PACIFIC JOURNAL OF MATHEMATICS
Vol. 120, No. 2, 1985

# SEMIPRIME ℵ-QF 3 RINGS

GIUSEPPE BACCELLA

**A ring $R$ (associative with identity) is called *right* ℵ-QF 3 if it has a faithful right ideal which is a direct sum of a family of injective envelopes of pairwise non-isomorphic simple right $R$-modules. A right QF 3 ring is just a right ℵ-QF 3 ring where the above family is finite. The aim of the present work is to give a structure theorem for semiprime ℵ-QF 3 rings. It is proved, among others, that the following conditions are equivalent for a given ring $R$: (a) $R$ is a semiprime right ℵ-QF 3 ring, (b) there is a ring $Q$, which is a direct product of right full linear rings, such that $\operatorname{Soc} Q \subset R \subset Q$, (c) $R$ is right nonsingular and every non-singular right $R$-module is cogenerated by simple and projective modules.**

A ring $R$ is called a *right* QF 3 ring if there is a minimal faithful module $U_R$, in the sense that every faithful right $R$-module contains a direct summand which is isomorphic to $U$; one proves that if there exists such a module $U$, then it is unique up to an isomorphism. It was proved by Colby and Rutter [**5**, Theorem 1] that $R$ is right QF 3 if and only if it contains a faithful right ideal of the form $E(S_1) \oplus \cdots \oplus E(S_n)$, where each $E(S_i)$ is the injective envelope of a simple module $S_i$, and the $S_i$'s are pairwise non-isomorphic. Following Kawada [**10**], we say that $R$ is a *right* ℵ-QF 3 ring if there is a family $(e_\lambda)_{\lambda\in\Lambda}$ of pairwise orthogonal and pairwise non isomorphic (in the sense that $e_\lambda R \neq e_\mu R$ whenever $\lambda \neq \mu$) idempotents of $R$ such that: (a) each $e_\lambda R$ is the injective envelope of a minimal right ideal, (b) the right ideal $W_R = \sum_{\lambda\in\Lambda} e_\lambda R$ is faithful; here ℵ stands for the cardinality of the set $\Lambda$. It is clear from Colby and Rutter's result that a right QF 3 ring is nothing other than a right ℵ-QF 3 ring where ℵ is a finite cardinal. By a ℵ-QF 3 ring we shall mean a ring which is both right and left ℵ-QF 3; similarly for QF 3 rings.

In [**4**] we studied those right ℵ-QF 3 rings which have zero right singular ideal. Our purpose in the present paper is to characterize the semiprime right ℵ-QF 3 rings. Our main result is that the following conditions are equivalent for a given ring $R$: (a) $R$ is a semiprime right ℵ-QF 3 ring, (b) $R$ is a semiprime ring with essential socle and every simple projective right $R$-module is injective. (c) $R$ is right nonsingular and every nonsingular right $R$-module is cogenerated by simple projective modules, (d) $R$ is (isomorphic to) a subring of a direct product $\prod_{\lambda\in\Lambda} Q_\lambda$ of right full linear rings and $\bigoplus_{\lambda\in\Lambda} \operatorname{Soc} Q_\lambda \subset R$. As a consequence we obtain that $R$ is a semiprime ℵ-QF 3 ring if and only if it satisfies one (and hence all) of the following conditions: (a) $R$ is a subring of the direct product of a family $(Q_\lambda)_{\lambda \in \Lambda}$ of simple artinian rings and contains the direct sum $\oplus_{\lambda \in \Lambda} Q_\lambda$, (b) $R$ is right nonsingular and every nonsingular injective right $R$-module is a direct product of pairwise independent semisimple and homogeneous modules (we say that two semisimple right $R$-modules $L$, $M$ are *independent* if $\mathrm{Hom}_R(L, M) = 0$, i.e. if $L$ does not contain a simple submodule which is isomorphic to some submodule of $M$).

