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BICONTACTUAL REGULAR MAPS

STEPHEN E. WILSON

In this paper, we will classify all rotary maps with the property that each face meets only one or two others. We will show that all such maps are in fact regular and that they are closed under the action of the operators D, P, opp and H_j. We will then use this information to prove this theorem: Every non-trivial rotary map whose number of edges is a power of 2 is orientable.

DEFINITIONS AND NOTATION. The following are used in this paper as defined in [6]: map; regular; rotary; chiral; Petrie path; jth order hole; symmetry; the symmetries α, β, γ, X, R, S, T; the operators D, P, H_j; $G(M)$. The generators of $G(M)$ satisfy these relations:

$$(*) \qquad I = \alpha^2 = \beta^2 = \gamma^2 = \alpha\beta\gamma = X^2 = RX\alpha = SX\beta = TX\gamma.$$

When we give defining relations for the group of a map, we will assume relations $(*)$ as given, and not mention them explicitly. The orders of R, S, T are p, q, r, respectively.

The trivial maps. An important class of maps are the "trivial" maps illustrated in Figure 1. The map ε_k is simply an equator of the sphere divided into k edges, and its dual, $D\varepsilon_k$, is a k-paneled "beachball." The maps δ_k and $D\delta_k$ on the projective plane are derived from ε_{2k} and $D\varepsilon_{2k}$ respectively by identification of antipodal points. The map M_k is one of the canonical representations of the orientable surface of genus $[k/2]$, and M_k' may be viewed as M_{k-1} with a diameter drawn in.

These maps are regular for every k, and are distinct for $k > 2$. They are exactly those maps in which R S, or T has order 2. The notation here was chosen for brevity, and differs from that in [1] and elsewhere. The correspondence with [1] is given by this table:

	ε_k	$D\varepsilon_k$	δ_k	$D\delta_k$	M_k	M_k'
k even:	$\{k,2\}_k$	$\{2,k\}_k$	$\{2k,2\}/2$	$\{2,2k\}/2$	$\{2k,2k\}_{1,0}$	$\{k,k\}_2$
k odd:	$\{k,2\}_{2k}$	$\{2,k\}_{2k}$	$\{2k,2\}_k$	$\{2,2k\}_k$	$\{2k,k\}_2$	$\{k,2k\}_2$

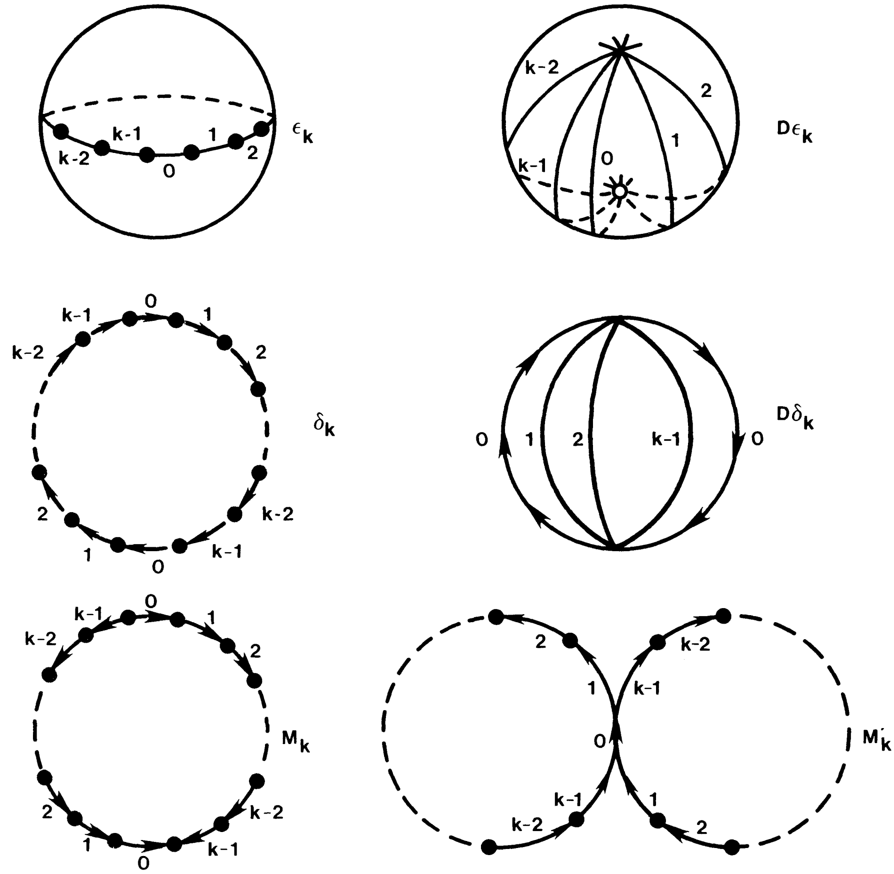


FIGURE 1: The trivial maps

It is not hard to check that the trivial maps are closed under the operators; the results are tabulated here:

If k is even:

M:	ε_k	$D\varepsilon_k$	δ_k	$D\delta_k$	M_k	M_k'
$D(M)$:	$D\varepsilon_k$	ε_k	$D\delta_k$	δ_k	M_k	M_k'
$P(M)$:	ε_k	M_k'	δ_k	M_k	$D\delta_k$	$D\varepsilon_k$
$\mathrm{opp}(M)$:	M_k'	$D\varepsilon_k$	M_k	$D\delta_k$	δ_k	ε_k.

and if k is odd:

M:	ε_k	$D\varepsilon_k$	δ_k	$D\delta_k$	M_k	M_k'
$D(M)$:	$D\varepsilon_k$	ε_k	$D\delta_k$	δ_k	M_k'	M_k
$P(M)$:	δ_k	M_k	ε_k	M_k'	$D\varepsilon_k$	$D\delta_k$
$\mathrm{opp}(M)$:	M_k'	$D\delta_k$	M_k	$D\delta_k$	δ_k	ε_k.

