{v}_{sj} v'_{jd} = \sum_{j=1}^n c_{js} v'_{jd} \\ &= \sum_{j=1}^n (-1)^{j+s} r^{[n(n-1)/2-(j-1)]} (\text{Det } \tilde{v}_{js}) (j-1) e^{i(j-1)\theta_d} r^{(j-2)} r' \\ &= r^{[n(n-1)/2]} r^{-1} r' \sum_{j=1}^n (-1)^{j+s} (j-1) e^{i(j-1)\theta_d} (\text{Det } \tilde{V}_{js}). \end{aligned}$$

We now let

$$(2.32) \quad k_{sd} = \sum_{j=1}^n (-1)^{j+s} (j-1) e^{i(j-1)\theta_d} (\text{Det } \tilde{V}_{js}),$$

and

$$(2.33) \quad b^{-1} := \left[\prod_{k < l} (e^{i\theta_l} - e^{i\theta_k}) \right]$$

and the result follows.

Throughout this work, we will assume that a suitable norm $\| \cdot \|$ defined on the set of n by n matrices is given.

We now plan to perform a second linear transformation. To this end, we need the following:

Assumption 2.4. The nonnegative mapping $h(x)$ given by

$$(2.34) \quad h(x) := \int_x^\infty |(r^{-2}r')'| dt$$

is finite on J . Also,

$$(2.35) \quad \lim_{x \rightarrow \infty} (r^{-2}(x)r'(x))^{(k)} = 0, \quad k = 0, 1,$$

and (therefore, without loss of generality) all eigenvalues of $r^{-1}A_1$ are distinct on J .

We can now provide the following needed lemma:

LEMMA 2.5. Consider the matrix $r^{-1}A_1$. Let assumptions 2.2, 2.4 hold. Then, there exists on J an n by n matrix function $T(x)$ with the following properties;

$T(x)$ is invertible and continuously differentiable on J .

$$(2.36) \quad T(x) = I + \Delta(x)$$

where

$$(2.37) \quad \|\Delta(x)\| = \mathcal{O}(h(x)), \quad x \rightarrow \infty.$$

Also,

$$(2.38) \quad \tilde{\Omega} := T^{-1}(x)[r^{-1}(x)A_1(x)]T(x) = \text{diag}\{\mu_1(x), \dots, \mu_n(x)\}$$

where $\mu_j(x)$, $j = 1, \dots, n$, are the n distinct eigenvalues of $r^{-1}A_1$ with the ordering

$$(2.39) \quad \lim_{x \rightarrow \infty} \mu_j(x) = e^{i\theta_j}, \quad j = 1, \dots, n.$$

Moreover,

$$(2.40) \quad \|T'(x)\| = \mathcal{O}(h'(x)), \quad x \rightarrow \infty.$$

Proof. A proof follows from the discussions in Gingold [6, 7].

We proceed to the next section.

3. Asymptotic decomposition. In this section, we intend to find a fundamental solution of (2.2) by use of an ‘‘asymptotic decomposition theorem.’’

THEOREM 3.1. Let assumptions 2.2, 2.4 hold. Assume that throughout J $r'(x)$ does not change sign if $\ln r(x)$ is unbounded and

$$(3.1) \quad \sum_{j \neq \nu} |\text{Re } b(k_{jj} - k_{\nu\nu})| \neq 0, \quad j, k = 1, \dots, n.$$

Let

$$(3.2) \quad \int_a^\infty |(r^{-3/2}r')^2| ds < \infty.$$

Then, there exists an interval $\hat{J} = [\hat{a}, \infty)$, $a \leq \hat{a}$, such that a fundamental solution of (2.2) on \hat{J} is given by

$$(3.3) \quad Y = V(I + P(x)) \exp \int^x D(s) ds.$$

The n by n matrix function $P(x)$ is continuously differentiable on \hat{J} and

$$(3.4) \quad \lim_{x \rightarrow \infty} \|P(x)\| = 0.$$

The matrix D is given by

$$(3.5) \quad D = \text{diag}[re^{i\theta_1} - b(\ln r)'k_{11}, \dots, re^{i\theta_n} - b(\ln r)'k_{nn}].$$