Throughout, all rings will be associative with identity, all modules will be unitary and all maps between modules will be module homomorphisms. For a given ring $R$, we shall denote with Mod-$R$ the category of all right $R$-modules. If $M$ is a given right $R$-module, we shall denote with $E(M)$, $Z(M)$, $J(M)$ and $\mathrm{Soc}\, M$ resp. the injective envelope, the singular submodule, the Jacobsen radical and the socle of $M$; if $\mathscr{A}$ is a set of pairwise non-isomorphic simple right $R$-modules, then $\mathrm{Soc}_{\mathscr{A}}(M)$ will denote the $\mathscr{A}$-homogeneous component of $\mathrm{Soc}\, M$ (we shall write $\mathrm{Soc}_P(M)$ in case $\mathscr{A} = \{P\}$); the notation $N \leq M_R$ (resp. $N \trianglelefteq M_R$) will mean that $N$ is an $R$-submodule (resp. an essential $R$-submodule) of $M$. Given a subset $X \subset M$, $r_R(X)$ will be the right annihilator of $X$ in $R$; similarly, if $M$ is a left $R$-module, then $l_R(X)$ will be the left annihilator of $X$ in $R$. We assume the reader familiar with elementary facts about torsion theories, in particular the Goldie torsion theory (see e.g. [**6**] and [**12**]).

We proceed to give first several preliminary results concerning the projective components of the socle of a ring; these results are mainly based on the following one, which was proved in [**2**, Proposition 1.4 and Corollary 1.5].

PROPOSITION 1. *Let $R$ be a given ring, let $\mathscr{P}$ be a set of representatives of the simple projective right $R$-modules and let $K$ be a two-sided ideal contained in* $\mathrm{Soc}\, R_R$. *Then the following conditions are equivalent*:

(1) $K^2 = K$.

(2) ${}_R(R/K)$ *is flat*.

(3) *There is a subset $\mathscr{A} \subset \mathscr{P}$ such that* $K = \mathrm{Soc}_{\mathscr{A}}(R_R)$.

*If these conditions hold, then for each module $M_R$ we have* $\mathrm{Soc}_{\mathscr{A}}(M) = MK$. □

By a *right full linear ring* we mean a ring which is isomorphic to the endomorphism ring of a right vector space over some division ring. It is well known that $R$ is a right full linear ring if and only if $R$ is a prime von Neumann regular right self-injective ring with essential socle (see [**12**, Ch. XII, Corollary 1.5, page 246]); if it is the case, then $R$ is a right QF 3 ring (see Tachikawa [**13**, page 43, 44]). The following proposition tells us that prime right QF 3 rings can be characterized as special subrings of right full linear rings (see however [**13**, Proposition 4.3]). We need a lemma.

LEMMA 2. *Let $P$ be a minimal right ideal of the ring $R$ and let $e$ be an idempotent such that $P \trianglelefteq eR_R$. Then either $eR = P$ or $P^2 = 0$.*

*Proof.* If $P^2 \neq 0$, then, by the modular law, $P$ is a direct summand of $eR$ and hence equals $eR$. □

PROPOSITION 3. *Given a ring $R$, the following conditions are equivalent*:

(1) *$R$ has a simple injective, projective and faithful right module.*

(2) *$R$ is a prime right* QF 3 *ring.*

(3) *$R$ is a subring of a right full linear ring $Q$ and* $\operatorname{Soc} Q \subset R$.

*Proof.* (1) ⇒ (2) is clear from [**5**, Theorem 1].

(2) ⇒ (3). It follows from (2) that $R$ has a nonzero homogeneous projective essential socle $S$. Moreover, since $R$ is right QF 3, there is an idempotent $e \in R$ such that $eR_R$ is faithful, injective with a simple essential socle $P$. Inasmuch as $P$ is prime, then $P^2 = P$ and hence $P = eR$ by Lemma 2, so all minimal right ideals of $R$ are injective. Let $Q$ be the maximal right quotient ring of $R$. It is well known that $Q \cong \operatorname{End} S_R \cong E(R_R)$ and $Q$ is a right full linear ring (see e.g. [**12**, page 249]). Now if $N$ is a minimal right ideal of $R$, then, by the above, $R \supset N = E(N_R) = NQ$. The latter equality tells us that $\operatorname{Soc} Q_Q = SQ \subset R$.