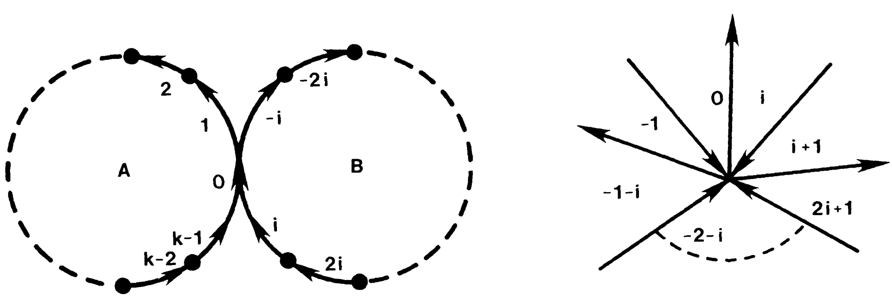
Let $|j|_k$ be the smallest positive integer g such that gj is a multiple of k; i.e., $|j|_k = k/(k, j)$. To calculate the effect of the operators H_j on the trivial maps, let $l = |j|_k$, $m = |j|_{2k}$. Then $H_j(D\varepsilon_k) = D\varepsilon_l$; if m is odd $H_j(D\delta_k) = D\varepsilon_m$; if m is even, $m = 2n$ for some n, then $H_j(D\delta_k) = D\delta_n$. The effect of H_j on M_k and M_k' can be determined using this information and $H_jP = PH_j$.

$M'_{k,i}$. It is not hard to see that the only possibilities for a map with one face are δ_k and M_k: however, ε_k and M_k' do not exhaust the possibilities for maps with two faces.

Suppose M is a rotary map with exactly two faces, A and B. Label each edge with a direction arrow so that, when viewed from the center of face A, all the arrows point in the same direction. If M is not orientable, then when viewed from the center of B, the arrows will not all point in the same direction. Then a rotation one step around B will be a symmetry of M which sends A onto itself and reverses the direction of at least one arrow. This must be a reflection about some axis of A, and so must have order 2. Thus each face must be a 2-gon and M must be $D\delta_2$.

So assume M is orientable and has k edges. Label the edges $0, 1, 2, \ldots, k-2, k-1, k = 0$, CCW around face A, and suppose that the first edge CCW from 0 in B is edge i. Then rotation one step CCW around B must be rotation i steps CCW around A, and the edges around B must be $0, i, 2i, 3i, \ldots, -2i, -i, 0 \pmod k$, as in Figure 2.

By the symmetry of the map, rotation one step around A must in turn be rotation i steps around B, which must be rotations i^2 steps around face A. Thus $i^2 \equiv 1 \pmod k$, and for each such pair k, i, the map, called $M'_{k,i}$ is rotary, in fact regular. In terms of the group of the map, the generators satisfy $I = R^k$, $\gamma R \gamma = R^i$, and these are a set of defining relations for $G(M)$. This is essentially the argument in [**1**, pp. 113–115].



a: $M'_{k,i}$

b: The neighborhood of a vertex in $M'_{k,i}$

FIGURE 2: The map $M'_{k,i}$

The properties of the maps $M'_{k,i}$ are easily determined by tracing through the diagram and using a few of the number-theoretic properties of the labelling. For instance, it is clear from Figure 2b that the valence of a vertex in $M'_{k,i}$ is $2|i+1|_k$; similarly, the length of a Petrie path is $2|i-1|_k$. Some of these results and special cases are listed here without proof:

1. $\mathrm{opp}(M'_{k,i}) = M'_{k,-i}$
2. $M_k{}' = M'_{k,1}$
3. $\varepsilon_k = M'_{k,-1}$
4. $H_{2j}M'_{k,i} = PD\varepsilon_g$, where $g = |j(i+1)|_k$
5. $H_{2j+1}M'_{k,i} = M'_{h,i}$, where $h = |ji + j + 1|_k$
6. $H_j DM'_{k,i} = DM'_{n,i}$, where $n|j|_k$
7. $M'_{k,i}$ is self-Petrie iff
   (a) k is even and $i = -1$ (so $M'_{k,i} = \varepsilon_k$)

or

   (b) $k = 8h$ and $i = 4h - 1$ for some integer h.
8. $M'_{k,i}$ is self-dual iff
   (a) k is even and $i = 1$ (so $M'_{k,i} = M_k{}'$)

or

   (b) $k = 8h$ and $i = 4h + 1$ for some integer h.

