PACIFIC JOURNAL OF MATHEMATICS
Vol. 120, No 2, 1985

SPECTRAL SETS AS BANACH MANIFOLDS

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Let A be a commutative Banach algebra, X its spectrum, and M a closed analytic submanifold of an open set in C^n. We may consider the set of germs of holomorphic functions from X to M, $\mathcal{O}(X, M)$. Now let ν be the functional calculus homomorphism from $\mathcal{O}(X, C^n)$ to A^n, and $A_M = \nu(\mathcal{O}(X, M))$.

It is proven that A_M is an analytic submanifold of A^n, modeled on projective A-modules of rank = dim M.

1. Introduction. Let A be a commutative complex Banach algebra with identity, and let X be the set of all characters of A, considered as a compact subset of the topological dual A' with the weak*-topology.

If U is an open neighborhood of X, and B a complex Banach space a map $f\colon U \to B$ will be called holomorphic if it is locally bounded and all its complex directional derivatives exist. The set of all such functions which are also bounded on U will be denoted by $H^\infty(U, B)$, or simply $H^\infty(U)$, when B is the complex field. These are locally convex spaces with the topology of uniform convergence. We define $\mathcal{O}(X, B)$ and $\mathcal{O}(X)$ to be the inductive limit of these spaces as U ranges over all open neighborhoods of X. $\mathcal{O}(X)$ is then a topological algebra. We recall (see [**2**] or [**7**]) that there exists a continuous algebra epimorphism, the holomorphic functional calculus

$$\nu\colon \mathcal{O}(X) \to A$$

such that: the composition of ν and the Gelfand map

$$\mathcal{O}(X) \to A \to C(X)$$

is the restriction map $f \mapsto f|_X$, and the composition of the linear map $a \mapsto \tilde{a}$ and ν

$$A \to \mathcal{O}(X) \to A$$

is the identity map of A. Here $\tilde{a}$ denotes the germ of the holomorphic map defined on A' by $\gamma \mapsto \gamma(a)$.

In [**6**], Raeburn has generalized previous results of Taylor and Novodvorskii ([**7**], [**5**]). He uses a generalization of the morphism ν, extending the holomorphic functional calculus to a linear map

$$S\colon \mathcal{O}(X, B) \to A \hat{\otimes} B.$$

If $M \subset B$ denotes a Banach submanifold, $\mathcal{O}(X, M)$ is defined and so is the set

$$A_M = \{S(f): f \in \mathcal{O}(X, M)\} \subset A \hat{\otimes} B.$$

Raeburn shows that if M is a discrete union of Banach homogeneous spaces the set A_M is locally path connected and the generalized Gelfand map

$$A_M \to C(X, M)$$

induces a bijection on the set of components

$$[A_M] \xrightarrow{\simeq} [X, M].$$

In this note, in §3, we take $B = \mathbf{C}^n$ and M a closed submanifold of an open set of $\mathbf{C}^n$, and prove that the set A_M is in fact an analytic submanifold of A^n. This was first stated by Taylor in [**8**]. A_M is modeled on projective A-modules of rank = dim M. We also prove that A_M and $A^M = \{a \in A^n\colon \operatorname{sp}(a) \subset M\}$ have the same homotopy type. Note that with $B = \mathbf{C}^n$, we have $S = \nu \times \cdots \times \nu$ and $A \hat{\otimes} B = A^n$.

In order to do this we first prove in §2 a version of the constant rank theorem.

2. The constant rank theorem. In this paragraph we give a version of the constant rank theorem valid for A-modules; the whole paragraph is an adaptation of the results in [**4**].

We will be dealing with submodules of the free module A^n, and A-module morphisms $T\colon A^n \to A^m$. A submodule E of A^n will be called *A-direct* if it is closed and there is another closed submodule E' of A^n such that $A^n = E \oplus E'$; obviously, this is equivalent to the fact: $E = \operatorname{Ker} p$ (resp: $E = \operatorname{Im} p$), for some continuous A-linear projector $p\colon A^n \to A^n$.

Note that in this case E is a projective module, but not necessarily free.

If $T\colon A^n \to A^m$ is an A-module morphism, we say that *T is A-direct* (also called "*split*") if Ker T and Im T are A-direct.

Assume that

$$A^n = E_1 \oplus E_2, \quad F_1 \oplus F_2 = A^m$$

for some closed submodules E_1, E_2, F_1, F_2; if $T\colon A^n \to A^m$ is an A-morphism we shall use the notation

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} : \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \to \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

with $T_{ij} \in \mathrm{Hom}_A(E_j, F_i)$ $(i, j = 1,2)$, meaning that if

$$x = x_1 + x_2 \qquad (x_1 \in E_1, x_2 \in E_2),$$

then

$$T(x) = [T_{11}(x_1) + T_{12}(x_2)] + [T_{21}(x_1) + T_{22}(x_2)]$$

is the expression of $T(x)$ as a sum of elements in F_1 and F_2.

We shall need the following elementary lemma, which we state without proof.

LEMMA 2.1. *Let P_1, P_2 be A-direct submodules of A^n of the same rank. Then $P_1 \subset P_2$ implies $P_1 = P_2$.*

THEOREM 1. *Suppose T_0: $A^n \to A^m$ is an A-direct morphism and let E_1 and F_2 be closed submodules of A^n and A^m respectively such that*

$$A^n = E_1 \oplus \mathrm{Ker}\, T_0, \quad \mathrm{Im}\, T_0 \oplus F_2 = A^m$$

If

$$T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : \begin{bmatrix} E_1 \\ \mathrm{Ker}\, T_0 \end{bmatrix} \to \begin{bmatrix} \mathrm{Im}\, T_0 \\ F_2 \end{bmatrix}$$

then the following are equivalent

(i) *T is A-direct, $A^n = E_1 \oplus \mathrm{Ker}\, T$ and $A^m = \mathrm{Im}\, T \oplus F_2$.*

(ii) *$\alpha \in \mathrm{Iso}(E_1, \mathrm{Im}\, T_0)$ and $\delta = \gamma\alpha^{-1}\beta$.*

(iii) *There exist A-linear automorphisms u: $A^n \to A^n$, v: $A^m \to A^m$ such that $T_0 = vTu$ and*

$$u|E_1 = \mathrm{id}_{E_1} \quad v|F_2 = \mathrm{id}_{F_2}.$$

(iv) *T is A-direct, $\alpha \in \mathrm{Iso}(E_1, \mathrm{Im}\, T_0)$ and* $\mathrm{rk}(\mathrm{Im}\, T_0) = \mathrm{rk}(\mathrm{Im}\, T)$.

