PACIFIC JOURNAL OF MATHEMATICS
Vol. 120, No. 1, 1985

A UNIFIED APPROACH TO CARLESON MEASURES AND A_p WEIGHTS. II

FRANCISCO J. RUIZ AND JOSÉ L. TORREA

In this note we find for each p, $1 < p < \infty$, a necessary and sufficient condition on the pair (μ, v) (where μ is a measure on $\mathbf{R}^{n+1}_+ = \mathbf{R}^n \times [0, \infty)$, and v a weight on $\mathbf{R}^n$) for the Poisson integral to be a bounded operator from $L^p(\mathbf{R}^n, v(x)\,dx)$ into $L^p(\mathbf{R}^{n+1}_+, \mu)$.

1. Introduction. In this note we find for each p, $1 < p < \infty$, a necessary and sufficient condition on the pair (μ, v) (where μ is a measure on $\mathbf{R}^{n+1}_+ = \mathbf{R} \times [0, \infty)$ and v a weight on $\mathbf{R}^n$) for the Poisson integral to be a bounded operator from $L^p(\mathbf{R}^n, v(x)\,dx)$ into $L^p(\mathbf{R}^{n+1}_+, \mu)$.

Our proof follows the ideas of Sawyer [7] and the condition we find is

$$(F_p) \quad \int_{\tilde{Q}} \left[\mathscr{M}\left(v^{1-p'}\chi_Q\right)(x,t)\right]^p d\mu(x,t) \le C\int_Q v^{1-p'}(x)\,dx < +\infty$$

for all cubes in $\mathbf{R}^n$ (cube will always means a compact cube with sides parallel to the coordinate axes).

For $\mathscr{M}$ we denote the maximal operator

$$(*) \qquad \mathscr{M}f(x,t) = \sup_Q \frac{1}{|Q|}\int_Q |f(x)|\,dx, \qquad x \in \mathbf{R}^n, t \ge 0,$$

where the supremum is taken over the cubes Q in $\mathbf{R}^n$, containing x and having side length at least t.

As usual $\tilde{Q}$ denotes the cube in $\mathbf{R}^{n+1}_+$, with the cube Q as its basis.

Carleson [1] showed that $\mathscr{M}$ is bounded from $L^p(\mathbf{R}^n, dx)$ into $L^p(\mathbf{R}^{n+1}_+, \mu)$ if and only if μ satisfies the so-called "Carleson condition"

$$(1) \qquad \mu(\tilde{Q}) \le C|Q| \quad \text{for each cube in } \mathbf{R}^n.$$

Afterwards, Fefferman and Stein [2] found that

$$(2) \qquad \sup_{x \in Q} \frac{\mu(\tilde{Q})}{Q} \le Cv(x) \quad \text{a.e. } x$$

is sufficient for $\mathscr{M}$ to be bounded from $L^p(\mathbf{R}^n, v(x)\,dx)$ into $L^p(\mathbf{R}^{n+1}_+, \mu)$.

Recently F. Ruiz [6] found the condition

$$(3)\qquad \frac{\mu(\tilde{Q})}{|Q|}\left(\frac{1}{|Q|}\int_Q v^{1-p'}(x)\,dx\right)^{p-1} \leq C$$

to be necessary and sufficient for the boundedness of the operator $\mathscr{M}$ from $L^p(\mathbf{R}^n, v(x)\,dx)$ into weak-$L^p(\mathbf{R}^{n+1}_+, \mu)$. The condition (3) will be denoted by (C_p) as in [6].

The paper is set out as follows: in §2 we give results and some consequences, whilst §3 contains detailed proofs.

2. **Results.** Throughout this paper, Q will denote a cube in $\mathbf{R}^n$ with sides parallel to the coordinate planes. For $r > 0$, rQ will denote the cube with the same centre as Q diameter r times that of Q. $|Q|_v$ will denote $\int_Q v(x)\,dx$.

We shall say that Q is a dyadic cube and we shall write $Q \in \mathscr{D}$, if Q is a subset of $\mathbf{R}^n$ of the form $\prod_{i=1}^n [x_i, x_i + 2^k)$, where $x \in 2^k\mathbf{Z}^n_+$, with k in $\mathbf{Z}$. We define the dyadic maximal operator $\mathscr{N}$ associated with the Poisson integral by

$$(**)\qquad \mathscr{N}f(x,t) = \sup_Q \frac{1}{|Q|}\int_Q |f(x)|dx, \qquad x \in \mathbf{R}^n, t \geq 0,$$

where the supremum is taken over the dyadic cubes in $\mathbf{R}^n$ containing x and having side length at least t.