Proof. First we show that on an interval \hat{J} , a fundamental solution of (2.2) can be found such that

$$(3.6) \quad Y = V(I + \Delta)(I + \tilde{P}) \exp \int^x \tilde{D}(s) ds.$$

In (3.6), $\tilde{P} = \tilde{P}(x)$ is a certain n by n matrix function such that for $x \in \hat{J}$

$$(3.7) \quad \|\tilde{P}(x)\| < 1 \text{ and } \lim_{x \rightarrow \infty} \|\tilde{P}(x)\| = 0.$$

The matrix \tilde{D} is a diagonal matrix defined as follows: Denote by s_{jk} , $j, k = 1, \dots, n$, the elements of $T^{-1}(x)T'(x)$. Denote

$$(3.8) \quad \hat{S} = \text{diag}[s_{11}, \dots, s_{nn}].$$

Then

$$(3.9) \quad \tilde{D} := r\tilde{\Omega} - \hat{S}$$

With the help of Lemma 2.5 we notice that the transformation

$$(3.10) \quad Z = TZ_2$$

takes the differential system (2.14) into

$$(3.11) \quad Z'_2 = (T^{-1}A_1T - T^{-1}T')Z_2.$$

The differential system (3.11) turns out to be an “almost diagonal system”. See Gingold [7]. Thus, there exists an interval \hat{J} such that a fundamental matrix solution of (3.11) is given by

$$(3.12) \quad Z_2 = (I + \tilde{P}(x)) \exp \int^x \tilde{D}(s) ds$$

on \hat{J} , where $\tilde{P}(x)$ satisfies (3.7). This implies (3.6).

We now intend to show that

$$(3.13) \quad \lim_{x \rightarrow \infty} \int_x^\infty \|D(s) - \tilde{D}(s)\| ds = 0.$$

Consider the determinant of $r^{-1}A_1 - \mu I$.

It is a simple exercise to verify that $p_n(\mu)$, the characteristic polynomial of $r^{-1}A_1$ has the following form:

$$(3.14) \quad p_n(\mu) = \prod_{j=1}^{j=n} (e^{i\theta_j} - bk_{jj}r^{-2}r' - \mu) + (r^{-2}r')^2 p_{n-2}(\mu)$$

where $p_{n-2}(\mu)$ is a polynomial of order $n - 2$ in μ . It possesses continuous coefficients in x on the closed interval $[\hat{a}, \infty]$.

Construct the new polynomial

$$(3.15) \quad q_n(\mu, \varepsilon) = \prod_{j=1}^{j=n} (e^{i\theta_j} - bk_{jj}r^{-2}r' - \mu) + \varepsilon p_{n-2}(\mu)$$

where ε is a complex number.

Without loss of generality, we may assume that $q_n(\mu, \varepsilon)$ possesses distinct eigenvalues on \hat{J} and $|\varepsilon| \leq \varepsilon_0$ where ε_0 is a small positive number.

Regard the roots of $q_n(\mu, \varepsilon) = 0$ as functions of ε (and the parameter x). Then by considering the initial value problems

$$(3.16) \quad \frac{d\mu}{d\varepsilon} = -\frac{p_{n-2}(\mu)}{\partial q_n / \partial \mu}, \quad \mu(0) = e^{i\theta_j} - bk_{jj}r^{-2}r', \quad j = 1, \dots, n,$$

we deduce that the n distinct roots of $q_n(\mu, \varepsilon)$ are holomorphic functions of ε for $|\varepsilon| \leq \varepsilon_0$.

Moreover,

$$(3.17) \quad \mu = e^{i\theta_j} - bk_{jj}r^{-2}r' + \mathcal{O}(\varepsilon), \quad j = 1, \dots, n,$$

where $\mathcal{O}(\varepsilon)$ is uniform with respect to x on \hat{J} .

In (3.15), by letting

$$(3.18) \quad \varepsilon = (r^{-2}r')^2$$

we conclude that

$$(3.19) \quad r^{-1}\mu_j = e^{i\theta_j} - bk_{jj}r^{-1}r' + \mathcal{O}((r^{-2}r')^2).$$

We have

$$(3.20) \quad \int_x^\infty \|D(s) - \tilde{D}(s)\| ds \leq \int_x^\infty \|D(s) - r(s)\tilde{\Omega}(s)\| ds + \int_x^\infty \|\hat{S}(s)\| ds.$$

This implies (3.13) by combining (3.2) and (2.34). Define now

$$(3.21) \quad P(x) := (I + \Delta(x))(I + \tilde{P}(x)) \exp \int_\infty^x [\tilde{D}(s) - D(s)] ds - I$$

and the result follows.