Proof: Suppose (i) and consider the diagram

$$\begin{array}{ccc} E_1 \times \mathrm{Ker}\, T & \xrightarrow{w} & \mathrm{Im}\, T \times F_2 \\ \phi \uparrow & & \downarrow \psi \\ A^n = E_1 \oplus \mathrm{Ker}\, T_0 & \xrightarrow{T} & \mathrm{Im}\, T_0 \oplus F_2 = A^m \end{array}$$

where ϕ is the isomorphism $v \to (v_1, v_2)$; here v_1 (resp: v_2) is the projection of v onto E_1 (resp: $\mathrm{Ker}\, T$) with kernel $\mathrm{Ker}\, T$ (resp. E_1). We define ψ

in a similar way. Then we have

$$\phi = \begin{bmatrix} 1 & \tau \\ 0 & \theta \end{bmatrix} : \begin{bmatrix} E_1 \\ \operatorname{Ker} T_0 \end{bmatrix} \to \begin{bmatrix} E_1 \\ \operatorname{Ker} T \end{bmatrix}$$

and

$$\psi = \begin{bmatrix} \mu & 0 \\ \nu & 1 \end{bmatrix} : \begin{bmatrix} \operatorname{Im} T \\ F_2 \end{bmatrix} \to \begin{bmatrix} \operatorname{Im} T_0 \\ F_2 \end{bmatrix}$$

with $\tau \in \operatorname{Hom}_A(\operatorname{Ker} T_0, E_1)$, $\nu \in \operatorname{Hom}_A(\operatorname{Im} T, F_2)$ and $\theta \in \operatorname{Iso}_A(\operatorname{Ker} T_0, \operatorname{Ker} T)$, $\mu \in \operatorname{Iso}_A(\operatorname{Im} T, \operatorname{Im} T_0)$. On the other hand we also have

$$w = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} E_1 \\ \operatorname{Ker} T \end{bmatrix} \to \begin{bmatrix} \operatorname{Im} T \\ F_2 \end{bmatrix}$$

with $\lambda \in \operatorname{Iso}_A(E_1, \operatorname{Im} T)$.

The commutativity of the diagram implies

$$\begin{bmatrix} \mu & 0 \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \tau \\ 0 & \theta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

hence $\mu\lambda = \alpha$ (which implies that α is an isomorphism) and $\delta = \nu\lambda\tau = \nu\lambda(\lambda^{-1}\mu^{-1})\mu\lambda\tau = \gamma\alpha^{-1}\beta$, and we have (ii). Now assume (ii): if

$$T_0 = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} E_1 \\ \operatorname{Ker} T_0 \end{bmatrix} \to \begin{bmatrix} \operatorname{Im} T_0 \\ F_2 \end{bmatrix}$$

with $\lambda \in \operatorname{Iso}_A(E_1, \operatorname{Im} T_0)$ we define

$$u = \begin{bmatrix} 1 & -\alpha^{-1}\beta \\ 0 & 1 \end{bmatrix} : \begin{bmatrix} E_1 \\ \operatorname{Ker} T_0 \end{bmatrix} \to \begin{bmatrix} E_1 \\ \operatorname{Ker} T_0 \end{bmatrix}$$

and

$$v = \begin{bmatrix} \lambda\alpha^{-1} & 0 \\ -\gamma\alpha^{-1} & 1 \end{bmatrix} : \begin{bmatrix} \operatorname{Im} T_0 \\ F_2 \end{bmatrix} \to \begin{bmatrix} \operatorname{Im} T_0 \\ F_2 \end{bmatrix}$$

and a routine calculation gives (iii).

Now suppose we have (iv) and define the automorphism $S\colon A^m \to A^m$ by

$$S = \begin{bmatrix} 1 & 0 \\ -\gamma\alpha^{-1} & 1 \end{bmatrix} : \begin{bmatrix} \operatorname{Im} T_0 \\ F_2 \end{bmatrix} \to \begin{bmatrix} \operatorname{Im} T_0 \\ F_2 \end{bmatrix}.$$

Then we have the composition

$$T' = ST = \begin{bmatrix} \alpha & \beta \\ 0 & \delta - \gamma\alpha^{-1}\beta \end{bmatrix} : \begin{bmatrix} E_1 \\ \operatorname{Ker} T_0 \end{bmatrix} \to \begin{bmatrix} \operatorname{Im} T_0 \\ F_2 \end{bmatrix}$$

which is also A-direct. Note that $\operatorname{Im}(T') = S(\operatorname{Im} T)$, hence $\operatorname{Im}(T')$ and $\operatorname{Im}(T)$ have the same rank; from this it follows that $\operatorname{rk}(\operatorname{Im} T') = \operatorname{rk}(\operatorname{Im} T_0)$.

But $\operatorname{Im}(T') \supset \alpha(E_1) = \operatorname{Im}(T_0)$; Lemma 2.1 gives $\operatorname{Im}(T') = \operatorname{Im}(T_0)$ and this fact implies $\delta - \gamma\alpha^{-1}\beta = 0$. This proves (ii)

(iii) $\Rightarrow$ (i) is simple; in fact, it is obvious that T is A-direct. It is also clear that $u(\operatorname{Ker} T_0) = \operatorname{Ker} T$, hence

$$A^m = v^{-1}(\operatorname{Im} T_0 \oplus F_2) = v^{-1}(\operatorname{Im} T_0) \oplus v^{-1}(F_2)$$

$$= v^{-1}T_0(A^n) \oplus F_2 = Tu(A^n) \oplus F_2 = \operatorname{Im} T \oplus F_2,$$

$$A^n = u(\operatorname{Ker} T_0 \oplus E_1) = u(\operatorname{Ker} T_0) \oplus E_1 = \operatorname{Ker} T \oplus E_1.$$

In order to complete the proof, we only need the inference (i) $\Rightarrow$ (iv): $\alpha \in \operatorname{Iso}(E_1, \operatorname{Im} T_0)$ as in (i) $\Rightarrow$ (ii). The rest is obvious, so the proof is complete.