The main results in this paper are the following:

THEOREM A. *Given a weight v in $\mathbf{R}^n$, a positive measure μ in $\mathbf{R}^{n+1}_+$, and p, $1 < p < \infty$, the following conditions are equivalent.*

(i) *The operator $\mathscr{M}$ defined in $(*)$ is bounded from $L^p(\mathbf{R}^n, v(x)\,dx)$ into $L^p(\mathbf{R}^{n+1}_+, \mu)$; i.e.*

$$\int_{\mathbf{R}^{n+1}_+} [\mathscr{M}f(x,t)]^p\,d\mu(x,t) \leq C\int_{\mathbf{R}^n} |f(x)|^p v(x)\,dx.$$

(ii) *The pair (μ, r) verifies (F_p).*

THEOREM B. *Given a weight v in $\mathbf{R}^n$, a positive measure μ in $\mathbf{R}^{n+1}_+$, and p, $1 < p < \infty$, the following conditions are equivalent.*

(i) *The operator $\mathscr{N}$ defined in $(**)$ is bounded from $L^p(\mathbf{R}^n, vx\,dx)$ into $L^p(\mathbf{R}^{n+1}, \mu)$.*

(ii) *The pair* (μ, v) *verifies*

$$\int_{\tilde{Q}} \left[\mathcal{N}\left(v^{1-p'}\chi_Q\right)(x,t)\right]^p d\mu(x,t) \le C\int_Q v^{1-p'}(x)\,dx < +\infty$$

for all dyadic cubes Q *in* $\mathbf{R}^n$.

The above results have certain consequences. (I) In the particular case in Theorem A where $v(x) \equiv 1$, the condition (F_p) reduces to

$$\int_{\tilde{Q}} \left[\mathcal{M}(\chi_Q)(x,t)\right]^p d\mu(x,t) \le C\int_Q dx = C|Q|$$

and since $\mathcal{M}(\chi_Q)(x,t) = 1$ for $(x,t) \in \tilde{Q}$, we see that Theorem A gives us Carleson's result mentioned in the introduction.

(II) If the measure μ in $\mathbf{R}^{n+1}_+ = \mathbf{R}^n \times [0,\infty)$ is of the form $d(x) = u(x)\,dx$ concentrated in $\mathbf{R}^n \times \{0\}$, then (F_p) is equivalent to Sawyer's condition

$$(S_p) \qquad \int_Q \left[M\left(v^{1-p'}\chi_Q\right)(x)\right]^p u(x)\,dx \le C\int_Q v^{1-p'}(x)\,dx < +\infty$$

where Mf denotes the Hardy Littlewood maximal operator.

Since $\mathcal{M}f(x,0) = Mf(x)$, $x \in \mathbf{R}^n$. Then from Theorem A we obtain

THEOREM (*Sawyer* [7]). *Let* $1 < p < \infty$. *Given weights* u *and* v *in* $\mathbf{R}^n$ *the following statements are equivalent*:

(i) (u, v) *satisfies the* (S_p) *condition*

(ii) $\int_{\mathbf{R}^n} (Mf(x))^p u(x)\,dx \le C_p \int_{\mathbf{R}^n} |f(x)|^p v(x)\,dx$.

(III) Hunt, Kurtz and Neugebauer [3] have shown by a direct proof that if a weight v belongs to the A_p class, $1 < p < \infty$, of Muckenhoupt, i.e.

$$\sup_Q \left(\frac{1}{|Q|}\int_Q v(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q v^{1-p'}(x)\right)^{p-1} \le C$$

then v satisfies the (S_p) condition in (II) with $u = v$.

In our case it can be shown, see [6], that if the pair (μ, v) satisfies the (C_p) condition, $1 < p < \infty$, and v belongs to the class A_p of Muckenhoupt, then the operator $\mathcal{M}$ is bounded from $L^p(\mathbf{R}^n, v(x)\,dx)$ into $L^p(\mathbf{R}^{n+1}_+, \mu)$ and this tells us that in particular (μ, v) will satisfy the (F_p) condition.

In the particular case considered in (II), this suggests that for a pair of weights (u, v) satisfying the A_p condition, $1 < p < \infty$, of Muckenhoupt, i.e.

$$\sup_Q \left(\frac{1}{|Q|} \int_Q u(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q v^{1-p'}(x)\, dx \right)^{p+1} \le C$$

the fact that $v \in A_p$ is sufficient for (u, v) to satisfy the S_p condition.