REMARKS. It is also possible to use a transformation of the independent variable x given by

$$(3.22) \quad \eta = \int^x r(s) ds$$

so that (3.6) would also be derived by Levinson’s theorem. See [3], p. 93. However, the setting of this theorem is more general.

For theoretical reasons, it seems immaterial to replace (3.6) by (3.3). However (3.3) points out that the eigenvalues of A_1 need not be calculated. While the explicit construction of $D(x)$ is a relatively simple task, the numerical calculation of $\tilde{D}(s)$ is more laborious. $\tilde{D}(s)$ cannot be found explicitly.

By following the proof in Gingold [7], finer bounds on $\|P(x)\|$ could be found.

For wide classes of mappings $c(s)$, it can be shown that

$$(3.23) \quad \|P(x)\| = \mathcal{O}(h(x)), \quad x \rightarrow \infty.$$

If we add the assumption that $c(x) \in C^3(J)$, then it can also be shown that

$$(3.24) \quad \|P(x)\| = \mathcal{O}\left(\frac{h'(x)}{r(x)}\right), \quad x \rightarrow \infty.$$

Theoretically, the bounds in (2.37), (3.23), (3.24) “may not be interesting”. However, for numerical purposes, accurate bounds may be crucial.

4. The location of zeroes. Thanks to Theorem 3.1 we are able to locate the zeroes of solutions of (1.1). Moreover, as a bonus other asymptotic properties of solutions will follow.

In the sequel, when we will refer to a solution of (1.1) we will have in mind a non-trivial solution of (1.1). Throughout this section we will assume that conditions of Theorem 3.1 are fulfilled.

Let us prove the following lemma.

LEMMA 4.1. *For $n = 1, 2, \dots$ and n is odd, the matrix A possesses $N = (n - 1)/2$ pairs of complex conjugate eigenvalues. If n is even and $c(x) > 0$, $A(x)$ possesses $N = (n - 2)/2$ pairs of complex conjugate eigenvalues but if $c(x) < 0$, it possesses $N = n/2$ pairs of complex conjugate eigenvalues.*

Proof. We let

$$(4.1) \quad \beta = \frac{\arg c(x)}{n}.$$

Then A possesses a real eigenvalue iff θ_k/π given by (2.8) is an integer. We will consider several cases.

Case I. n is odd and $c(x) > 0$. It is obvious by (2.8) that

$$(4.2) \quad \beta = 0, \quad \frac{\theta_k}{\pi} = \frac{2(k-1)}{n}, \quad k = 1, \dots, n.$$

Then for $k = 1$ we obtain the only real eigenvalue

$$(4.3) \quad \lambda_1(x) = r(x).$$

Case II. n is odd and $c(x) > 0$. Then,

$$(4.4) \quad \beta = \frac{\pi}{n}, \quad \frac{\theta_k}{\pi} = \frac{2(k-1)}{n}, \quad k = 1, \dots, n,$$

which implies that the only real eigenvalue is

$$(4.5) \quad \lambda_k(x) = -r(x), \quad k = \frac{n+1}{2}.$$

Case III. n is even and $c(x) > 0$.

Then by (2.8) the only real eigenvalues are

$$(4.6) \quad \lambda_1(x) = r(x), \quad \lambda_{n/2+1}(x) = -r(x)$$

Case IV. n is even and $c(x) < 0$.

Then by (4.4) it follows that no real eigenvalue is possible at all. Lemma 4.1 helps us to adopt the following notations.

DEFINITION 4.2. We say that the differential equation (1.1) belongs to class I, II, III, or IV if $c(x)$ pertains to one of the four cases described in Lemma 4.1.