We shall be concerned now with a generalization of the results in §1 of [**6**], we shall follow the definitions of this reference.

Let Ω be an open set in A^n, F: $\Omega \to A^m$ an holomorphic map, and $a \in \Omega$; we denote the differential of F at a by $DF(a)$.

A *linear representation of* F in a is an object (u, U, v, V, T) where

(i) U is a neighborhood of $0 \in A^n$, u is biholomorphic from U onto $u(U)$, a neighborhood of a contained in Ω, and $u(0) = a$.

(ii) V is a neighborhood of $0 \in A^m$, v is biholomorphic from V onto $v(V)$, a neighborhood of $F(a)$ and $v(0) = F(a)$

(iii) T: $U \to A^m$ is the restriction of an A-linear map, and $v^{-1}Fu = T$.

(iv) $Du(x)$ and $Dv(y)$ are A-linear maps if $x \in U, y \in V$.

We will say that the holomorphic map F: $\Omega \to A^m$ is *locally A-direct* at $a \in \Omega$ if there are closed sub-modules $E_1 \subset A^n$, $F_2 \subset A^m$ and a neighborhood U of a such that, for all $x \in U$,

(i) $DF(x)$ is A-linear

(ii) $A^n = E_1 \oplus \operatorname{Ker} DF(x)$

(iii) $A^m = \operatorname{Im} DF(x) \oplus F_2$.

We have now the following:

LEMMA 2.2. *Let Ω be an open set in A^n, F: $\Omega \to A^m$ holomorphic and $a \in \Omega$. If F is locally A-direct at a, then there is a linear representation (u, U, v, V, T) of F in a, with T A-direct.*

Proof. Without loss of generality we can assume that $a = 0$ and $F(a) = 0$; then there exist a neighborhood $\Omega_0 \subset \Omega$ of $0 \in A^n$ and closed submodules $E_1 \subset A^n$, $F_2 \subset A^m$ such that

$$A^n = E_1 \oplus \operatorname{Ker} DF(x), \qquad A^m = \operatorname{Im} DF(x) \oplus F_2$$

for all $x \in \Omega_0$. Also, $DF(x)$ is A-linear if $x \in \Omega_0$.

Let $E_2 = \operatorname{Ker} DF(0)$, $F_1 = \operatorname{Im} DF(0)$; we denote x_1, x_2 (resp: y_1, y_2) the components of $x \in A^n$ (resp: $y \in A^m$) in the decomposition $E_1 \oplus E_2$ (resp: $F_1 \oplus F_2$). In a similar way we write $F(x) = f_1(x) + f_2(x)$, with $f_1(x) \in F_1$ and $f_2(x) \in F_2$.

We have

$$DF(x) = \begin{bmatrix} D_1 f_1(x) & D_2 f_1(x) \\ D_1 f_2(x) & D_2 f_2(x) \end{bmatrix} : \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \to \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

and so we can simplify the notation writing $\alpha_{ij}(x) = D_i f_j(x)$ $(i, j = 1, 2)$. Recall that Theorem 1 gives

(a) $\alpha_{11}(x)\colon E_1 \to F_1$ is an isomorphism, and

(b) $\alpha_{22}(x) = \alpha_{12}(x)\alpha_{11}(x)^{-1}\alpha_{21}(x)$ for all $x \in \Omega_0$.

Define the following A-linear maps

$$\begin{aligned}
&S\colon E_1 \to F_1, && S = \alpha_{11}(0), \\
&T\colon A^n \to A^m, && T(x) = S(x_1), \\
&c\colon A^m \to A^n, && c(y) = S^{-1}(y_1), \\
&P\colon A^n \to A^n, && P(x) = x_2, \\
&Q\colon A^m \to A^m, && Q(y) = y_2.
\end{aligned}$$

Now define the holomorphic map $h\colon \Omega_0 \to A^n$ by

$$h = cF + P.$$

We have: $Dh(x)$ is an A-linear map if $x \in \Omega_0$. In fact,

$$\begin{aligned}
Dh(x) &= \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{11}(x) & \alpha_{21}(x) \\ \alpha_{12}(x) & \alpha_{22}(x) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} S^{-1}\alpha_{11}(x) & S^{-1}\alpha_{21}(x) \\ 0 & 1 \end{bmatrix},
\end{aligned}$$

hence by the inverse function theorem $h\colon \Omega_1 \to \Omega_2$ is biholomorphic for suitable neighborhoods of $0 \in A^n$.

Note that the differential of the map $Fh^{-1}P\colon P^{-1}(\Omega_2) \to A^m$ vanishes identically, that is

$$D(Fh^{-1}P)(x) = 0 \qquad \left(x \in P^{-1}(\Omega_2)\right).$$

In fact we can compute this differential as the composition $DF(h^{-1}P(x))Dh(h^{-1}P(x))^{-1}P$; the calculation leads (with $x' = h^{-1}P(x)$) to

$$\begin{bmatrix} \alpha_{11}(x') & \alpha_{21}(x') \\ \alpha_{12}(x') & \alpha_{22}(x') \end{bmatrix} \begin{bmatrix} \alpha_{11}(x')^{-1}S & -\alpha_{11}(x')^{-1}\alpha_{21}(x) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} S & 0 \\ \alpha_{12}(x')\alpha_{11}(x')^{-1}S & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0,$$

where we use the identity $\alpha_{22} = \alpha_{12}\alpha_{11}^{-1}\alpha_{21}$.

Hence we have proved

(c) $Fh^{-1}P$ vanishes identically in a neighborhood of 0 (for instance, in the connected component of 0 in $P^{-1}(\Omega_2)$).

Finally we define the holomorphic mapping $g\colon c^{-1}(\Omega_2) \to A^m$

$$g = Fh^{-1}c + Q.$$

Then if $x = h^{-1}c(y)$ we compute

$$Dg(y) = \begin{bmatrix} 1 & 0 \\ \alpha_{12}(x)\alpha_{11}(x)^{-1} & 1 \end{bmatrix}$$

and this shows that $g\colon \Omega_1' \to \Omega_2'$ is a biholomorphic map, where Ω_1' and Ω_2' are small enough neighborhoods of $0 \in A^m$. Also $Dg(y)$ is A-linear for every $x \in \Omega_1'$.

