PACIFIC JOURNAL OF MATHEMATICS
Vol. 120, No. 2, 1985

ON BANACH SPACES HAVING A RADON-NIKODYM DUAL

C. DEBIEVE

The purpose of this paper is to prove a new characterisation of Banach spaces having a Radon-Nikodym dual, namely that if E is a Banach space, then E' has the Radon-Nikodym property if and only if there exists an equivalent norm on E such that for each E-valued measure m of bounded variation, there exists an E'-valued function f with norm 1 $|m|$-a.e. such that $|m| =$ f.m.

1. Introduction. In [**1**], we have proved that if E is a Banach space, m an E-valued measure defined on a σ-algebra $\mathscr{A}$ of subsets of a set T, with bounded variation $|m|$, and if ε is any positive number, then there exists an E'-valued strongly measurable function f defined on the set T, such that $\|f\| < 1 + \varepsilon$ and $|m|(A) = \int_A f\, dm$ for each A in $\mathscr{A}$.

A very natural question which arises is the following: Does there always exist an E'-valued strongly measurable function with norm 1 such that $|m|(A) = \int_A f\, dm$ for each A in $\mathscr{A}$? Following the example given in [**1**], this seems to be possible.

Finally, an answer to that question was provided by F. Delbaen who proved the following unpublished theorem: If E is a Banach space, the following are equivalent:

(a) E' has the Radon-Nikodym property

(b) For each equivalent norm on E, for each E-valued measure m of bounded variation defined on a σ-algebra $\mathscr{A}$ of subsets of a set T, there exists a $|m|$-strongly measurable function f from T to E' such that $\|f\| = 1$ $|m|$-a.e. and $|m|(A) = \int_A f\, dm$ for each A in $\mathscr{A}$.

The purpose of this paper is to provide a positive answer to the following question: Is it possible to weaken assertion (b) by requiring the existence of an equivalent norm on the space having the property instead of assuming it for each equivalent norm on E.

2. Proof of the theorem. Before proving our theorem let us recall the Mazur density theorem and prove two lemmas.

THEOREM (*Mazur density theorem* [5] *p.* 171). *If E is a separable Banach space, then for each equivalent norm on E, the set of smooth points of the unit sphere of E is dense in the unit sphere.*

LEMMA 1. *Let E be a Banach space such that E' is not separable, B a dense subset of $S(E) = \{x | x \in E, \|x\| = 1\}$ and $\varepsilon > 0$. If we denote by Ω the first uncountable ordinal and by S the set $\{i | i < \Omega\}$, then for each i in S, there exists x_i in B and x_i' in $S(E')$, the unit sphere of E' such that $x_i'(x_i) = 1$ and $\|x_i' - x_j'\| > 1 - \varepsilon$ if $i \neq j$.*

Proof. Let i in S and suppose that the families (x_j) and (x_j') are chosen for $j < i$.

As E' is not separable, $\bigcap_{j<i} \operatorname{Ker} x_j' \neq \{0\}$.

Let $x \in S(E) \cap \bigcap_{j<i} \operatorname{Ker} x_j'$ and choose x_i in B such that $\|x - x_i\| < \varepsilon$.

Now, if we choose x_i' in $S(E')$ such that $x_i'(x_i) = 1$ it is easy to see that we are done.

$\|x_i' - x_j'\| > 1 - \varepsilon$ follows from the fact that if $j < i$, $(x_i' - x_j')(x_i) > 1 - \varepsilon$.

LEMMA 2. *For the same set S as in Lemma 2, there exists a positive scalar measure μ on the σ-algebra $\mathscr{P}(S)$ of the subsets of S such that $\mu(S) = 1$ and $\mu(A) = 0$ if A is countable.*

Proof. Let i in S and define μ_i as the evaluation measure at the point i. As the set of measures on the σ-algebra of the subsets of S is the dual of the space of continuous bounded functions on S for a locally convex topology, the family of measures has a cluster point which is a measure satisfying our requirement.

We are now ready for the proof of the following

THEOREM. *For any Banach space E, the following are equivalent:*

(1) *E' has the Radon-Nikodym property.*

(2) *For each equivalent norm on E, for each E-valued measure m of bounded variation defined on a σ-algebra $\mathscr{A}$ of subsets of a set T, there exists a function f from T into E' $|m|$-strongly measurable such that $\|f(t)\| = 1$ $|m|$-a.e. and $|m|(A) = \int_A f\,dm$ for each A in $\mathscr{A}$.*

(3) *There exists an equivalent norm on E such that for each E-valued measure m of bounded variation defined on a σ-algebra $\mathscr{A}$ of subsets of a set T, there exists a function f from T into E' $|m|$-strongly measurable such that $\|f(t)\| = 1$ $|m|$-a.e. and $|m|(A) = \int_A f\,dm$ for each A in $\mathscr{A}$.*

Proof. (1) $\Rightarrow$ (2) It follows from the theorem we proved in [**1**] that for each integer n, there exists a function f_n from T into E' such that f_n is $|m|$-strongly measurable, $1 \leq \|f_n(t)\| < 1 + 1/n$ and $|m|(A) = \int_A f_n \, dm$ for each A in $\mathscr{A}$.