Notation 4.3. By *relabeling* indices we may assume without loss of generality that (from the outset, in §1)

$$(4.7) \quad \operatorname{Re} \lambda_1 > \operatorname{Re} \lambda_2 > \dots > \operatorname{Re} \lambda_N, \quad \lambda_{j+N} = \bar{\lambda}_j, \quad j = 1, \dots, N,$$

where N is the number of pairs of complex conjugate eigenvalues guaranteed by Lemma 4.1. Moreover, we let

(I)

$$(4.8) \quad \operatorname{Re} \lambda_n = \lambda_n = r(x) > \operatorname{Re} \lambda_1$$

if (1.1) belongs to class I.

(II)

$$(4.9) \quad \operatorname{Re} \lambda_N > \operatorname{Re} \lambda_n = \lambda_n = -r(x)$$

if (1.1) belongs to class II.

(III)

$$(4.10) \quad \begin{aligned} r(x) &= \lambda_n = \operatorname{Re} \lambda_n > \operatorname{Re} \lambda_1 > \operatorname{Re} \lambda_N > \operatorname{Re} \lambda_{n-1} \\ &= -r(x) = \lambda_{n-1} \end{aligned}$$

if (1.1) belongs to class III. (The case $N = 0$ is not excluded from this notation.)

Notation 4.4. We now define the real functions $A_\nu(t), B_\nu(t)$ as follows:

$$(4.11) \quad A_\nu(x) + iB_\nu(x) := \int^x [r(t)e^{i\theta_\nu} - r^{-1}(t)r'(t)k_{\nu\nu}] dt, \quad \nu = 1, \dots, n.$$

Thus

$$(4.12) \quad A_\nu(x) = \int^x (r(t) \cos \theta_\nu - r^{-1}(t)r'(t)a_{1\nu}) dt$$

$$(4.13) \quad B_\nu(x) = \int^x (r(t) \sin \theta_\nu - r^{-1}(t)r'(t)a_{2\nu}) dt$$

where

$$(4.14) \quad a_{1\nu} := \operatorname{Re} k_{\nu\nu},$$

$$(4.15) \quad a_{2\nu} := \operatorname{Im} k_{\nu\nu}, \quad \nu = 1, \dots, n.$$

By the convention just made one has the following:

Case I:

$$(4.16) \quad A_n = \int_{x_1}^x (r(t) - r^{-1}(t)r'(t)a_{1n}) dt, \quad x_1 \in J,$$

$$(4.17) \quad B_n = -a_{2n} \int_{x_1}^x r^{-1}(t)r'(t) dt = -a_{2n} \ln[r(x)r^{-1}(x_1)],$$

$$(4.18) \quad A_j = \int_{x_1}^x \left(r(t) \cos \frac{2j}{n}\pi - r^{-1}(t)r'(t)a_{1j} \right) dt,$$

$$(4.19) \quad B_j = \int_{x_1}^x \left(r(t) \sin \frac{2j}{n}\pi - r^{-1}(t)r'(t)a_{2j} \right) dt, \quad j = 1, \dots, N.$$

Case II:

$$(4.20) \quad A_j := \int_{x_1}^x \left(r(t) \cos \frac{2j-1}{n}\pi - r^{-1}(t)r'(t)a_{1j} \right) dt,$$

$$(4.21) \quad B_j := \int_{x_1}^x \left(r(t) \sin \frac{2j-1}{n}\pi - r^{-1}(t)r'(t)a_{2j} \right) dt,$$

$$j = 1, \dots, N.$$

$$(4.22) \quad A_n = - \int_{x_1}^x (r(t) + r^{-1}(t)r'(t)a_{1n}) dt$$

$$(4.23) \quad B_n = -a_{2n} \ln[r(x)r^{-1}(x_1)].$$

Case III. A_n, B_n , are given by (4.16), (4.17). A_j, B_j are given by (4.18), (4.19). A_{n-1}, B_{n-1} are obtained from (4.22), (4.23) by substituting $n - 1$ in place of n .

Case IV. A_j, B_j are given by (4.20), (4.21).

For the sake of uniformity we add the following assumption:

Assumption 4.5. If

$$(4.24) \quad \sum_{\nu=1}^{\nu=n} |a_{2\nu}|^2 \neq 0,$$

then the function $r(x)$ is monotone increasing or monotone decreasing on \hat{J} .

Since we have Assumption 4.5, we may add the following assumption without loss of generality.