For the proofs of these facts, see [4].

$B(k,2l)$ and $B^*(k,2l)$. Let us agree to call a map *bicontactual* provided that each face of the map meets exactly two others. Two special cases of such maps are provided by the following construction: Consider a rectangle k squares high by $2l$ squares long. Darken alternate horizontal edges in each row and each column; label these with ordered pairs, (i, j)

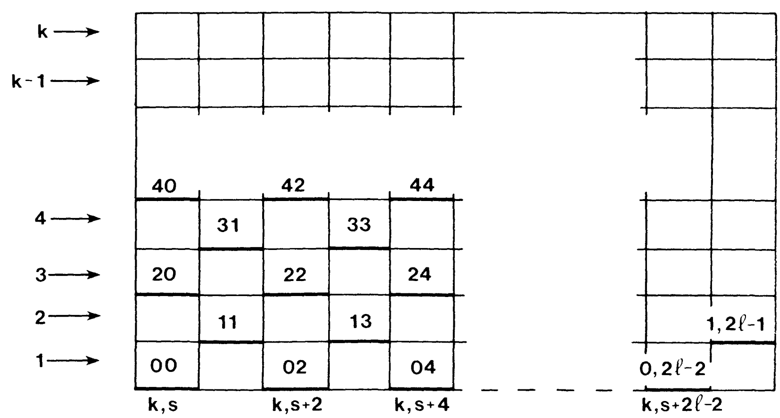


FIGURE 3: A scheme for some bicontactual maps

labelling the edge in the ith row and jth column when $i \equiv j$, (mod 2) as in Figure 3. This diagram is considered to be lying on the torus, so that the left and right vertical edges are identified directly, and the top row identified with the bottom so that the edge (k, s) is identified with $(0, 0)$ for some s. Note that this implies that $k \equiv s \pmod 2$.

To make a map from this scheme, use k faces, each of them a $2l$-gon. Number the faces $1, 2, \ldots, k$, and number the horizontal rows of squares $1, 2, \ldots, k$, as in Figure 3. Label the edges of face i with the labels of the darkened edges in row i of the scheme, in order CW as they are encountered reading left to right. Thus the edges of face 1 are $(0,0), (1,1), (0,2), (1,3), (0,4), \ldots, (1, 2l-1)$ in order clockwise. Identify orientably pairs of edges with the same label; also identify $(0, j)$ with $(k, j+s)$, as in the scheme.

The result of this construction will be a map of some sort. When will it be rotary? A rotation one step CW around face i must take face i to face $k+2-i \pmod k$, and must take the edge (i, j) to $(k+1-i, j+1+s)$ for $i \neq 0$. Then that same rotation would send $(k+1-i, j+1+s)$ to $(i, j+2s+2)$. But rotation two steps CW around face i clearly shifts the whole diagram two columns to the right; i.e., it sends (i, j) to $(i, j+2)$. Thus, $2s$ must be $0 \bmod 2l$. Conversely, if $2s \equiv 0$, $\pmod{2l}$, the resulting map is not only rotary but regular. We prove this by displaying the following functions:

$$\begin{array}{ll} \alpha: & (i, j)\alpha = (i, -j) \\ \beta: & (i, j)\beta = (k-i, s-j) \\ X: & (i, j)X = (k+1-i, s-1-j) \quad \text{if } i \neq 0 \\ & (0, j)X = (1, -1-j). \end{array}$$

These permutations of the edges are easily seen to be symmetries of the scheme and so symmetries of the map.

The equation $2s \equiv 0 \pmod{2l}$ has two solutions, $s \equiv 0$ or $s \equiv l \pmod{2l}$. If $s \equiv 0$, we call the map $B^*(k, 2l)$, and in that case k must be even, since $k \equiv s \pmod 2$. If $s \equiv l$, we call the map $B^*(k, 2l)$, and then $k \equiv l \pmod 2$. Figure 4 illustrates some of the possibilities, showing three schemes and the corresponding maps. Figure 4a is $B(4, 8)$ and Figure 4b is $B^*(4, 8)$; note the difference in the identification of faces 1 and 4 in the scheme and in the map. Figure 4c shows a case with k odd (and so l is odd also), $B^*(5, 6)$.

Comparing the scheme to the map diagram, one can see that a set of edges which form a vertex in the map appear in the scheme as a diagonal, e.g., $00, 11, 22, \ldots$ or $31, 22, 12, 04, \ldots$, while the edges that form a Petrie