In order to complete the proof, set $u = h^{-1}$ and $v = g$; we must show that the identity

$$gTh = F$$

holds in some neighborhood of $0 \in A^n$; but this follows from (c) and the computation

$$gTh = (Fh^{-1}c + Q)T(cF + P) = Fh^{-1}cQF$$

$$= Fh^{-1}cF = Fh^{-1}(h - P) = F - Fh^{-1}P.$$

THEOREM 2. *Let Ω be an open subset of A^n, and $F\colon \Omega \to A^n$ an holomorphic retraction that is locally A-direct at x for all $x \in \Omega$. Then* Im F *is a Banach analytic manifold, and for all $x \in$* Im F *the tangent space T_x*(Im F) *at x is* Im $DF(x)$.

Proof. For every $F(x) \in \operatorname{Im} F$ there is, by Lemma 2.2, a linear representation $(u_x, U_x, v_x, V_x, T_x)$ of F with T_x A-direct.

For all $x' \in U_x$,

$$T_x = DT_x(x') = Dv_x^{-1}\big(Fu_x(x')\big) \cdot DF\big(u_x(x')\big) \cdot Du_x(x')$$
$$= \big[Dv_x\big(T_x(x')\big)\big]^{-1} \cdot DF\big(u_x(x')\big) \cdot Du_x(x').$$

$Dv_x(Z)$ and $Du_x(Z')$ are A-linear isomorphisms, so $\operatorname{Im} T_x \simeq \operatorname{Im} DF(u_x(x'))$, for all $x' \in U_x$. But F is A-direct at x, so there is a neighborhood of x where $\operatorname{Im} DF(a) \simeq \operatorname{Im} DF(b)$, for a, b in this neighborhood. Hence the $\operatorname{Im} T_z$ for z in this neighborhood are all A-isomorphic to a fixed A-module P. Call $h_z\colon \operatorname{Im} T_z \to P$ these A-isomorphisms. For every $x \in \operatorname{Im} F$, $x = F(x)$, and U_x, V_x may be chosen so that $u_x(U_x) = v_x(V_x)$. Then $v_x\colon V_x \cap \operatorname{Im} T_x \to v_x(V_x) \cap \operatorname{Im} F$ is a bijection: it is one to one over all of V_x, and if $v_x(z) \in \operatorname{Im} F$, say $v_x(z) = u_x(z')$,

$$v_x(z) = Fv_x(z) = Fu_x(z') = v_x T_x u_x^{-1}\big(u_x(z')\big) = v_x\big(T_x(z')\big)$$

so $v_x(z) \in v_x(V_x \cap \operatorname{Im} T_x)$.

Now define the chart near $x \in \operatorname{Im} F$: $(v_x(V_x) \cap \operatorname{Im} F, h_x v_x^{-1})$. These charts are compatible. To see this, suppose

$$U_{xy} = v_x(V_x) \cap v_y(V_y) \cap \operatorname{Im} F \neq \varnothing$$

we then have

$$\big(h_y v_y^{-1}\big)\big(h_x v_x^{-1}\big)^{-1}\colon h_x v_x^{-1}\big(U_{xy}\big) \to h_y v_y^{-1}\big(U_{xy}\big).$$

But $(h_y v_y^{-1})(h_x v_x^{-1})^{-1} = h_y v_y^{-1} v_x h_x^{-1}$ is holomorphic. The same goes for the other composition. The tangent space $T_x(\operatorname{Im} F)$ is given by

$$\operatorname{Im}\big(Dv_x(0) h_x^{-1}\big) = Dv_x(0)(\operatorname{Im} T_x) = \operatorname{Im}\big(Dv_x(0) T_x\big) = \operatorname{Im} D(v_x T_x)(0)$$
$$= \operatorname{Im} D(Fu_x)(0) = \operatorname{Im}\big(DF(u_x(0)) Du_x(0)\big) = \operatorname{Im} DF(x).$$

3. A_M as an analytic manifold. Here we will apply the results in the preceding paragraph to Taylor's A_M [7] where M is a closed submanifold of an open set of $\mathbf{C}^n$.

For $a \in A^n$, let $\hat{a}$ denote the function $A' \to \mathbf{C}^n$ defined by $\hat{a}(\gamma) = (\gamma(a_1), \ldots, \gamma(a_n))$ for all $\gamma \in A'$. Note that with the supremum norm in both A^n and $\mathbf{C}^n$, $|\hat{a}(\gamma)| \leq \|\gamma\|\,\|a\|$. We will sometimes write ϕ^n for $\phi \times \cdots \times \phi$. We denote by θ_a the classical holomorphic functional calculus of Arens and Calderón [1]. All other functional calculus morphisms and their restrictions will be denoted by ν.

We will need the following lemma.

LEMMA 3.1. *Let W be an open subset of $\mathbf{C}^n$. Then A_W is an open subset of A^n.*

Proof. Let $a \in A_W$, and $f \in \mathcal{O}(X, W)$ such that $a = \nu(f)$. Since $f(X)$ is a compact subset of W, there is an $\varepsilon > 0$ such that for every $\phi \in X$, the polydisc $\{z \in \mathbf{C}^n: |f(\phi) - z| < \varepsilon\}$ is contained in W. Now let $U = \{b \in A^n: \|a - b\| < \varepsilon\}$. $\hat{b}(X) \subseteq W$, because

$$|f(\phi) - \hat{b}(\phi)| = |\widehat{a - b}(\phi)| \leq \|a - b\| < \varepsilon.$$

Then $\hat{b}^{-1}(W)$ is a neighborhood of X in A', so $\hat{b} \in \mathcal{O}(X, W)$, and $b \in A_W$.

The sets A_W, with W open, are now appropriate domains for holomorphic functions. We will need to lift holomorphic functions in $\mathbf{C}^n$ to holomorphic functions in A^n. This will be done as follows. Let h: $W \to \mathbf{C}^m$ be holomorphic, and define A_h: $A_W \to A^m$ by $A_h(a) = \nu(h \circ f)$, if $a = \nu(f)$.

LEMMA 3.2. *A_h is a well-defined holomorphic function. For all $a = \nu(f) \in A_W$, $DA_h(a)$ is an A-module homomorphism given by $\nu(Dh(f))$.*

Proof. First, we will see that $\nu(f) = \nu(g)$ implies $\nu(h \circ f) = \nu(h \circ g)$.