(3) ⇒ (1). Suppose that $\operatorname{Soc} Q_Q \subset R \subset Q$, where $Q$ is a right full linear ring. Then $R$ is right primitive, $\operatorname{Soc} R = \operatorname{Soc} Q_Q$ and $Q$ is the maximal right quotient ring of $R$. If $N$ is a minimal right ideal of $R$, then $N_R$ is faithful, projective, and, as in the proof of the implication (2) ⇒ (3), $E(N_R) = NQ$, therefore $N$ is essential in $NQ_R$. Since the latter is semi-simple, it follows that $N = NQ$ and hence $N_R$ is injective. □

COROLLARY 4. *A ring $R$ is a prime* QF 3 *ring if and only if $R$ is simple artinian.*

*Proof.* The "if" part is obvious. Assume that $R$ is prime and QF 3. Then $R$ has both a right and a left simple injective, projective and faithful module by Proposition 3. It follows from Jans [**9**, corollary 2.2] that $R$ is simple artinian. □

In what follows we fix a simple projective right $R$-module $P$ and we set $L = l_R(\mathrm{Soc}_P(R_R))$. Then, in view of Proposition 1, we have $\mathrm{Soc}_P(R_R) \cdot L \subset \mathrm{Soc}_P(R_R) \cap L = L \cdot \mathrm{Soc}_P(R_R) = 0$, so that $P$ may be regarded as a simple right $R/L$-module. The proof of the following lemma is left to the reader.

LEMMA 5. *With the above notations, $R/L$ is a right nonsingular ring with essential and homogeneous right socle; to be precise, the canonical map $R \to R/L$ induces an isomorphism* $\mathrm{Soc}_P(R_R) \cong \mathrm{Soc}(R/L)_{R/L}$. □

LEMMA 6. *With the above notations, the following conditions are equivalent*:

(1) *$P_R$ is injective.*

(2) *$P_{R/L}$ is injective.*

(3) *$R/L$ is a prime right* QF 3 *ring.*

*If any of the above conditions holds, then $L = r_R(P)$.*

*Proof.* (1) ⇒ (2) is obvious.

(2) ⇒ (3). It follows from Lemma 5 that $\mathrm{Soc}(R/L)_{R/L}$ is homogeneous and essential in $R/L$ and, since $P_{R/L}$ is injective, we have $J(R/L) = 0$. Thus $R/L$ is primitive and $P_{R/L}$ is a simple faithful, injective and projective module, therefore $R/L$ is right QF 3 by Proposition 3.

(3) ⇒ (1). If $R/L$ is a prime right QF 3 ring, then, again by Proposition 3, $R/L$ is primitive with $P$ as a simple faithful injective and projective right $R/L$-module. This implies $J(R) \subset L$ and, taking Proposition 1 into account, we get $J(R) \cap \mathrm{Soc}_P(R_R) = J(R)\mathrm{Soc}_P(R_R) = 0$. We may now apply [**3**, Theorem 1.3, equivalence of conditions (1) and (8)] and we infer that $E((\mathrm{Soc}_P(R_R))_{R/L})$ is $R$-injective. From that, since $P_{R/L}$ is injective, we conclude that $P_R$ is injective.

Finally, the arguments in the proof of the last implication together with [**3**, Theorem 1.3], show the last part of our lemma. □

If $P_R$ is injective, then $J(R) \cap \mathrm{Soc}_P(R_R) = 0$ and [**3**, Theorem 1.3] implies that $\mathrm{Soc}_P(R_R) = \mathrm{Soc}_{P'}({}_RR)$, where $P'$ is some simple projective left $R$-module (to be precise, $P' = \mathrm{Hom}_R(P, R)$); moreover $L = r_R(P) = l_R(P')$. The condition that ${}_RP'$ also is injective is very sharp, as it is shown by the following corollary.