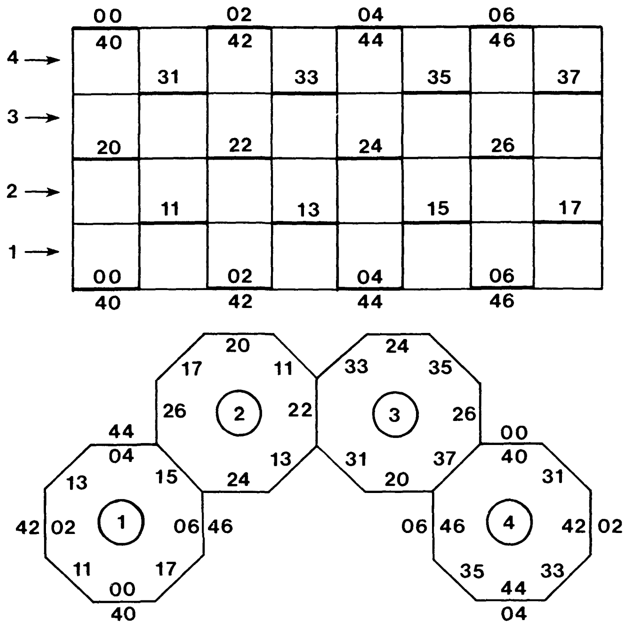


FIGURE 4a: $B(4,8)$

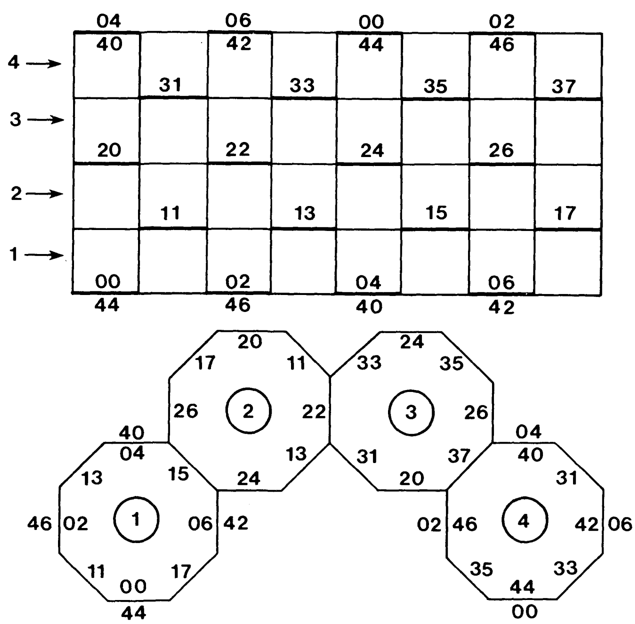


FIGURE 4b: $B*(4,8)$

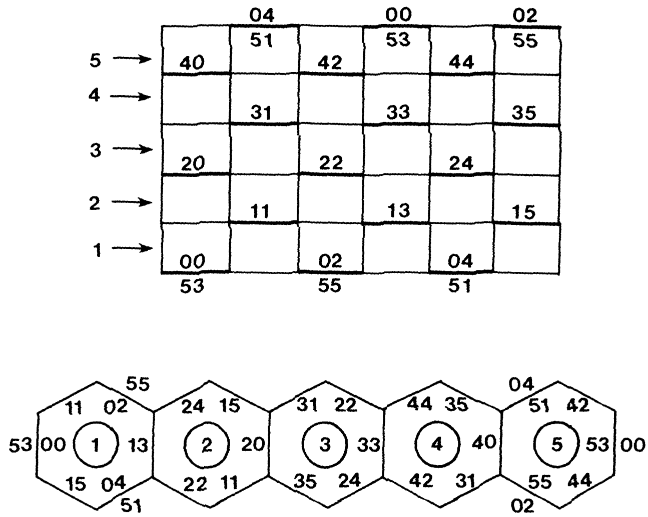


FIGURE 4c: $B*(5,6)$

path appear in adjacent columns, for instance $02, 11, 22, 31, \ldots$. The tractability of the system of labels, together with these observations make it easy to analyze the maps $B(k,2l)$ and $B^*(k,2l)$ by simply tracing through the corresponding schemes. We list below some of these results and special cases. Again, the proofs (where proofs are needed) are in [**4**].

1. $D\varepsilon_k = B(k,2)$ if k is even,
   $B^*(k,2)$ if k is odd.
2. $B^*(1,2l) = M_l$ if l is odd.
3. $B(2,2l) = M'_{2l}$.
4. $B^*(2,2l) = M'_{2l,l+1}$ if l is even.
5. $H_2\,\mathrm{opp}\{4,4\}_{k,0} = H_2\,\mathrm{opp}\{3,6\}_{k,0} = B(k,k)$ if k is even,
   $B^*(k,2k)$ if k is odd,
   $H_2\,\mathrm{opp}\{4,4\}_{k,k} = B^*(k,2k)$,
   $H_2\,\mathrm{opp}\{3,6\}_{k,k} = B^*(k,6k)$.
6. $PB(k,2l) = B(2l,k)$,
   $PB^*(k,2l) = B^*(l,2k)$.
7. Let $k_1 = |j|_k$, $k_2 = \mathrm{LCM}(2,k_1)$,
   $l_1 = |j|_l$, $l_2 = |j|_{2l}$. Then $H_jB(k,2l) = B(k_2,2l_1)$ if
   k_1l_2 is even, $B^*(k_1,2l_1)$ if k_1l_2 is odd.
8. If n is a positive integer, let $t(n)$ be that positive integer k such that $n/2^k$ is an odd integer. Let $\mathrm{sgn}(n) = 1, 0, -1$ according as n is positive, zero, or negative. With k_1, l_1 as before, $H_j(B^*(k,2l)) = B^*(k_1,2l_1)$ if $\mathrm{sgn}(t(j)-t(k)) = \mathrm{sgn}(t(j)-t(l))$, $B(2k_1,2l_1)$ otherwise.