Assumption 4.6. For $t \in \hat{J}$ in cases I, III

$$(4.25) \quad \sin \frac{2\pi}{n} - r^{-n}(t)r'(t)a_{1\nu} \geq 0, \quad \nu = 1, \dots, n,$$

and in cases I, IV

$$(4.26) \quad \sin \frac{\pi}{n} - r^{-2}(t)r'(t)a_{1\nu} \geq 0, \quad \nu = 1, \dots, n.$$

This assumption is given in order to guarantee that the functions $B_\nu(x)$, found above, possess inverses.

Indeed, the following lemma (whose trivial proof is omitted) can be proven.

LEMMA 4.7. *With Assumptions 4.5 and 4.6 the functions $B_n(x)$, $B_{n-1}(x)$ and $B_j(x)$, $j = 1, \dots, N$, are monotone functions of x on \hat{J} .*

We will need to find n real linearly independent solutions of (1.1). We first define

$$(4.27) \quad W = (w_{jk}) = V(I + P), \quad P = (p_{jk}), j, k = 1, \dots, n,$$

and we write down the elements of the first row of W by

$$(4.28) \quad w_{1\nu} = 1 + \sum_{\nu}, \quad \nu = 1, 2, \dots,$$

with

$$(4.29) \quad \sum_\nu := \sum_{j=1}^n p_{j\nu}.$$

Similarly, we have

$$(4.30) \quad w_{2\nu} = \sum_{j=1}^n \lambda_j (\delta_{j\nu} + p_{j\nu}) = \lambda_\nu (1 + p_{\nu\nu}) + \sum_{j \neq \nu}^n \lambda_j p_{j\nu}$$

and

$$(4.31) \quad w_{m\nu} = \sum_{j=1}^n \lambda_j^{m-1} (\delta_{j\nu} + p_{j\nu}) \\ = \lambda_\nu^{m-1} (1 + p_{\nu\nu}) + \sum_{j \neq \nu}^n \lambda_j^{m-1} p_{j\nu}, \quad m = 2, \dots, n.$$

We choose n real linearly independent solutions of (1.1) as follows: By (4.28) via (3.3) we have each of the n linearly independent solutions of (1.1) to be

$$\left| 1 + \sum_\nu \right| \exp(A_\nu(x) + i\theta_{1\nu}(x) + iB_\nu(x))$$

with

$$\theta_{1\nu} = \arg(1 + \sum_\nu), \quad \nu = 1, \dots, n.$$

From now on the letter y with lower indices will denote *something other than* the entries of the matrix function Y which appears in (3.3).

Thus, we adopt

Notation 4.8. We denote n real solutions of (1.1) by $y_\nu(x)$, $\nu = 1 \cdots n$ and we define them as follows:

If (1.1) belongs to classes I, II or III

$$y_n(x) = \operatorname{Re} w_{1n} \exp(A_n + iB_n) \\ = \left| 1 + \sum_n \right| (\exp A_n(x)) \cos(B_n(x) + \theta_{1n}(x)) \\ y_{1j}(x) = \operatorname{Re} w_{1j} \exp(A_j + iB_j) \\ = \left| 1 + \sum_j \right| (\exp A_j(x)) \cos(B_j(x) + \theta_{1j}(x)) \\ y_{2j}(x) = \operatorname{Im} w_{1j} \exp(A_j + iB_j) \\ = \left| 1 + \sum_j \right| (\exp A_j(x)) \sin(B_j(x) + \theta_{1j}(x)), \quad j = 1, \dots, N.$$

If (1.1) belongs to class III, we pick

$$(4.32) \quad y_{n-1}(x) = \operatorname{Re} w_{1,n-1} \exp(A_{n-1} + iB_{n-1}) \\ = \left| 1 + \sum_{n-1} \right| (\exp A_{n-1}(x)) \cos(B_{n-1}(x) + \theta_{1,n-1}(x))$$

and the rest $n - 1$ real solutions according to (4.29), (4.30), (4.31). If (1.1) belongs to class IV we pick the $2N = n$ real solutions according to (4.30), (4.31).

It is simple to verify that in all cases I–IV the set of n real solutions form a basis for the linear space of solutions of (1.1).

Notation 4.9. We denote by S_n the linear subspace generated by $y_n(x)$ if (1.1) belongs to class I, II, or III. If (1.1) belongs to class IV, S_n will denote an empty set.