COROLLARY 7. *With the above notations, the following conditions are equivalent*:

(1) *$P_R$ and its dual ${}_RP' = \mathrm{Hom}_R(P, R)$ are injective.*

(2) *$R/L$ is a simple artinian ring.*

(3) $\mathrm{Soc}_P(R_R) = eR$ *for a central idempotent $e \in R$.*

*Proof*. (1) ⇒ (2). As we observed before, the injectivity of $P_R$ implies that $L = r_R(P) = l_R(P')$. Thus, according to Lemma 6, (1) implies that $R/L$ is a prime QF 3 ring; hence $R/L$ is simple artinian by Corollary 4.

(2) ⇒ (3). If (2) holds, then $J(R) \subset L$ and hence $J(R) \cap \mathrm{Soc}_P(R_R) = 0$. According to the above remarks, there is a simple projective left $R$-module $P'$ such that $\mathrm{Soc}_P(R_R) = \mathrm{Soc}_{P'}({}_RR)$. Taking Lemma 5 into account, we see that $R/L \cong \mathrm{Soc}_P(R_R) = \mathrm{Soc}_{P'}({}_RR)$, therefore $R/L$ is projective both as a right and a left $R$-module. We conclude that $L = fR$ for a central idempotent $f \in R$ and (3) holds with $e = 1 - f$.

(3) ⇒ (1) is a consequence of [**2**, Theorem 2.7]. □

Recall that the ring $R$ is *semiprime* if it has no non-zero nilpotent right (and hence left) ideals. Without any hypothesis on $R$, if $N$ is a minimal right ideal of $R$, then either $N^2 = 0$ or $N = eR$ for some idempotent $e \in R$. Thus, if $R$ is semiprime, it follows from Proposition 1 that $\mathrm{Soc}\, R_R = \mathrm{Soc}_{\mathscr{P}}(R_R)$ and every two-sided ideal contained in $\mathrm{Soc}\, R_R$ is of the form $\mathrm{Soc}_{\mathscr{A}}(R_R)$ for some subset $\mathscr{A} \subset \mathscr{P}$; moreover, it was proved by Jacobson (see [**8**, Ch. IV, n. 3, Theorem 1, page 65]) that every homogeneous component of $\mathrm{Soc}\, R_R$ is also a homogeneous component of $\mathrm{Soc}_R R$ and conversely, so that $\mathrm{Soc}\, R_R = \mathrm{Soc}_R R$.

LEMMA 8. *Let $Q$ be a ring with essential and projective right socle $S$ and let $R$ be a subring of $Q$ containing $S$. Then the following are true*:

(1) $S = \mathrm{Soc}\, R_R = \mathrm{Soc}\, Q_R$.

(2) $S_R$ *is projective.*

(3) $S \trianglelefteq R_R \trianglelefteq Q_R$.

*Moreover, if $Q$ is semiprime, then $R$ is semiprime as well.*

*Proof*. Let $U$ be a minimal right ideal of $Q$ and let $0 \neq x \in U$. Taking Proposition 1 into account we have $U = xQ = xS \subset xR \subset U$, hence $xR = U$. This shows that $S \subset \mathrm{Soc}\, R_R$. Since $S \trianglelefteq Q_Q$, then $xS \neq 0$ for each non-zero $x \in Q$ and therefore $S \trianglelefteq R_R$. We infer that $S = \mathrm{Soc}\, R_R$ and $S_R$ is projective since $S^2 = S$. Moreover $S \trianglelefteq Q_R$, so $S = \mathrm{Soc}\, Q_R$. If $Q$ is semiprime, then every minimal right ideal of $Q$ is generated by an idempotent. This fact, together with $S \trianglelefteq R_R$, implies easily that $R$ is semiprime. □

Following L. Levy [**11**], we say that the ring $R$ is an *irredundant subdirect product* of a family $(R_\lambda)_{\lambda \in \Lambda}$ of rings if:

(a) $R$ is a subdirect product of the $R_\lambda$'s,

(b) canonically identifying $R$ with a subring and each $R_\lambda$ with a two-sided ideal of $\prod_{\lambda \in \Lambda} R_\lambda$, we have $R \cap R_\lambda \neq 0$.