For this, let $b_1, \ldots, b_k \in A$ be elements that finitely determine f and g, in other words, there is an open set Ω in $\mathbf{C}^k$ and there are F and G in $\mathcal{O}(\Omega, W)$ such that the following diagram commutes

$$\begin{tikzcd} \hat{b}^{-1}(\Omega) \arrow[r, "f(\text{resp. } g)"] \arrow[d, "\hat{b}"'] & W \arrow[r, "h"] & \mathbf{C}^m \\ \Omega \arrow[ur, "F(\text{resp. } G)"'] & & \end{tikzcd}$$

$\nu(f) = \nu(g)$ means that $\theta_b(F) = \theta_b(G)$, so $\mathrm{sp}(\theta_b(F)) = \mathrm{sp}(\theta_b(G)) \subseteq W$. Since $h \in \mathcal{O}(W, \mathbf{C}^m)$, we may write $\theta_{\theta_b(F)}(h) = \theta_{\theta_b(G)}(h)$. Then $h(F(b)) = h(G(b))$, so $\theta_b(h \circ F) = \theta_b(h \circ G)$ and $\nu(h \circ f) = \nu(h \circ g)$.

To prove that A_n is holomorphic, let $a \in A_W$, and $b \in A^n$. It will be enough to prove the existence of

$$\frac{\partial A_h}{\partial b}(a) = \lim_{\lambda \to 0} \frac{1}{\lambda}[A_h(a + \lambda b) - A_h(a)]. \tag{1}$$

Let $a = \nu(f)$, $b = \nu(g)$. Then $a + \lambda b = \nu(f + \lambda g)$, and (1) is $\lim_{\lambda \to 0} \lambda^{-1}[\nu(h \circ (f + \lambda g) - h \circ f)]$. Since the functional calculus is continuous, the limit (1) will exist if $\lim_{\lambda \to 0} \lambda^{-1}[h \circ (f + \lambda g) - h \circ f]$ exists in $\mathcal{O}(X, C^m)$. We must see that $\lambda^{-1}[h \circ (f + \lambda g) - h \circ f]$ converges uniformly over a neighborhood of X as $\lambda \to 0$. For this, set $\varepsilon > 0$, and if $\lambda \in C$ with $|\lambda| < \varepsilon$ and $\gamma \in X$, let

$$S(\lambda, \gamma) = \begin{cases} \dfrac{1}{\lambda}[h(f(\gamma) + \lambda g(\gamma)) - h(f(\gamma))] - \dfrac{\partial h}{\partial g(\gamma)} f(\gamma), & \text{if } \lambda \neq 0. \\ 0 & \text{if } \lambda = 0. \end{cases}$$

h is holomorphic, so $\lim_{\lambda\to 0} S(\lambda,\gamma)=0$ for each $\gamma\in X$. Then there are $\delta_\gamma>0$ and neighborhoods V_γ of γ such that $|S(\lambda,\phi)|<\varepsilon$ for $\lambda\in\mathbf{C}$ with $|\lambda|<\delta_\gamma$ and all $\phi\in V_\gamma$. Being X compact, there are $\gamma_1,\ldots,\gamma_p\in X$ such that V_{γ_i}, $i=1,\ldots,p$, cover X. Let $\delta=\min\{\delta_{\gamma_i}\colon 1\le i\le p\}$, and $V=\bigcup_{i=1}^p V_{\gamma_i}$. Then for all $\lambda\in C$ with $|\lambda|<\delta$ and all $\gamma\in V$, $S(\lambda,\gamma)<\varepsilon$, so A_h is holomorphic. We shall denote the limit of $\lambda^{-1}[h\circ(f+\lambda g)-h\circ f]$ as $\lambda\to 0$, by $Dh(f)(g)$.

$DA_h(a)$ is more than just a linear morphism. It is A-linear. To prove this we must show that the diagram

$$\begin{array}{ccccc} \mathcal{O}(X,\mathbf{C})^{m\times n} & \times & \mathcal{O}(X,\mathbf{C})^{n} & \to & \mathcal{O}(X,\mathbf{C})^{m} \\ \nu\downarrow & & \nu\downarrow & & \nu\downarrow \\ A^{m\times n} & \times & A^{n} & \to & A^{m} \end{array}\qquad \text{commutes.}$$

Here the horizontal arrows indicate matrix multiplication.

As all the arrows are continuous, and $P(\hat{A})^k$ is dense in $\mathcal{O}(X,\mathbf{C})^k$ for all k, where $P(\hat{A})$ is the algebra of polynomials in Gelfand transforms of elements of A, it will be enough to show that $\nu(p\cdot q)=\nu(p)\cdot\nu(q)$, where $p_{ij}, q_j\in P(\hat{A})$. Let

$$p_{ij}=\sum_{(k)}\widehat{a^{ij}}(k),\quad \text{where } \widehat{a^{ij}}(k)=\widehat{a^{ij}k_1}^{k_1}\cdots\widehat{a^{ij}k_r}^{k_r}$$

$$q_{j}=\sum_{(k')}\widehat{a^{j}}(k'),\quad \text{where } \widehat{a^{j}}(k')=\widehat{a^{j}k_1'}^{k_1'}\cdots\widehat{a^{j}k_s'}^{k_s'}.$$

$$\begin{aligned}\nu(p\cdot q)&=\nu\left(\sum_{j=1}^n p_{1j}q_j,\ldots,\sum_{j=1}^n p_{mj}q_j\right)\\ &=\nu\left(\sum_{j=1}^n\sum_{(k)}\widehat{a^{1j}}(k)\sum_{(k')}\widehat{a^{j}}(k'),\ldots,\sum_{j=1}^n\sum_{(k)}\widehat{a^{mj}}(k)\sum_{(k')}\widehat{a^{j}}(k')\right)\\ &=\left(\sum_{j=1}^n\sum_{(k)}a^{1j}(k)\sum_{(k')}a^{j}(k'),\ldots,\sum_{j=1}^n\sum_{(k)}a^{mj}(k)\sum_{(k')}a^{j}(k')\right).\end{aligned}$$

On the other hand,

$$\nu(p)\cdot\nu(q)=\left(\sum_{j=1}^n\nu(p)_{1j}\nu(q)_j,\ldots,\sum_{j=1}^n\nu(p)_{mj}\nu(q)_j\right). \tag{2}$$

But

$$\nu(p)_{lj}=\nu(p_{lj})=\nu\left(\sum_{(k)}\widehat{a^{lj}}(k)\right)=\sum_{(k)}a^{lj}(k),$$

and

$$\nu(q)_j = \nu(q_j) = \nu\left(\sum_{(k')} \widehat{a^j}(k')\right) = \sum_{(k')} a^j(k').$$

So

$$(2) = \left(\sum_{j=1}^{n} \sum_{(k)} a^{1j}(k) \sum_{(k')} a^j(k'),\dots, \sum_{j=1}^{n} \sum_{(k)} a^{mj}(k) \sum_{(k')} a^j(k')\right) = \nu(p \cdot q).$$

Then

$$DA_h(a)(b) = \nu(Dh(f)(g)) = \nu(Dh(f)) \cdot \nu(g) = \nu(Dh(f))(b).$$

So that $DA_h(a) = \nu(Dh(f)) \in A^{m\times n}$ is an A-module morphism, for all $a \in A_W$.