9. If $i^2 \equiv 1 \pmod{k}$, let $K = (k, i+1)$, $L = k/K$. Then if $L = (k, i-1)$, $DM'_{k,i} = B^*(K, 2L)$, and if $L = \frac{1}{2}(k, i-1)$, $DM'_{k,i} = B(K, 2L)$.
10. The generators of the groups for $B^*(k,2l)$ and $B^*(k,2l)$ both satisfy $I = R^{2l}$, $\gamma R^2 \gamma = R^2$. $B(k,2l)$ also satisfies $I = T^k$, and $B^*(k,2l)$ satisfies $T^k = R^l$ if k is even, $I = T^{k-l}S^l$ if k is odd; further, these are a complete set of defining relations for the respective groups.

Multiple contact. Suppose a map M has the property that each face meets exactly h others, l times apiece. Let A, B be adjacent faces and let L be the surface formed by the union of the interiors of A and B and the relative interiors of their common edges. Call M *locally orientable* if L is an orientable surface. An argument essentially identical to that at the beginning of our analysis of $M'_{k,i}$ shows that if M is not locally orientable, $l = 2$, and if M is locally orientable, there is a number ρ relatively prime to l so that rotation h steps CW around A is the same as rotation ρh steps CW around face B; by the symmetries $\rho^2 \equiv 1 \pmod{l}$. If $h = 1$, we have the map $M'_{l,\rho}$.

It is the case, $h = 2$, bicontactual maps, with which we are concerned in this paper. If such a map has k faces, they may be numbered $1, 2, 3, \ldots, k-1, k$ so that face 1 meets only faces $i-1$ and $i+1$ (mod k). Then the faces occur in the order $1, 2, 3, \ldots, k, 1, 2, 3, \ldots, k-1, k$ around each vertex, so every face meets every vertex, k divides q and so V, the number of vertices, divides $2l = p$.

If $l = 1$, M either $D\varepsilon_k$ or $D\delta_k$. If $l = 2$, there are three cases to consider:

I. $V = 1$. Then M is M_2

II. $V = 2$. Then M is $DM'_{k,k/2-1}$

III. $V = 4$. If M is locally orientable, it is not hard to show that k must be even; if $\frac{1}{2}k$ is even, M is $B(k,4)$ while if $\frac{1}{2}k$ is odd, M must be $B^*(k,4)$. If M is not locally orientable, the first few faces must be arranged as in Figure 5. Since each face meets each vertex exactly once, the valence of a vertex must be exactly k; on the other hand, from the arrangement of the vertices, b, c, d around a, it is clear that $q = k$ must be divisible by three, $k = 3n$ for some integer n. Conversely, the map Γ_n so constructed is regular for each n; its defining relations are $I = R^4 = S^{3n} = RSRT$. The map Γ_1 is the hemi-cube, a map on the projective plane derived from the cube by identification of antipodal points, and Γ_n is constructed from Γ_1 by covering it with an n-sheeted covering branched at

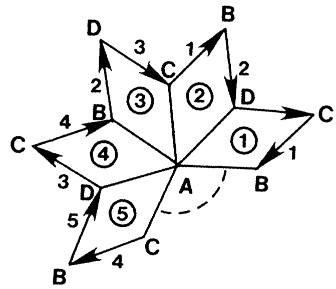


FIGURE 5: Some faces of Γ_n

the four vertices. See [5]. The dual of Γ_2 was first discovered by Grek; see [3], p. 30, his map $(4,6)_2$.

Henceforward, then, we assume that $l > 2$, so M is locally orientable. We can then choose an orientation of face 1, i.e., decide which is CW and CCW there, and let that force an orientation on face 2 by extension. We can continue that forcing through faces $3, 4, \ldots, k$ and let CW* and CCW* be the extensions of CW and CCW respectively that would be forced on face 1 by the choices on face k. Thus M is orientable iff CW = CW*. Let Q be the symmetry of M which rotates face 1 by two steps CW. Then Q sends each face onto itself, and moves face 2 by 2ρ steps CW, face 3 by 2 steps CW, 4 and 2ρ steps CW, and so on. It is not hard to see that if $\rho = 1$, the edges may be represented in a rectangular scheme, and so the map must be a $B(k, 2l)$ or $B^*(k, 2l)$. There are a few cases to consider:

A. k is odd so Q moves face k by 2 steps CW and moves face 1 by 2ρ steps CW*.

A1. M is orientable, CW* is CW, so $\rho = 1$ and M is $B^*(k, 2l)$.

A2. M is not orientable, CW* is CCW, so $\rho = -1$. Thus, opp M is a bicontactual map with $\rho = 1$, and so M is opp $B^*(k, 2l)$.

B. k is even, Q moves face k by 2ρ steps CW, and moves face 1 by 2 steps CW*, so CW* = CW, and M is orientable. This is the case we wish to look at in detail.