LEMMA 9. *Given a ring $R$, the following conditions are equivalent*:

(1) *$R$ is semiprime with essential socle.*

(2) *$R$ is an irredundant subdirect product of a family $(R_\lambda)_{\lambda\in\Lambda}$ of prime rings each with a non-zero socle $S_\lambda$.*

(3) *$R$ is a subdirect product of a family $(R_\lambda)_{\lambda\in\Lambda}$ of prime rings, each with a non-zero socle $S_\lambda$, and, canonically identifying $R$ with a subring and each $R_\lambda$ with a two-sided ideal of $\prod_{\lambda\in\Lambda} R_\lambda$, the equality $\operatorname{Soc} R_R = \bigoplus_{\lambda\in\Lambda} S_\lambda$ holds.*

*Proof.* (1) $\Rightarrow$ (3). Inasmuch as $R$ is semiprime, $\operatorname{Soc} R_R$ is projective. Let $(P_\lambda)_{\lambda\in\Lambda}$ be a family of representatives of all simple projective right $R$-modules and, for each $\lambda \in \Lambda$, let us write $L_\lambda = r_R(P_\lambda)$ and $R_\lambda = R/L_\lambda$. It follows from [**3**, Theorem 1.3] that $L_\lambda = l_R(\operatorname{Soc}_{P_\lambda}(R_R))$, hence $R$ is a subdirect product of the family $(R_\lambda)_{\lambda\in\Lambda}$ by Gordon [**7**, Theorem 2.3]; moreover each $R_\lambda$ has essential right socle $S_\lambda$ and is prime by the above. Let us identify $R$ with a subring and each $R_\lambda$ with a two-sided ideal of the ring $\prod_{\lambda\in\Lambda} R_\lambda$ and let $p_\lambda\colon R \to R_\lambda$ be the canonical projection. Then $\operatorname{Soc}_{P_\lambda}(R_R)$ is canonically identified with $S_\lambda$ via $p_\lambda$ (see Lemma 5). It follows that $S_\lambda \subset R \cap R_\lambda$ and hence $\operatorname{Soc} R = \bigoplus_{\lambda\in\Lambda} S_\lambda$.

(3) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Let us write $Q = \prod_{\lambda\in\Lambda} R_\lambda$. We may again assume that $R$ is a subring and each $R_\lambda$ is a two-sided ideal of $Q$. For each $\lambda \in \Lambda$, since $R_\lambda$ is prime, every non-zero two-sided ideal of $R_\lambda$ is essential, thus $S_\lambda$ is a minimal two-sided ideal of $R$; since $R \cap R_\lambda \neq 0$, then $S_\lambda \subset R \cap R_\lambda \subset R$, whence $\operatorname{Soc} Q_Q = \bigoplus_{\lambda\in\Lambda} S_\lambda \subset R$. Inasmuch as $Q$ is semiprime, it follows from Lemma 8 that $R$ is semiprime with essential socle. □

We are now in position to state and prove our structure theorem on semiprime $\aleph$-QF 3 rings. Recall that $R$ is a *right* QF 3′ ring if $E(R_R)$ is torsionless. A torsion theory $(\mathscr{T}, \mathscr{F})$ is *jansian* (or "TTF") if $\mathscr{T}$ is closed by direct products; this happens if and only if there is an idempotent two-sided ideal $I$ of $R$ such that $\mathscr{T} = \{L_R \mid LI = 0\}$.