Note that A_h could have been well-defined by putting $A_h(a) = \nu(h \circ \hat{a})$, but this definition will not do for our later purposes.

Now let M be a closed submanifold of an open set of $\mathbf{C}^n$, of dimension k. We recall that by [**3**; Ch. VIII, C] there is an open neighborhood W of M and an holomorphic retraction r: $W \to M$. Hence we also have A_r: $A_W \to A_M$, the image of A_r being contained in A_M because $r \circ f \in \mathcal{O}(X, M)$ for all $f \in \mathcal{O}(X, W)$. Of course the image of A_r is exactly A_M, for if $a \in A_M$, then $A_r(a) = \nu(r \circ f)$ where $f \in \mathcal{O}(X, M)$ so $r \circ f = f$, and $A_r(a) = \nu(r \circ f) = \nu(f) = a \in \operatorname{Im} A_r$. Now we obtain our main theorem.

THEOREM 3. *If M is a closed submanifold of an open set of $\mathbf{C}^n$, of dimension k, then A_M is a Banach manifold modeled on projective A-modules of rank k.*

Proof. By Theorem 2, it will clearly be enough to verify that A_r is A-direct at a for all a in a neighborhood of A_M.

Since r is a retraction, $Dr(r(z)) \circ Dr(z) = Dr(z)$ for all $z \in W$. Therefore $\operatorname{Im} Dr(z) \subseteq \operatorname{Im} Dr(r(z))$, but the rank of the matrix $Dr(z)$ is at least that of $Dr(r(z))$ for z near $r(z)$, so that actually $\operatorname{Im} Dr(z) = \operatorname{Im} Dr(r(z))$ for z in an open neighborhood of M. This means that $\dim \operatorname{Im} Dr(z) = k$, and $\dim \operatorname{Ker} Dr(z) = n - k$ near M. $\mathbf{C}^n$ can be written as the direct sum

$$\mathbf{C}^n = \operatorname{Im} Dr(r(z)) \oplus \operatorname{Ker} Dr(r(z)) = \operatorname{Im} Dr(z) \oplus \operatorname{Ker} Dr(r(z)).$$

Because of the continuity of Dr, we may also write $\mathbf{C}^n = \operatorname{Im} Dr(z) \oplus \operatorname{Ker} Dr(z)$, for z near M. Note also that $Dr(r(z))|\operatorname{Im} Dr(r(z))$ is the identity, so that $Dr(z)|\operatorname{Im} Dr(z)$ is an automorphism of $\operatorname{Im} Dr(z)$ near M. We may suppose the neighborhood of M where all this is true to be W;

just discard the old W. For all $z \in W$,

$$\alpha_z = \begin{bmatrix} Dr(z) & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & I \end{bmatrix} : \begin{bmatrix} \operatorname{Im} Dr(z) \\ \\ \operatorname{Ker} Dr(z) \end{bmatrix} \to \begin{bmatrix} \operatorname{Im} Dr(z) \\ \\ \operatorname{Ker} Dr(z) \end{bmatrix},$$

is an automorphism of $\mathbf{C}^n$. Define α: $W \to \mathrm{GL}_n(\mathbf{C})$ by $\alpha(z) =$ the matrix of α_z in the canonical basis of $\mathbf{C}^n$. We will show that α is an holomorphic function. For this, let $z_0 \in W$. There is a neighborhood U of z_0 and there are holomorphic functions v_i: $U \to \mathbf{C}^n$, $1 \leq i \leq n$, such that $v_1(z), \ldots, v_k(z)$ is a basis of $\operatorname{Im} Dr(z)$ and $v_{k+1}(z), \ldots, v_n(z)$ is a basis of $\operatorname{Ker} Dr(z)$ for all $z \in U$. Let $\beta_z \in \mathbf{C}^{k \times k}$ be the matrix of $Dr(z) | \operatorname{Im} Dr(z)$ in the basis $v_1(z), \ldots, v_k(z)$ and let $c(z)$ be the matrix which changes the canonical basis of $\mathbf{C}^n$ to $v_1(z), \ldots, v_n(z)$. Then

$$\alpha(z) = c(z)^{-1} \cdot \begin{bmatrix} \beta_z & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & I \end{bmatrix} \cdot c(z)$$

and it will be enough to verify that β_z is an holomorphic function of z in U, but this follows from the equations

$$Dr(z)(v_i(z))_t = \sum_{s=1}^{k} \beta_{z_{ts}} v_i(z)_s, \qquad i \leq i, t \leq k.$$

We therefore have A_α: $A_W \to A_{\mathrm{GL}_n(\mathbf{C})} = \mathrm{GL}_n(A)$. But

$$A_\alpha(x)|_{\operatorname{Im} DA_r(x)} = DA_r(x)|_{\operatorname{Im} DA_r(x)}$$

for all $x \in A_W$. To see this, let $b = \nu(Dr(g)(h)) \in \operatorname{Im} DA_r(x)$, where $x = \nu(g)$. Now $A_\alpha(x)(b) = \nu(\alpha \circ g) \cdot \nu(Dr(g)(h)) = \nu(\alpha \circ g \cdot Dr(g)(h))$, but for all γ near X,

$$\alpha(g(\gamma))|_{\operatorname{Im} Dr(g(\gamma))} = Dr(g(\gamma))|_{\operatorname{Im} Dr(g(\gamma))},$$

so

$$\begin{aligned} A_\alpha(x)(b) &= \nu(Dr(g) \cdot Dr(g)(h)) \\ &= \nu(Dr(g)) \cdot \nu(Dr(g)(h)) = DA_r(x)(b). \end{aligned}$$