(IV) If a weight v is given and we call $F_p(v)$ (respectively $C_p(v)$) the set of measures μ on $\mathbf{R}_+^{n+1}$ such that (μ, v) satisfies the F_p condition (respectively the C_p condition) we can state that for $1 < p \le q$

$$C_1(v) \subset F_p(v) \subset C_p(v) \subset \cdots \subset F_q(v) \subset C_q(v) \subset \cdots.$$

The inclusion $C_p(v) \subset C_q(v)$ is proved in [**6**]. To see that $F_p(v) \subset C_p(v)$ let us observe that for $(x, t) \in \tilde{Q}$

$$\mathcal{M}\left(v^{1-p'}\chi_Q\right)(x, t) \ge \frac{1}{|Q|} \int_Q v^{1-p'}(y)\, dy$$

and this implies for $\mu \in F_p(v)$

$$\left(\frac{1}{|Q|} \int_Q v^{1-p'}(x)\, dx \right)^p \mu(\tilde{Q}) \le C \int_Q v^{1-p'}(x)\, dx.$$

So $\mu \in C_p(v)$.

Now, given $p < q$, and $\mu \in C_p(v)$, using the Marcinkiewicz interpolation theorem between the boundedness of $\mathcal{M}$ from $L^p(\mathbf{R}^n, v(x)\, dx)$ into weak-$L^p(\mathbf{R}_+^{n+1}, \mu)$ and the trivial L^∞ boundedness, we obtain $\mu \in F_q(v)$.

REMARK. If, for a given p, v belongs to the A_p class of Muckenhoupt, then it can be shown that $C_p(v) = C_q(v)$, $p \le q \le \infty$, see [**6**]. This fact and the fact that for $v \in A_p$ there exists $\varepsilon > 0$ such that $v \in A_{p-\varepsilon}$ allows us to obtain that

$$F_p(v) = C_p(v) = C_q(v) = F_q(v), \qquad p \le q \le \infty.$$

3. Detailed proofs. The proof of the implication (i) $\Rightarrow$ (ii) is the same in both Theorems A and B, the only difference being the use of non dyadic or dyadic cubes.

Firstly, let us see that $\int_Q v^{1-p'}(x)\, dx < +\infty$ for all cubes. If

$$\int_Q v^{1-p'}(x)\, dx = \int_Q \left(v^{-1}(x)\right)^{p'} v(x)\, dx = \infty$$

this would imply the existence of a function $f \in L^p(v)$ such that

$$\int_Q f(x)\, dx = \int_Q f(x) v^{-1}(x) v(x)\, dx = +\infty,$$

and in particular $\mathcal{M}f(x,t) = +\infty$ for $(x,t) \in \mathbf{R}^{n+1}_+$ which contradicts the hypothesis:

$$\int_{\mathbf{R}^{n+1}} [\mathcal{M}f(x,t)]^p \, d\mu(x,t) \le C \int_{\mathbf{R}^n} f^p(x) v(x) \, dx < +\infty.$$

To show the inequality in (ii) it is sufficient to choose $f(x) = \chi_Q(x) v^{1-p'}(x)$ in the hypothesis.

Proof of (ii) $\Rightarrow$ (i) *in Theorem B.* In order to handle a Calderón-Zygmund decomposition we introduce the operators

$$\mathcal{N}^R f(x,t) = \sup_Q \frac{1}{Q} \int_Q |f(x)| dx, \qquad x \in \mathbf{R}^n, t > 0,$$

the supremum being taken over all dyadic cubes in $\mathbf{R}^n$ containing x and having side length at least t and at most R.

Observe that $\mathcal{N}^R f(x,t) = 0$ for $t > R$ and that

$$\lim_{R \to \infty} \mathcal{N}^R f(x,t) = \mathcal{N} f(x,t)$$

with increasing limit.

Let Ω_k be the set

$$\Omega_k = \{(x,t) : \mathcal{N}^R f(x,t) > 2^k\}, \qquad k \in \mathbf{Z}.$$

LEMMA. *For each $k \in \mathbf{Z}$ there exists a family $\{Q_j^k\}$, $j \in J_k$, of dyadic cubes in $\mathbf{R}^n$ such that*

(i) $1/Q_j^k \int_{Q_j^k} |f(x)| \, dx > 2^k$.
(ii) *The interiors of $\tilde{Q}_j^k$ are disjoints*
(iii) $\Omega_k = \bigcup_{j \in J_k} \tilde{Q}_j^k$.