We denote by S_{n-1} the linear subspace generated by $y_{n-1}(x)$ if (1.1) belongs to class III. If (1.1) belongs to class I, II or IV, S_{n-1} will denote an empty set.

We denote by $S_j, j = 1, \dots, N$, the N two-dimensional linear spaces spanned by $y_{1j}(x), y_{2j}(x)$. If $N = 0$, we let S_j above denote empty sets.

We also denote by S the linear space of solutions of (1.1).

Let us first turn to the location of zeroes of a non-trivial solution $y_j(x) \in S_j, j = 1, \dots, N (N > 0)$.

Let c_{1j}, c_{2j} be two real variables. Then, it can be easily verified that

$$(4.33) \quad y_j(x) = c_{1j}y_1(x) + c_{2j}y_2(x) \\ = \left[M_j(x) \cos(B_j(x) + \theta_{1j}(x) + \Psi_{1,2,j}) \right] \sqrt{c_{1j}^2 + c_{2j}^2}$$

$$(4.34) \quad M_j(x) := \left| 1 + \sum_j \right| (\exp A_j(x))$$

and with

$$(4.35) \quad \tan \Psi_{1,2,j} = \frac{-c_{2j}}{c_{1j}}, \quad j = 1, \dots, N.$$

In order to better understand the process of locating the zeroes of $y(x)$, we define an “ideal solution” of (1.1), $y_L(x)$ by letting in (4.33), (4.34).

$$(4.36) \quad P = 0.$$

For the particular solution (4.33) we then get

$$(4.37) \quad \theta_{1j}(x) \equiv 0, \quad \sum_j \equiv 0$$

to obtain

$$(4.38) \quad y_L(x) = (\exp A_j(x)) \sqrt{c_{1j}^2 + c_{2j}^2} \cos(B_j(x) + \Psi_{1,2,j}).$$

We do not claim that $y_L(x)$ is a solution of (1.1). However, the location of its zeroes on \hat{J} are easily observed to be at x_l such that

$$(4.39) \quad B_j(x_l) + \Psi_{1,2,j} = (2l + 1) \frac{\pi}{2} \quad (l \text{ an integer}),$$

or if $B_j(x)$ has an inverse on \hat{J}

$$(4.40) \quad x_l = B_j^{-1}\left((2l + 1) \frac{\pi}{2} - \Psi_{1,2,j}\right).$$

We expect the zeroes of $y(x)$ for x large enough, to be close to x_l given by (4.39) or by (4.40).

Indeed, we are now able to apply the results of Gingold [5], §5, to locate with high accuracy the zeroes of $y_j(x), j = 1, \dots, N$.

It will be superfluous to repeat all the arguments of Gingold [5]. Therefore, we state our results in a form such that the reader who is interested in the details will be able to easily reproduce them as an exercise.

The next theorems provide information on solutions of (1.1) beyond that given by Kim [12, 13], Willet [24], Kreith [15], Swanson [21]. We did not come across any other method in the past which also provided with high accuracy the *location of zeroes of solutions* of (1.1). It clearly demonstrates an edge of asymptotic techniques.

THEOREM 4.10. *Let assumptions of Theorem 3.1 and Assumptions 4.5 and 4.6 hold. Then, the linear subspace $S_n, (S_{n-1})$ contains nonoscillatory solutions of (1.1) if either*

$$(4.41) \quad a_{2n} = 0 \quad (a_{2,n-1} = 0).$$

or (without restriction on $a_{2n} (a_{2,n-1})$) if

$$(4.42) \quad \sup_{x \in J} |\ln r(x)| < \infty.$$

moreover, $y_n \in S_n$ has an exponential growth in cases I and III and it has an exponential decay in case II. $y_{n-1} \in S_{n-1}$ has exponential decay in case III.