$B(k, 2l, \rho, \sigma)$. Label the edges CW around face i by ordered triples

$$(i, U, 0), (i, L, 0), (i, U, 1), (i, L, 1), (i, U, 2), \ldots, (i, U, l-1),$$
$$(i, L, l-1), (i, U, l) = (i, U, 0),$$

and choose the starting point in each face so that $(i, U, 0) = (i+1, L, 0)$ for $1 \leq i < k$. This will force $(i, U, j) = (i+1, L, \rho j)$ for $1 \leq i < k$, and

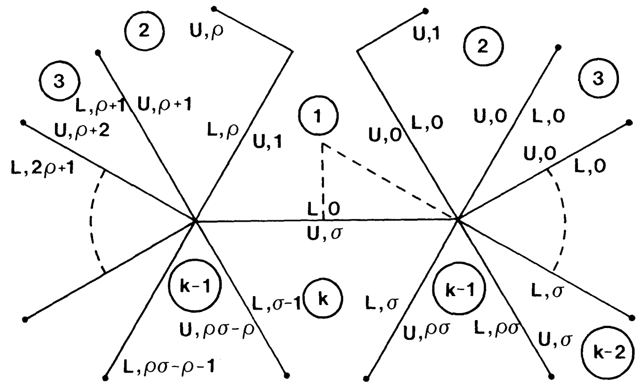


FIGURE 6: Neighborhood of the edge $(1, L, 0) = (k, U, \sigma)$ in $B(k, 2l, \rho, \sigma)$.

it will force $(k, U, j + \sigma) = (1, L, \rho j)$ for some constant σ. Thus, the map determines the parameters k, l, ρ, σ. On the other hand, given any k, l, ρ, σ, we can make a map of k faces, each a $2l$-gon, labelling the edges and identifying them according to these rules. Call this map $B(k, 2l, \rho, \sigma)$. We wish to find necessary and sufficient condition on the parameters for the map to be rotary. We already have that if the map is to be rotary, k must be even, say $k = 2\kappa$, and that $\rho^2 \equiv 1 \pmod{l}$.

In Figure 6, we show some of the edges around edge $(1, L, 0)$. If M is to be rotary, there must be a symmetry which acts as rotation one step CCW around face 1. This symmetry must send face i onto face $2 - i \pmod{k}$, and for $2 \le i \le k$, there must be integers $r_i \bmod k$ so that edge (i, U, j) is sent to $(k + 2 - i, U, j + r_i + 1)$. From the diagram, $r_2 = \sigma - 1$, $r_3 = \rho\sigma - \rho - 1$. It is not hard to see that for $2 \le i < k$, $r_{i+1} = \rho r_i - 1$. By induction, $r_{2n+1} = \rho\sigma - n(\rho + 1)$, $r_{2n+2} = \sigma - n(\rho + 1) - 1$. Then $r_k = r_{2\kappa} = \sigma - \kappa(\rho + 1) + \rho$; but from the diagram, $r_k = \rho - \sigma$. These two expressions must be equivalent mod l. Equating them and solving, we get $2\sigma \equiv \kappa(\rho + 1) \pmod{l}$. Similarly, there must be a symmetry which rotates one step around the left vertex of $(1, L, 0)$, and a necessary condition for that to exist is $\rho\sigma \equiv \sigma \pmod{l}$.

Thus, if $B(k, 2l, \rho, \sigma)$ is to be rotary, the parameters must satisfy these conditions;

$$\begin{aligned} &k = 2\kappa \\ &\rho^2 \equiv 1 && \pmod{l} \\ &2\sigma \equiv \kappa(\rho + 1) && \pmod{l} \\ &\rho\sigma \equiv \sigma && \pmod{l}. \end{aligned}$$

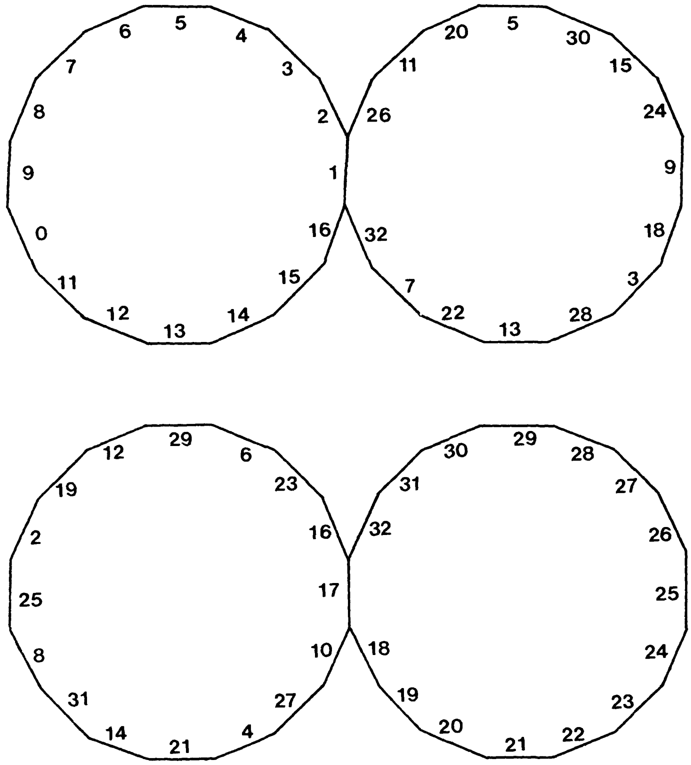


FIGURE 7: $B(4, 16, 3, 0)$

On the other hand, see [**4**] for a proof that these conditions are sufficient for the map to be not only rotary but regular. The generators of the group satisfy $I = R^{2l}$, $\gamma R^2 \gamma = R^{2\rho}$, $S^k = R^{2\sigma}$, and these form a complete set of defining relations for the group. Figure 7 shows $B(4, 16, 3, 0)$. This self-Petrie map and its opposite, $B(4, 16, 5, 2)$ are the smallest bicontactual maps which are not direct derivates of some $B(k, 2l)$ or $B^*(k, 2l)$.