THEOREM 10. *Let $R$ be a given ring, let $(P_\lambda)_{\lambda\in\Lambda}$ be a family of representatives of all simple projective right $R$-modules and let $\aleph$ be a non-zero cardinal number. Then the following conditions are equivalent*:

(1) *$R$ is a semiprime right $\aleph$-QF 3 ring.*

(2) *$R$ is a semiprime QF 3′ ring with essential socle and $\operatorname{Card}(\Lambda) = \aleph$.*

(3) *$R$ is a right $\aleph$-QF 3 ring without nilpotent minimal right ideals.*

(4) *$R$ is a semiprime ring with essential socle, every simple projective right $R$-module is injective and $\operatorname{Card}(\Lambda) = \aleph$.*

(5) *$R$ is an irredundant subdirect product of a family $(R_\lambda)_{\lambda\in\Lambda}$ of prime right* QF 3 *rings and* $\mathrm{Card}(\Lambda) = \aleph$.

(6) *$R$ is (isomorphic to) a subring of the direct product of a family $(Q_\lambda)_{\lambda\in\Lambda}$ of right full linear rings, with* $\mathrm{Card}(\Lambda) = \aleph$, *and* $\oplus_{\lambda\in\Lambda} \mathrm{Soc}\, Q_\lambda \subset R$.

(7) *$R$ is right nonsingular,* $\mathrm{Card}(\Lambda) = \aleph$ *and every nonsingular right $R$-module is cogenerated by simple projective modules.*

(8) $\mathrm{Card}(\Lambda) = \aleph$ *and a module $M_R$ is singular if and only if* $\mathrm{Hom}_R(M, P_\lambda) = 0$ *for each* $\lambda \in \Lambda$.

*Proof.* (1) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). Assume that (3) holds. By the definition of a right $\aleph$-QF 3 ring and taking [**4**, Proposition 2.3] into account, we may assume that each $P_\lambda$ is a minimal right ideal and there is a family $(e_\lambda)_{\lambda\in\Lambda}$ of idempotents of $R$, with $W_R = \sum_{\lambda\in\Lambda} e_\lambda R$ faithful, such that $e_\lambda R = E(P_\lambda)$ for each $\lambda \in \Lambda$. Our assumption, together with Lemma 2, implies that $e_\lambda R = P_\lambda$ for each $\lambda \in \Lambda$, so that every simple projective right $R$-module is injective. Moreover $e_\lambda R \cap J(R) = P_\lambda \cap J(R) = 0$, hence $e_\lambda J(R) = 0$ for each $\lambda \in \Lambda$. We infer that $WJ(R) = 0$ and then $J(R) = 0$, being $W_R$ faithful. Thus $R$ is semiprime and has essential socle by [**4**, Theorem 2.4].

(4) $\Rightarrow$ (5). It follows from Lemma 9 that $R$ is an irredundant subdirect product of the family $(R_\lambda)_{\lambda\in\Lambda}$, where $R_\lambda = R/l_R(\mathrm{Soc}_{P_\lambda}(R_R))$ for each $\lambda \in \Lambda$. Moreover every $R_\lambda$ is a prime right QF 3 ring by Lemma 6.

(5) $\Rightarrow$ (6). Suppose that (5) holds. It follows then from Lemma 8 and 9 that $R$ is a semiprime ring with essential socle and $\mathrm{Soc}\, R = \oplus_{\lambda\in\Lambda} \mathrm{Soc}\, R_\lambda$. Now Proposition 3 tells us that each $R_\lambda$ is (isomorphic to) a subring of a right full linear ring $Q_\lambda$ and $\mathrm{Soc}\, Q_\lambda \subset R_\lambda$. This is enough to conclude that $R$ has the properties stated in (6).

(6) $\Rightarrow$ (1). If (6) holds, then it follows from Lemma 9 that $R$ is semiprime and $\mathrm{Soc}\, R = \oplus_{\lambda\in\Lambda} \mathrm{Soc}\, Q_\lambda$. Moreover $E(R_R) = \prod_{\lambda\in\Lambda} Q_\lambda$ (see [**12**, Ch. XII, Proposition 2.4, page 247]). There is a family $(e_\lambda)_{\lambda\in\Lambda}$ of pairwise orthogonal and pairwise non-isomorphic idempotents of $R$ such that $e_\lambda Q_\lambda = e_\lambda R$ is simple and injective. Since $\sum_{\lambda\in\Lambda} e_\lambda Q_\lambda$ is faithful as a right ideal of $\prod_{\lambda\in\Lambda} Q_\lambda$, then it is faithful as a right ideal of $R$ and therefore $R$ is right $\aleph$-QF 3.