If (1.1) belongs to class III, then

$$(4.43) \quad \lim_{x \rightarrow \infty} \frac{M_{n-1}}{M_n} = 0,$$

where

$$(4.44) \quad M_n := M_n(x) = \left| 1 + \sum_n \right| \exp A_n(x),$$

$$(4.45) \quad M_{n-1} := M_{n-1}(x) = \left| 1 + \sum_{n-1} \right| \exp A_{n-1}(x).$$

Proof. This is an immediate consequence of the definitions.
We have

THEOREM 4.11. *Let assumptions of Theorem 4.10 hold. If S_n is not the empty set and $N > 0$ then in cases I, III,*

$$(4.46) \quad \lim_{x \rightarrow \infty} \frac{y_j(x)}{M_n(x)} = 0, \quad j = 1, \dots, N.$$

If $N > 1$,

$$(4.47) \quad \lim_{x \rightarrow \infty} \frac{M_j(x)}{M_{j+1}(x)} = 0, \quad j = 1, \dots, N - 1,$$

where M_j are given by (4.34).

If S_{n-1} is not the empty set and $N > 0$ then in case III

$$(4.48) \quad \lim_{x \rightarrow \infty} \frac{M_{n-1}(x)}{M_n(x)} = 0.$$

Proof. The trivial proof is omitted.

Next we have

THEOREM 4.12. *If assumptions of Theorem 4.10 hold. Let $n \geq 3$, or let (1.1) belong to the class IV.*

Then S contains N linear subspaces S_j of oscillatory solutions y_j given by (4.33), $j = 1, \dots, N$.

The location of the zeroes of y_j are given at the points x_{m_j} on \hat{J} such that

$$(4.49) \quad \lim_{|m_j| \rightarrow \infty} \left[x_{m_j} - B_j^{-1} \left((2m_j + 1) \frac{\pi}{2} - \Psi_{1,2,j} \right) \right] = 0$$

and also

$$(4.50) \quad x_{m_j} \sim B_j^{-1} \left((2m_j + 1) \frac{\pi}{2} - \Psi_{1,2,j} \right), \quad |m_j| \rightarrow \infty.$$

$|m_j| \rightarrow \infty$ means that m_j attains successive integral values and either $m_j \rightarrow +\infty$ or $m_j \rightarrow -\infty$.

Moreover, let $a_{2n} \neq 0$ and let

$$(4.51) \quad \lim_{x \rightarrow \infty} |\ln r(x)| = \infty.$$

Then in cases I, II, $y_n \in S_n$ is an oscillatory solution with zeroes x_{m_n}

$$(4.52) \quad \lim_{|m_n| \rightarrow \infty} \left[x_{m_n} - B_n^{-1} \left((2m_n + 1) \frac{\pi}{2} \right) \right] = 0,$$

$$x_{m_n} \sim B_n^{-1} \left((2m_n + 1) \frac{\pi}{2} \right), \quad |m_n| \rightarrow \infty.$$

In cases II, III, with (4.51) and $a_{2,n-1} \neq 0$, $y_{n-1} \in S_{n-1}$ is an oscillatory solution with zeroes $x_{m_{n-1}}$ satisfying (4.52) with $n - 1$ replacing n .

Proof. See Gingold [5] §5.

We state a theorem which summarizes the preceding oscillation results.

THEOREM 4.13. *Let assumptions of Theorem 4.10 hold. Let $y(x)$ be the general solution of (1.1) given by*

$$(4.53) \quad y(x) = c_n y_n(x) + \left(\sum_{j=1}^N c_{1j} y_{1j}(x) + c_{2j} y_{2j}(x) \right) + c_{n-1} y_{n-1}(x).$$

If S_n or S_{n-1} or both are zero we choose correspondingly c_n or c_{n-1} or both to be zero.

(i) *If*

$$(4.54) \quad a_{2n} \neq 0, \quad a_{2,n-1} \neq 0$$

and (4.51) holds then each non-trivial solution of (1.1) is oscillatory. In particular, all solutions in case IV are oscillatory.

(ii) *Let $c_n \neq 0$. Then, (in cases I, III) the zeroes of x_{m_j} of $y(x)$ are such that with $j = n$ they satisfy (4.52) and*

$$(4.55) \quad \left| \frac{y(x) - c_n y_n(x)}{M_1(x)} \right| \leq C, \quad x \in \hat{J},$$

for a certain constant C .