Proof of the lemma. If $(x,t) \in \Omega_k$ it means that there exists a dyadic cube with $x \in Q$, $l(Q) \ge t$, $l(Q) \le R$ and $1/|Q| \int_Q |f(x)| \, dx > 2^k$. This implies the existence of a dyadic maximal Q_j^k such that $Q \subset Q_j^k$, $l(Q_j^k) \le R$, $l(Q_j^k) \ge t$ and

$$\frac{1}{|Q_j^k|} \int_{Q_j^k} |f(x)| dx > 2^k.$$

In particular, $(x,t) \in \tilde{Q}_j^k$. The fact that the interiors of $\tilde{Q}_j^k$ are disjoint is an obvious consequence of the same property for the $\tilde{Q}_j^k$'s.

Now let us consider the sets

$$E_j^k = \tilde{Q}_j^k \setminus \{(x,t) : \mathcal{N}^R f(x,t) \ge 2^{k+1}\}.$$

Then we have a family of sets $\{E_j^k\}_{j,k}$ with disjoint interiors and

$$\int_{\mathbf{R}^{n+1}} [\mathcal{N}^R f(x,t)]^p \, d\mu(x,t) \le \sum_{k,j} \int_{E_j^k} [\mathcal{N}^R f(x,t)]^p \, d\mu(x,t)$$

$$\le \sum_{j,k} 2^{(k+1)p} \mu\left(E_j^k\right) \le 2^p \sum_{j,k} \mu\left(E_j^k\right) \frac{1}{|Q_j^k|} \left(\int_{Q_j^k} |f(x)| dx \right)^p .$$

Following the ideas of Sawyer [7] and Jawerth [4], we introduce the following notations:

$$\sigma(x) = v^{1-p'}(x), \qquad \sigma(Q) = \int_Q \sigma(x)\,dx$$

$$\gamma_{jk} = \mu\left(E_j^k\right) \left(\frac{\sigma\left(Q_j^k\right)}{Q_j^k} \right)^p ,$$

$$g_{jk} = \left(\frac{1}{\sigma\left(Q_j^k\right)} \int_{Q_j^k} \frac{|f(x)|}{\sigma(x)} \sigma(x)\,dx \right)^p$$

$X = \{(k,j) : k \in \mathbf{Z}, j \in J_k\}$ with atomic measure γ_{jk}.

$\Gamma(\lambda) = \{(k,j) \in X : g_{jk} > \lambda\}$.

Then we can write

$$\int_{\mathbf{R}^{n+1}} [\mathcal{N}^R f(x,t)]^p \, d\mu(x,t) \le 2^p \sum_{j,k} \gamma_{jk} g_{jk}$$

$$= 2^p \int_0^\infty \gamma\{(k,j) : g_{jk} > \lambda\}\, d\lambda = 2^p \int_0^\infty \left\{ \sum_{(k,j)\in\Gamma(\lambda)} \gamma_{jk} \right\} d\lambda$$

$$= 2^p \int_0^\infty \sum_{(k,j)\in\Gamma(\lambda)} \mu\left(E_j^k\right) \left(\frac{\sigma\left(Q_j^k\right)}{|Q_j^k|} \right)^p d\lambda$$

$$= 2^p \int_0^\infty \sum_{(k,j)\in\Gamma(\lambda)} \int_{E_j^k} \left(\frac{\sigma\left(Q_j^k\right)}{|Q_j^k|} \right)^p d\mu(x,t)\, d\lambda,$$

calling Q_i the maximal cubes of the family $\{Q_j^k : (k,j) \in \Gamma(\lambda)\}$. This is equal to

$$2^p \int_0^\infty \sum_j \sum_{\substack{(k,j)\in\Gamma(\lambda) \\ Q_j^k \subset Q_i}} \int_{E_j^k} \left(\frac{\sigma\left(Q_j^k\right)}{|Q_j^k|} \right)^p d\mu(x,t)\, d\lambda$$

$$\le 2^p \int_0^\infty \sum_i \sum_{\substack{(k,j)\in\Gamma(\lambda) \\ Q_j^k \subset Q_i}} \int_{E_j^k} \left(\mathcal{N}^R(\sigma\chi_{Q_i})(x,t) \right)^p d\mu(x,t)\, d\lambda$$

by the disjointness of the E_j^k's. This is less than

$$2^p \int_0^\infty \sum_i \int_{\tilde{Q}_i} \left(\mathcal{N}^R(\sigma\chi_{Q_i})(x,t)\right)^p d\mu(x,t)\, d\lambda.$$

Following hypothesis (i) this is less than

$$2^p \int_0^\infty \sum_i \left(\int_{Q_i} \sigma(x)\, dx\right) d\lambda = 2^p \int_0^\infty \sigma\left(\bigcup Q_i\right) d\lambda$$

$$= 2^p \int \sigma\left(\bigcup_{(k,j)\in\Gamma(\lambda)} Q_j^k\right) d\lambda.$$

The definition of $\Gamma(\lambda)$ states that

$$\bigcup_{(k,j)\in\Gamma(\lambda)} Q_j^k \subset \left\{x\colon N_\sigma\left(\frac{|f|}{\sigma}\right)(x) > \lambda^{1/p}\right\}$$

where

$$N_\sigma(x) = \sup \frac{1}{\sigma(Q)} \int_Q g(x)\sigma(x)\, dx,$$

the supremum being taken over all dyadic cubes in $\mathbf{R}^n$ containing x.