(4) $\Rightarrow$ (7). Inasmuch as $R$ is semiprime with essential socle, $R$ must be right (and left) nonsingular. Thus the Lambek torsion theory and the Goldie torsion theory on Mod-$R$ coincide, so that every nonsingular (= torsionfree) right $R$-module is cogenerated by $E(R_R)$. It follows from the equivalence of conditions (4) and (6) that $E(R_R) = \prod_{\lambda\in\Lambda} Q_\lambda$, where

each $Q_\lambda$ is a right full linear ring. Since $Q_\lambda$ is isomorphic to the direct product $P_\lambda^{\Gamma_\lambda}$ for some $\Gamma_\lambda$, we infer that the family $(P_\lambda)_{\lambda\in\Lambda}$ cogenerates $E(R_R)$, hence it cogenerates every nonsingular right $R$-module.

(7) $\Rightarrow$ (8). This implication is clear, taking into account that, since $R$ is right nonsingular, the Goldie torsion class in Mod-$R$ consists of all singular modules.

(8) $\Rightarrow$ (4). Assume that (8) holds and let us prove first that $Z(R_R) = 0$. Let us denote by $S$ the projective component of Soc $R_R$. Since $\aleph \neq 0$, (8) implies that $S \neq 0$ and ${}_R(R/S)$ is flat by Proposition 1, so that we may consider the jansian torsion theory $(\mathscr{T}, \mathscr{F})$ associated with the idempotent two-sided ideal $S$: $\mathscr{T} = \{L_R | LS = 0\}$, $\mathscr{F} = \{M_R | MS \trianglelefteq M\}$ (for the last equality see [**1**, Proposition 1.3]). Now (8) implies that a module $M_R$ is nonsingular iff it has projective and essential socle and, since the latter is given by $MS$ (see Proposition 1), we infer that $(\mathscr{T}, \mathscr{F})$ coincides with the Goldie torsion theory. Moreover (8) implies that the class of all singular right $R$-modules is a (hereditary) torsion class, whence it must coincide with $\mathscr{T}$. From this we conclude that the Gabriel topology $\{I \leq R_R | S \subset I\}$ associated with $\mathscr{T}$ consists of all essential right ideals, whence $S \trianglelefteq R_R$ and so $Z(R_R) = 0$. Let us prove now that each $P_\lambda$ is injective. Indeed, since $E(P_\lambda)$ is nonsingular, it follows from (8) that there is a non zero homomorphism $E(P_\lambda) \to P_\mu$ for some $\mu \in \Lambda$. Thus, since $P_\mu$ is projective, $E(P_\lambda)$ has a direct summand isomrophic to $P_\mu$, which implies $\lambda = \mu$ and $E(P_\lambda) = P_\lambda$. We conclude from the above that every minimal right ideal of $R$ is idempotent and, since $S = \text{Soc } R_R \trianglelefteq R_R$, $R$ must be semiprime.

(1) $\Leftrightarrow$ (2). By the equivalence of conditions (1) and (4), a semiprime right $\aleph$-QF 3 ring has essential socle. Since a semiprime ring with essential socle is nonsingular, the equivalence of (1) and (2) follows from [**4**, Theorem 2.11 ]. □

In the following corollary we characterize those semiprime rings which are $\aleph$-QF 3.

COROLLARY 11. *With the same hypothesis as in Theorem* 10, *the following conditions are equivalent*:

(1) *$R$ is a semiprime* $\aleph$-QF 3 *ring.*

(2) Soc $R_R \trianglelefteq R_R$, *there is a family* $(f_\lambda)_{\lambda\in\Lambda}$ *of idempotents of $R$ such that the $f_\lambda R$'s are the homogeneous components of* Soc $R_R$ *and* Card$(\Lambda) = \aleph$.