(iii) *Let*

$$(4.56) \quad c_n = c_{1j} = c_{2j} = 0, \quad j = 1, \dots, j_0 - 1, j_0 \leq N.$$

Then the zeroes of $y(x)$ are located at points $x_{m_{j_0}}$ such that

$$(4.57) \quad \lim_{m_{j_0} \rightarrow \infty} \left[x_{m_{j_0}} - B_{j_0}^{-1} \left((2m_{j_0} + 1) \frac{\pi}{2} - \Psi_{1,2,j_0} \right) \right] = 0$$

and

$$(4.58) \quad x_{m_{j_0}} \sim B_{j_0}^{-1} \left((2m_{j_0} + 1) \frac{\pi}{2} - \Psi_{1,2,j_0} \right), \quad |m_{j_0}| \rightarrow \infty.$$

Also

$$(4.59) \quad \left| \frac{y(x) - y_{j_0}(x)}{M_l(x)} \right| \leq C, \quad x \in \hat{J},$$

with $l = n - 1$ if $j_0 = N$ and $l = j_0 + 1$ if $l < N$, and C a certain constant. (If S_{n-1} is empty put $M_{n-1} \equiv 1$.)

(iv) If in case III

$$(4.60) \quad c_n = c_{1j} = c_{2j} = 0, \quad j = 1, \dots, N,$$

then the zeroes of $y(x)$ are located at $x_{m_{n-1}}$. This satisfies (4.52) with $n - 1$ replacing n .

Proof. The inequalities (4.55), (4.59) follow directly from the definitions of $A_\nu(x)$, $M_\nu(x)$. Thus, in (4.55) we take

$$(4.61) \quad C := \sup_{x \in \hat{j}} \left| \frac{y(x) - c_n y_n(x)}{M_1(x)} \right| < \infty$$

and in (4.59) we take

$$(4.62) \quad C := \sup_{x \in \hat{j}} \left| \frac{y(x) - y_{j_0}(x)}{M_j(x)} \right| < \infty.$$

For the approximation of the zeroes of $y(x)$, we “identify in (4.53) a leading term” and use the details in Gingold [5] §5.

Next we obtain a “non-oscillation” theorem.

THEOREM 4.14. *Let assumptions of Theorem 4.10 hold.*

(i) *Let $a_{2n} = 0$ or let $a_{2n} \neq 0$ and let (4.42) hold.*

If $c_n \neq 0$ when S_n is a non-empty set, then in cases I, III $y(x)$ is non-oscillatory and (4.61) holds.

(ii) *If $a_{2,n-1} = 0$ or if $a_{2,n-1} \neq 0$ and (4.42) holds and*

$$(4.63) \quad c_n = c_{1j} = c_{2j} = 0, \quad j = 1, \dots, N,$$

then in case III $y(x)$ is nonoscillatory if S_{n-1} is not an empty set.

If in case III, $a_{2n} = 0$ or $a_{2n} \neq 0$ and (4.42) hold, and if

$$c_{1j} = c_{2j} = 0, \quad j = 1, \dots, N.$$

Then $y(x)$ is nonoscillatory.

Proof. We omit the trivial proof.

REMARKS. A similar analysis carries over to n th order linear homogeneous differential equations whose characteristic polynomials possesses n distinct eigenvalues. In this case the location of zeroes of oscillatory solutions can be provided in terms of the elements on the main diagonal of a certain matrix derived from the coefficients of the linear differential equation.

If

$$(4.64) \quad \lim_{x \rightarrow \infty} r^{-2} r' \neq 0, \quad \infty$$

then a different analysis is needed. It is an easy exercise to verify that for

$$(4.65) \quad c(x) = \theta x^m, \quad m > -n, \theta \text{ real},$$

all conditions of Theorem 3.1 are satisfied.

The same is true for (4.66) and (4.67).

$$(4.66) \quad c(x) = \theta e^{-x^\alpha}, \quad \theta \text{ real and } \alpha > 0.$$

It is worth noticing that in case (4.66) or in case

$$(4.67) \quad c(x) = x^\alpha (\ln x)^\beta, \quad \alpha > -n, \beta \text{ real},$$

the “analytic theory” of asymptotic expansions cannot be applied.

This article also shows how to derive Pruffer type formulas for solutions of higher order linear differential equations. This question has been asked by several authors. See Swanson [21], pg. 95.

The relation between oscillation and certain eigenvalue problems of (1.1) is well known. Thus, our method provides approximations to eigenvalues in certain eigenvalue problems.

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