Then

$$DA_r(x)|_{\operatorname{Im} DA_r(x)} \colon \operatorname{Im} DA_r(x) \to \operatorname{Im} DA_r(x) \text{ is an automorphism.}$$

We prove that $A^n = \operatorname{Im} DA_r(x) \oplus \operatorname{Ker} DA_r(x)$ for all $x \in A_W$:

$$0 = \operatorname{Ker}\left(DA_r(x)|_{\operatorname{Im} DA_r(x)}\right) = \operatorname{Im} DA_r(x) \cap \operatorname{Ker} DA_r(x).$$

If $c \in A^n$, there exists $b \in \operatorname{Im} DA_r(x)$ such that $DA_r(x)(b) = DA_r(x)(c)$. Then $c = b + (c - b)$, with $b \in \operatorname{Im} DA_r(x)$ and $c - b \in \operatorname{Ker} DA_r(x)$. $\operatorname{Ker} DA_r(x)$ is closed, so the direct sum is topological.

We now know that Im $DA_r(x)$ is a projective A-module.

We shall see that its rank is k.

First we must prove that for all $x \in A_W$ and $\phi \in X$,

$$\phi^n(Im\ DA_r(x)) = \text{Im } Dr(\phi^n(x))$$

and

$$\phi^n(Ker\ DA_r(x)) = \text{Ker } Dr(\phi^n(x)).$$

Take

$$\begin{aligned} DA_r(x)(b) \in \text{Im } DA_r(x) \cdot \phi^n(DA_r(x)(b)) &= \widehat{\nu(Dr(\hat{x})(\hat{b}))}(\phi) \\ &= (Dr(\hat{x})(\hat{b}))(\phi) = Dr(\phi^n(x))(\phi^n(b)) \in \text{Im } Dr(\phi^n(x)). \end{aligned}$$

Now take $b \in \text{Ker } DA_r(x)$.

$$Dr(\phi^n(x))(\phi^n(b)) = \phi^n(DA_r(x)(b)) = \phi^n(0) = 0,$$

so $\phi^n(b) \in \text{Ker } Dr(\phi^n(x))$, and we have proven both left-to-right inclusions. We have $A^n = \text{Im } DA_r(x) \oplus \text{Ker } DA_r(x)$, and ϕ^n is surjective, so

$$\mathbf{C}^n = \phi^n(\text{Im } DA_r(x)) + \phi^n(\text{Ker } DA_r(x)),$$

but because of the inclusions we have just proven, this sum is direct. Then

$$\begin{aligned} \mathbf{C}^n &= \phi^n(\text{Im } DA_r(x)) \oplus \phi^n(\text{Ker } DA_r(x)) \\ &= \text{Im } Dr(\phi^n(x)) \oplus \text{Ker } Dr(\phi^n(x)), \end{aligned}$$

so the inclusions are actually equalities.

Now let $x \in A_W$, $P = \text{Im } DA_r(x)$, $Q = \text{Ker } DA_r(x)$, and $\phi \in X$. Then $\text{rk}_\phi P = \text{rk}_{A_\phi} P_\phi = \text{rk}_{A_\phi}(A_\phi \otimes_A P)$ is, by Nakayama's Lemma the same as $\dim_{\mathbf{C}}[(A_\phi \otimes_A P) \otimes_{A_\phi} \mathbf{C}]$, when $\mathbf{C}$ (and also $\phi^n(P)$) has the A_ϕ-module structure induced by ϕ. We then have the A_ϕ-module morphism

$$q\colon (A_\phi \otimes_A P) \otimes_{A_\phi} \mathbf{C} \to \phi^n(P);$$

$$q\left(\sum_j \left(\sum_i \frac{a_{ij}}{b_{ij}} \otimes p_{ij}\right) \otimes \lambda_j\right) = \sum_j \sum_i \lambda_j \frac{\phi(a_{ij})}{\phi(b_{ij})} \phi^n(p_{ij}).$$

Let $v_1, \ldots, v_k$ has a basis for $\phi^n(P) = \text{Im } Dr(\phi^n(x))$, and let $b_1, \ldots, b_k \in P$ such that $\phi^n(b_i) = v_i$ for $i = 1, \ldots, k$. Then $(1/1 \otimes b_i) \otimes 1$, $i = 1, \ldots, k$, are $\mathbf{C}$-linearly independent: if $0 = \sum_{i=1}^k \lambda_i (1/1 \otimes b_i) \otimes 1$, then

$$0 = q(0) = \sum_{i=1}^k \lambda_i \phi^n(b_i) = \sum_{i=1}^k \lambda_i v_i$$

and $\lambda_i = 0$ for $i = 1, \ldots, k$.

Therefore $\text{rk}_\phi P = \dim_{\mathbf{C}}[(A_\phi \otimes_A P) \otimes_{A_\phi} \mathbf{C}] \geq k$.

In a similar manner, and since $\phi^n(Q) = \text{Ker } Dr(\phi^n(x))$, $\text{rk}_\phi Q \geq n - k$. But $\text{rk}_\phi P + \text{rk}_\phi Q = n$, so $\text{rk}_\phi P = k \ \forall \phi \in X$. Then $\text{rk } P = k$.

To complete our proof, let $a \in A_M$ and write:

$$DA_r(x) = \begin{bmatrix} P(x) & Q(x) \\ R(x) & S(x) \end{bmatrix} : \begin{bmatrix} \operatorname{Im} DA_r(a) \\ \operatorname{Ker} DA_r(a) \end{bmatrix} \to \begin{bmatrix} \operatorname{Im} DA_r(a) \\ \operatorname{Ker} DA_r(a) \end{bmatrix}.$$

Since $DA_r(a)$ is an indempotent, $DA_r(a)|_{\operatorname{Im} DA_r(a)}$ is the identity, and $P(a) = I$. But $\operatorname{Im} DA_r(a)$ is a Banach space, so by the continuity of P, $P(x)$ is an automorphism of $\operatorname{Im} DA_r(a)$ for all x in a neighborhood U of a.

We have then verified conditions (iv) of Theorem 1 for all $x \in U$. Therefore, A_r is A-direct at x for all x in a neigborhood of A_M.