Then we have

$$\int_{\mathbf{R}^{n+1}} [\mathcal{N}^R f(x,t)]^p\, d\mu(x,t) \le 2^p \int_0^\infty \sigma\left\{x\colon \left(N_\sigma\left(\frac{|f|}{\sigma}\right)(x)\right)^p > \lambda\right\} d\lambda$$

$$= 2^p \int_{\mathbf{R}^n} \left(N_\sigma\left(\frac{|f|}{\sigma}\right)(x)\right)^p \sigma(x)\, dx$$

$$\le 2^p \int_{\mathbf{R}^n} \frac{|f(x)|^p}{\sigma(x)^p} \sigma(x)\, dx$$

since the dyadic maximal operator with respect to any positive measure ν maps $L^p(d\nu)$, $1 < p < \infty$, into itself.

The proof ends by applying Fatou's lemma and observing that $\sigma^{1-p} = v$.

Proof of (ii) $\Rightarrow$ (i) *in Theorem A*. The proof of this part follows easily from the ensuing lemma due to Sawyer [7].

LEMMA 2. *Define for each* $y \in \mathbf{R}^n$

$${}^y\mathcal{N} f(x,t) = \sup \frac{1}{|Q|} \int_Q |f(u)|du,$$

the supremum being taken in all cubes Q with $x \in Q$, side length less than t and such that the set $Q - y = \{u - y \colon u \in Q\}$ is a dyadic cube. Then,

$$\mathcal{M}^{2^k} f(x,t) \le C \int_{[-2^{k+2}, 2^{k+2}]^n} {}^y\mathcal{N} f(x,t) \frac{dy}{2^{n(k+3)}}$$

where the constant C depends only on the dimension.

By $\mathcal{M}^R$ we mean the maximal operator obtained considering cubes with side length less than R.

Observe that the proof of Theorem B can be repeated for the operator ${}^\lambda\mathcal{N}$ where the dyadic cubes are now of the type $\prod_{i=1}^n [x_i, x_{i+} + 2^k)$ with $x - y \in 2^k \mathbf{Z}^n$.

Then, by Lemma 2 we have

$$\begin{aligned}
&\int_{\mathbf{R}_+^{n+1}} \left[\mathcal{M}^{2^k} f(x,t)\right]^p d\mu(x,t) \\
&\le C \int_{\mathbf{R}_+^{n+1}} d\mu(x,t) \left[\int_{[-2^{k+2}, 2^{k+2}]^n} {}^y\mathcal{N} f(x,t) \frac{dy}{2^{n(k+3)}}\right]^p \\
&\le C \left[\int_{[-2^{k+2}, 2^{k+2}]^n} \frac{dy}{2^{n(k+3)}} \left(\int |f(x)|^p v(x)\, dx\right)^{1/p}\right]^p \\
&= C \int |f(x)|^p v(x)\, dx.
\end{aligned}$$

By letting $k \to \infty$ we conclude (i) in Theorem A.

The proof of Lemma 2 follows along the lines of the corresponding result in [7] and is therefore omitted.

Acknowledgments. The authors are grateful to José L. Rubio for his helpful suggestions.

REFERENCES

[1] L. Carleson, *Interpolation by bounded analytic functions and the corona problem*, Annals of Math., **76** (1962), 547–559.

[2] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math., **93** (1971), 107–115.

[3] R. Hunt, D. Kurtz and C. Neugebauer, *A note on the equivalence of Ap and Sawyer's condition for equal weights*, Conference A. Zygmund, Chicago 1981.

[4] B. Jawerth, *Weighted inequalities for maximal operators: Linearization, localization and factorization*, preprint.

[5] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., **165** (1972), 115–121.

[6] F. Ruiz, *A unified approach to Carleson measures and Ap weights*, to appear in Pacific J. Math.

[7] E. Sawyer, *A characterization of a two-weight norm inequality for maximal operators*, Studia Math., **75** (1982), 1–11.

Received November 21, 1983 and in revised form April 4, 1984.

UNIVERSIDAD DE ZARAGOZA
ZARAGOZA, SPAIN

AND

UNIVERSIDAD AUTONOMA DE MADRID
MADRID-34, SPAIN