While the system of edge labels for $B(k, 2l, \rho, \sigma)$ is not as tidy as that for the B and B^* maps, it is sufficiently manageable to enable one to determine desired information about the maps without actually drawing the picture. Some special cases and general results are gathered here:

1. The valence of a vertex in $B(k, 2l, \rho, \sigma)$ is $k|\sigma|_l$; the length of a Petrie path is $k|\kappa - \sigma|_l$.
2. $B(k, 2l) = B(k, 2l, 1, \kappa)$.
3. If k is even, $B^*(k, 2l) = B(k, 2l, 1, \kappa + \frac{1}{2}l)$.
4. $\operatorname{opp} B(k, 2l, \rho, \sigma) = B^*(k, 2l, -\rho, \kappa - \sigma)$.
5. $B(2, 2l, \rho, \sigma) = M'_{2l, 2\sigma - 1}$.

6. $PB(k, 2l, \rho, \sigma)$ is bicontactual iff $\rho - 1$ is a multiple of $\kappa - \sigma$ (mod l). In that case, let $\rho - 1 \equiv a(\kappa - \sigma)$ (mod l), $n = (\kappa - \sigma, l)$, $\sigma - \kappa \equiv nm$ (mod l); let j solve $jm \equiv \rho$ (mod l/n) and let $\bar{\sigma} = n - \kappa j$. Then $PB(k, 2l, \rho, \sigma) = B(2n, kl/n, a\kappa + l, \bar{\sigma})$.

7. $DB(k, 2l)$ is bicontactual iff $(k, 2l) = 4$, and then $DB(k, 2l) = B(4, \frac{1}{2}kl, \rho, \sigma)$, where ρ and σ satisfy

$$\begin{aligned} \rho &\equiv -1 && \left(\text{mod } \tfrac{1}{2}k\right) \\ \rho &\equiv 1 && (\text{mod } l) \\ \sigma &\equiv 0 && \left(\text{mod } \tfrac{1}{2}k\right) \\ \sigma &\equiv 2 && (\text{mod } l). \end{aligned}$$

In particular, $B(k, 2l)$ is self-dual iff $k = 4$ and l is even.

8. $DB^*(k, 2l)$ is bicontactual iff $(k + l, 2l) = 4$, and then $DB^*(k, 2l) = B(4, \frac{1}{2}kl, \rho, \sigma)$, where ρ and σ satisfy

$$\begin{aligned} \rho(k + l)/4 &\equiv (k - l)/4 && \left(\text{mod } \tfrac{1}{4}kl\right) \\ \sigma(k + l)/4 &\equiv (k/2) && \left(\text{mod } \tfrac{1}{4}kl\right). \end{aligned}$$

9. Let $s = \rho + 1$, $d = (s, l)$, $h = l/d = |s|_l$. There are only three possibilities for $H_2B(k, 2l, \rho, \sigma)$:

I. $B(k, 2h)$

II. $B(\kappa, 2h)$

III. $B^*(\kappa, 2h)$, as determined by the following table:

Cases	*Result*
1. l is odd	
A. κ is odd	III
B. κ is even	II
2. l is even	
A. κ is even	
i. $\sigma \equiv (\kappa s)/2$ (mod l)	II
ii. $\sigma \equiv (\kappa s + l)/2$ (mod l)	
a. $t(s) < t(l)$	III
b. $t(s) \geq t(l)$	I
B. κ is odd	
i. $\sigma \equiv (\kappa s)/2$ (mod l)	
a. $t(s) > t(l)$	III
b. $t(s) \leq t(l)$	I
ii. $\sigma \equiv (\kappa s + l)/2$ (mod l)	
a. $t(s) = t(l)$	III
b. $t(s) \neq t(l)$	I

For the purposes of this result, ρ and σ should be regarded as integers instead of numbers mod l, so that an expression like $(\kappa s + l)/2$ will make sense. The reader should convince himself that the outcome of the table is independent of the choice of representatives of the classes of ρ and σ mod l (e.g., we can write $B(14, 60, 11, 27)$ as $B(14, 60, 41, 87)$ if we wish, but in either case, H_2 of the map is $B(14, 10)$).

10. Let $n = 2m + 1$, $s = m(\rho + 1) + 1$, $h = |s|_l$, $v = |n|_k$, $u = |n|_\kappa$.
If l is odd, let $\bar{\sigma} = \frac{1}{2}(h + 1)u(\rho + 1)$.
If l is even, and $\sigma vn/k \equiv us(\rho + 1)/2$, let $\bar{\sigma} = u(\rho + 1)/2$.
If l is even, and $\sigma vn/k \equiv \frac{1}{2}l + us(\rho + 1)/2$, let

$$\bar{\sigma} = \tfrac{1}{2}h + u(\rho + 1)/2.$$

Then $H_n B(k, 2l, \rho, \sigma) = B(v, 2k, \rho, \bar{\sigma})$.