Let G be the Banach subspace of E' generated by $\bigcup_{n=1}^{\infty} f_n(T)$.

As G is separable and E' has Radon-Nikodym property, there exists a Banach space F such that F' is separable and $G \subseteq F'$ ([**3**]). Let f be a pointwise $\sigma(F', F)$-cluster point of the sequence (f_n). f is G-valued, thus E'-valued.

It is clear that $\|f\| \leq 1$ and that f is $\sigma(F', F)$-measurable. As $|m|(A) = \int_A f \, dm$ for each A in $\mathscr{A}$, if we prove that f is strongly measurable, the norm of f will be greater than 1 and our assertion will be proved.

Let m_0 from $\mathscr{A}$ into F' defined by $m_0(A)(y) = \int_A \langle f, y \rangle \, d|m|$.

It is clear that m_0 is a measure with finite variation and that $|m_0| = |m|$.

As F has the Radon-Nikodym property, there exists a measurable function g from T into F' such that $m_0(A) = \int_A g \, d|m|$ for each A in $\mathscr{A}$.

It follows that if $y \in F$, $m_0(A)(y) = \int_A \langle g, y \rangle \, d|m|$ which shows that $\langle g, y \rangle = \langle f, y \rangle$, $|m|$-a.e. for each y in F.

As F is separable, it follows that $f = g$ $|m|$-a.e. and that f is strongly measurable which proves the first assertion.

As (2) $\Rightarrow$ (3) is obvious, it remains to show that

(3) $\Rightarrow$ (1) It is easy to prove that if property (3) is satisfied for E it is also satisfied for each Banach subspace of E. Now as we have to prove that each separable subspace of E has a separable dual, we only have to prove that if a separable Banach space satisfies (3), it has a separable dual.

Let us suppose that there exists a separable Banach space E satisfying property (3) and such that E' is not separable. Let B be the set of smooth points of the unit sphere $S(E)$ of E which is dense in $S(E)$ by Mazur density theorem, $\varepsilon = 1/4$ and apply Lemma 1.

We define the function f from S to E by $f(i) = x_i$. If $\mathscr{A}$ is defined as the set of inverse images by f of the open subsets of $S(E)$, the function f is strongly measurable. Let us choose on $\mathscr{A}$ a positive scalar measure μ such that $\mu(S) = 1$ and $\mu(A) = 0$ if A is countable. Such a μ exists by Lemma 2. Now we define m from $\mathscr{A}$ to E by $m(A) = \int_A f \, d\mu$.

m is clearly a measure of bounded variation and $|m| = \mu$. So there exists a function g from S into E' which is μ-strongly measurable, $\|g\| = 1$ μ-a.e. and $\mu(A) = \int_A g \, dm$ for each A in $\mathscr{A}$.

It follows that $\mu(A) = \int_A \langle f, g \rangle \, d\mu$ for each A in $\mathscr{A}$ and that $\langle f, g \rangle = 1$ μ-a.e.

So there exists a μ-negligible subset N of S such that $g(i)(f(i)) = 1$ if $i \notin N$ and $g(S \setminus N)$ is separable If $i \notin N$, $g(i)(f(i)) = g(i)(x_i) = 1$.

As x_i is a smooth point and $\|g(i)\| = 1$, $g(i) = x_i$,

It follows that $\|g(i) - g(j)\| \geq 1 - \varepsilon = 3/4$ for $i \neq j$ in $S \setminus N$ which shows that $g(S \setminus N)$ is discrete.

As it is separable, it has to be countable. So $S \setminus N$ has to be countable which is impossible.

Acknowledgment. The author would like to express his deep gratitude to Professor D. H. Tucker for many helpful conversastions during the preparation of this paper while the author was in the mathematical department of the University of Utah, Salt Lake City, U.S.A.

References

[1] C. Debieve, *On a Radon-Nikodym problem for vector-valued measures*, Pacific J. Math. 2, **107** (1983), 335–339.

[2] F. Delbaen, *Private communicationo*.

[3] J. Diestel and J. J. Uhl, *Vector measures*, Amer. Math. Soc. Mathematical Surveys no. 15.

[4] N. Dinculeanu, *Vector Measures*, Pergamon Press, New York.

[5] Holmes, *Geometrical Functional Analysis and its Applications*, Graduate Texts in Mathematics no. 24, Springer-Verlag.

Received March 10, 1984.

INSTITUT DE MATHÉMATIQUES
PURES ET APPLIQUÉES
2, CHEMIN DU CYCLOTRON
B 1348 LOUVAIN-LA-NEUVE
BELGIUM