(3) *$R$ is (isomorphic to) a subring of the direct product of a family* $(Q_\lambda)_{\lambda\in\Lambda}$ *of simple artinian rings, with* Card$(\Lambda) = \aleph$, *and* $\oplus_{\lambda\in\Lambda} Q_\lambda \subset R$.

(4) *$R$ is right nonsingular, every non-zero injective nonsingular right $R$-module is a direct product of pairwise independent semisimple and homogeneous modules, and* Card$(\Lambda) = \aleph$.

*Proof*. (1) ⇒ (3). In view of Theorem 10, (1) implies that every simple projective right or left $R$-module is injective; hence it follows from Corollary 6 that $R/l_R(\mathrm{Soc}_{P_\lambda}(R_R))$ is a simple artinian ring for each $\lambda \in \Lambda$. Thus (3) holds with $Q_\lambda = R/l_R(\mathrm{Soc}_{P_\lambda}(R_R))$ (see the proof of the implications (4) ⇒ (5) ⇒ (6) of Theorem 10).

(3) ⇒ (2) is straightforward.

(2) ⇒ (4). It follows from (2) that $\mathrm{Soc}\, R_R$ is projective and, taking [**2**, Theorem 2.7] into account, every semisimple, projective and homogeneous right $R$-module is injective. Assume that $M_R \neq 0$ is injective and nonsingular. Then $\mathrm{Soc}\, M = M(\mathrm{Soc}\, R_R) \trianglelefteq M$ and it follows from (2) that the homogeneous components of $\mathrm{Soc}\, M$ are the $Mf_\lambda$ $(\lambda \in \Lambda)$. Moreover $\bigoplus_{\lambda\in\Lambda} MF_\lambda$ is essential in $\prod_{\lambda\in\Lambda} Mf_\lambda$; indeed, if $0 \neq (x_\lambda) \in \prod_{\lambda\in\Lambda} Mf_\lambda$, then $x_\lambda f_\lambda \neq 0$ for some $\lambda \in \Lambda$, so that $0 \neq (x_\lambda)_\lambda(\bigoplus_{\lambda\in\Lambda} f_\lambda R) \subset \bigoplus_{\lambda\in\Lambda} Mf_\lambda$. Since all $Mf_\lambda$'s are injective, we conclude that $M = \bigoplus_{\lambda\in\Lambda} Mf_\lambda$.

(4) ⇒ (1). Assume that (4) holds. Then one easily checks that every non-singular $R$-module is cogenerated by simple projective modules, hence $R$ is a semiprime right ℵ-QF 3 ring by Theorem 10. Also, (4) implies that every projective semisimple and homogeneous right $R$-module is injective, whence every homogeneous component of $\mathrm{Soc}\, R_R$ is generated by a central idempotent (see [**2**, Theorem 2.7]). Inasmuch as $R$ is semiprime, then every homogeneous component of $\mathrm{Soc}\, R_R$ is also a homogeneous component of $\mathrm{Soc}\, {}_R R$ and conversely. From this and again by [**2**, Theorem 2.7] we infer that each simple projective left $R$-module is injective. Finally, since $\mathrm{Soc}\, R$ is essential both as a right and a left ideal, it follows from Theorem 10 that $R$ is left ℵ-QF 3 as well. □

REMARK. The assumption that $R$ is right nonsingular in condition (7) of Theorem 10 and condition (4) of the last corollary cannot be omitted. In fact, if $R = S \times T$, where $S$ is a quasi-Frobenius ring with essential singular ideal and $T$ is a semisimple ring, then $R$ is QF 3 and every nonsingular $R$-module is semisimple and injective, but $R$ is not semiprime.

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Received March 19, 1984 and in revised form August 27, 1984.

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DELL'AQUILA
VIA ROMA, 33
67100 L'AQUILA, ITALY