Observe that the tangent space $T_a(A_M)$ at a is $\operatorname{Im} DA_r(a)$. These are of course projective A-modules of rank k, but they need not be isomorphic on different connected components of A_M. In fact, some of these modules may be free while others may not.

Now consider for any Banach algebra A, the category $\underline{M(A)}$ whose objects are analytic manifolds modeled on projective A-modules, with morphisms holomorphic functions whose differentials are A-module morphism, and the ordinary composition. Let $\underline{M}$ be the category of closed analytic submanifolds of open subsets of finite products of $\mathbf{C}$. Then we have:

PROPOSITION 3.3. *$A_{(\cdot)}$ is a covariant functor from $\underline{M}$ to $\underline{M(A)}$.*

Proof. A_M is defined for every object in $\underline{M}$ and is an object of $\underline{M(A)}$, by Theorem 3. Now let M and N be two objects of $\underline{M}$ and $h: M \to N$ an holomorphic function. h can be extended to an open neighborhood W of M for example by $h \circ r$. If $\bar{h}$ is such an extension, then we can define $A_{\bar{h}}$ as before Lemma 3.2. Now define A_h to be the restriction of $A_{\bar{h}}$ to A_M, for any extension $\bar{h}$ of h. Obviously, $\operatorname{Im} A_h = A_{\bar{h}}(A_M) \subseteq A_N$, and if h_1 and h_2 are two extensions of h, and $a \in A_M$, $a = \nu(f)$ with $f \in \mathcal{O}(X, M)$, then

$$A_{h_1}(a) = \nu(h_1 \circ f) = \nu(h \circ f) = \nu(h_2 \circ f) = A_{h_2}(a),$$

so A_h is well defined. The rest of the Proposition is easily verified.

There are many holomorphic functions in A^n whose differentials are A-module morphisms, but which are not of the form A_h for any h. As an example, take $a \in A$ such that there are $x \in A$, and ϕ, $\psi \in X$ with $\phi(x) = \psi(x) \neq 0$ and $\phi(a) \neq \psi(a)$; and consider $L_a: A \to A$ defined by $L_a(y) = ay$. L_a is A-linear, but $L_a \neq A_h$ for all h: if L_a were A_h, $ax = L_a(x) = A_h(x) = \nu(h \circ \hat{x})$, so over X, $\hat{a}\hat{x} = h \circ \hat{x}$, and then

$$\phi(a) \cdot \phi(x) = h(\phi(x)) = h(\psi(x)) = \psi(a)\psi(x).$$

Hence, $\phi(a) = \psi(a)$, contrary to our assumptions.

Finally, we wish to compare A_M and A^M.

PROPOSITION 3.4. $A^M = A_M + \mathrm{Rad}(A)^n$.

Proof. Let $\mathcal{N} = \{f \in \mathcal{O}(X, \mathbf{C}): f|_X = 0\}$. Then $\nu(\mathcal{N}) = \mathrm{Rad}(A)$: if $f \in \mathcal{N}$, $\widehat{\nu(f)}|_X = f|_X = 0$, so $\nu(\mathcal{N}) \subseteq \mathrm{Rad}(A)$; on the other hand, if $a \in \mathrm{Rad}(A)$, $a = \nu(\hat{a})$ with $\hat{a}|X = 0$. We identify also $\mathrm{Rad}(A)^n$ with $\nu(\mathcal{N}^n)$. Note that $A^M \subseteq A_W$, for if $\hat{a}(X) = \mathrm{sp}(a) \subseteq M$, then $\hat{a} \in \mathcal{O}(X, W)$. Now take $a \in A^M$, and put $a = A_r(a) + (a - A_r(a))$. $A_r(a) \in A_M$, and

$$a - A_r(a) = \nu(\hat{a}) - \nu(r \circ \hat{a}) = \nu(\hat{a} - r \circ \hat{a}) \in \mathrm{Rad}(A)^n,$$

because $\hat{a} - r \circ \hat{a} \in \mathcal{N}^n$. For the other inclusion, let $b \in A_M$ and $c \in \mathrm{Rad}(A)^n$. $c = \nu(g)$, with $g \in \mathcal{N}^n$. Then

$$\begin{aligned} \mathrm{sp}(b + c) &= \widehat{b + c}(X) = \big(\hat{b} + \widehat{\nu(g)}\big)(X) \\ &= (\hat{b} + g)(X) = \hat{b}(X) = \mathrm{sp}(b) \subseteq M. \end{aligned}$$

COROLLARY 3.5. *A^M and A_M have the same homotopy type. If A is semisimple, then $A^M = A_M$. (See also* [7; 2.8].)

Proof. Let $\iota: A_M \to A^M$ denote the inclusion. $A_r \circ \iota$ is the identity on A_M and it is easily verified that $\iota \circ A_r$ is homotopic to the identity on A^M.

4. An example. We wish to consider briefly an example of a spectral set. Suppose A is semisimple, and the manifold M is given as the zero set of a holomorphic function

$$W \xrightarrow{F} \mathbf{C}^k.$$

It has been established in the last paragraph that A_M is a Banach manifold. This would have been a much simpler matter in this particular case, but a bit more can be said. Lift F to an analytic function

$$A_W \xrightarrow{A_F} A^k$$

and the zero set of A_F is exactly A_M. To see this, let $a \in A_M$; then $a = \nu(f)$ with $f \in \mathcal{O}(X, M)$, and $A_F(a) = \nu(F \circ f) = \nu(0) = 0$, so $a \in A_F^{-1}(0)$. Now if $A_F(a) = 0$, $\nu(F \circ \hat{a}) = 0$ and $F \circ \hat{a} = 0$ over X. Hence $F(\mathrm{sp}(a)) = \{0\}$, and $\mathrm{sp}(a) \subset M$. We then have $A_M \subset A_F^{-1}(0) \subset A^M$, but since A is semisimple, all three are the same.

Now take $W = \mathrm{GL}_n(\mathbf{C})$, and G a Lie subgroup of W which is the zero set of analytic functions, for instance an algebraic group. Then the corresponding zero set of the same functions in $\mathrm{GL}_n(A)$ is a Lie subgroup of $\mathrm{GL}_n(A)$.

It can in fact be shown that all Lie groups give rise to Banach Lie groups, and that these have tangent spaces which are free A-modules.

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Received March 8, 1984.

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