11.

$$H_2 DB(k, 2l, \rho, \sigma) = DM_{l,\rho};$$

$$H_{2j+1} B(k, 2l, \rho, \sigma) = DB(k, 2|2j + 1|_l, \rho, \sigma).$$

Summary of defining relations. It is often handy to have the defining relations in a standard form consisting of a number of words set equal to the identity, each of which is a product of positive powers of R, S, and T. Below are the standard forms of defining relations of each of the families we have discussed, together with the necessary conditions of the parameters.

Family	E	Defining relations	Conditions
$M'_{k,i}$	k	$I = R^k = T^2R^{i-1}$	$i^2 \equiv 1 \pmod{k}$
$B(k,2l)$	kl	$I = R^{2l} = T^k = (RT)^2$	k is even
$B^*(k,2l)$	kl	$I = R^{2l} = T^kR^l = (RT)^2$	k, l both even
		$I = R^{2l} = S^kR^{l-k} = (RT)^2$	k, l both odd
$B(k,2l,\rho,\sigma)$	kl	$I = R^{2l} = S^{k-2}TR^{2\sigma-2}T = TRTR^{2\rho-1}$	$k = 2\kappa$
			$\rho^2 \equiv 1 \pmod{l}$
			$2\sigma \equiv \kappa(\rho + 1) \pmod{l}$
			$\rho\sigma \equiv \sigma \pmod{l}$

In [2], Garbe determines all maps with 5 or fewer faces, and many of these are bicontactual. The maps of his Theorem 5.1 are

$$B^*(2m - 1, (2m - 1)(2n - 1)).$$

The maps of the first lemma in §6 are $M'_{p,r}$, and those of the lemma for Theorem 6.4 are $B(4, 2s, r, r + 1 + \frac{1}{2}k)$.

Regular maps with 2^n edges. [4] contains a catalogue of all non-trivial regular maps with no more than 100 edges (the list is complete except perhaps at $E = 84$), grouped according to the number of edges. It was noticed in compiling this catalogue that every entry under $E = 2^n$ was orientable for every n, and no other easily definable family of values for E had this property.

In order to prove that this happens in general, we will need the following results from [**5**]: let B be a set of words in R, S, and T, and let H be the normal closure of B in G, the full group of symmetries of a regular map M. B is called *allowable* if H contains none of α, β, γ, or X. If B is allowable, G/H is the group of some regular map L, M is a branched covering of L, and the factor homomorphism ψ: $G \to G/H$ may be considered as a projection ψ of the surface of M onto the surface of L. In this case, we write $L = M/B$. Moreover, if $\langle S^h \rangle$ is normal in G for some H, then each vertex of M meets h_1 others, where h_1 is some factor of h, $\langle S^{h_1} \rangle$ is also normal in G, and the projection ψ: $M \to M/S^h$ is one-to-one at each vertex.

Further, since R and S preserve orientation while T reverses orientation, it is not hard to see that the following criterion holds:

(**) A regular map M is orientable (i.e. it lies on an orientable surface) iff each word of the defining relations in standard form contains an even number of T's.

THEOREM. *If M is a non-trivial rotary map with 2^n edges for some n, then M is orientable.*

Proof. First note that a non-orientable rotary map is regular ([**1**, p. 102]); thus, we can assume that M is regular. Also note that the phrase "non-trivial" cannot be deleted from the hypothesis, since the regular maps δ_k and $D\delta_k$ on the projective plane exist for every k.

We proceed by induction on M. The base for the induction is easily established by noting that for $n = 1$ and $n = 2$, every regular map with 2^n edges is trivial. Now suppose that M is a non-trivial regular map with 2^n edges and that the theorem has been established for all smaller values of n. The statement that M is non-trivial is equivalent to the statement that neither α, β, nor γ commutes with X. Thus, neither α, β, γ nor X is in Z, the center of G. On the other hand, the order of G is 2^{n+2}; since it is a non-trivial 2-group, it has a non-trivial center. Let Q be any involution in Z; then $\{Q\}$ is an allowable set. Let $L = M/Q$. Then L has 2^{2n-1} edges and there are two cases to consider:

Case I. L is non-trivial. Then L is orientable by the induction hypothesis, and M, as a covering of L, must also be orientable.

Case II. L is trivial. Then one of R, S, or T must have order 2. Let L^* be a direct derivate of L in which S has order 2 and let M^* be the corresponding derivate of M. Since the order of S is not 2 in M^*, $Q = S^2$ in $G(M^*)$. Since S has order 2 in L^*, each vertex meets only two others there; since the projection is one-to-one on vertices, the same must be true in M^*, so M^* is the dual of a bicontactual regular map and so M is a direct derivate of a bicontactual map. However, if we examine the defining relations for bicontactual maps and the criterion $(**)$, we see that any direct derivate of a bicontactual regular map with 2^n edges must be orientable.

Thus, in any case, M must be orientable.

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MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824

Author's current address: Department of Mathematics
Northern Arizona University
Flagstaff, AZ 86011