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THE SPECTRUM OF AN INTERPOLATED OPERATOR AND ANALYTIC MULTIVALUED FUNCTIONS

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Let $[B_0, B_1]$ be a complex interpolation pair and T: $B_0 + B_1 \to B_0 + B_1$ be a linear map whose restriction to each interpolation space $[B_0, B_1]_s$ is a bounded operator on that space with spectrum $\mathrm{Sp}_s T$. Under mild conditions on T it is shown that the set-valued map $\lambda \to \mathrm{Sp}_{(\mathrm{Re}\,\lambda)} T$ is an analytic multivalued function. This fact is used to unify and generalise a number of previously known results about the spectrum of an interpolated operator, and also to motivate some new ones.

Introduction. Let $[B_0, B_1]$ be a complex interpolation pair and $B_s = [B_0, B_1]_s$ $(0 \leq s \leq 1)$ be the corresponding interpolation spaces. If T: $B_0 + B_1 \to B_0 + B_1$ is a linear map whose restriction to each B_s is a bounded operator on B_s, then its spectrum $\mathrm{Sp}_s T$ in $L(B_s)$ can vary with s. This phenomenon has been investigated in a wide variety of special cases. Examples include: certain sorts of matrices on l_p-spaces [**15, 16, 27, 47**], Cesaro-type operators on L_p-spaces [**10, 26, 35**], multipliers on the L_p-spaces of a locally compact Abelian group [**28, 36, 50, 51, 52**], and certain singular integral operators on L_p-spaces [**29**], and even H_p-spaces [**14**]. The main theoretical results have been of three kinds: conditions ensuring that $\mathrm{Sp}_s T$ is independent of s [**17, 21, 22, 36, 48**], establishment of bounds for $\mathrm{Sp}_s T$ in terms of $\mathrm{Sp}_0 T$ and $\mathrm{Sp}_1 T$ [**46, 48**], and investigation of the continuity properties of the set-valued map $s \to \mathrm{Sp}_s T$ [**43, 44, 45, 52**].

This last is the starting point for the present paper. Although the upper semicontinuity of $s \to \mathrm{Sp}_s T$ for $s \in (0,1)$ is a purely topological statement, its proof in [**44**] depends upon properties of analytic functions. This state of affairs seems somewhat unsatisfactory: surely from such a proof it should be possible to draw *analytic* conclusions. We do just that, using the recently developed tools from the theory of analytic multivalued functions. This theory was first applied by Z. Słodkowski in [**38**] to describe the spectrum of an analytically varying operator on a Banach space; by contrast we keep the operator fixed and allow the space to vary. One of our two main results (2.7) asserts that the map $\lambda \to \mathrm{Sp}_{(\mathrm{Re}\,\lambda)} T$ is an analytic multivalued function on $\{\lambda;\ 0 < \mathrm{Re}\,\lambda < 1\}$. Unfortunately, technical problems force us to impose a mild condition on T for the proof to go through (so mild, in fact, that the author knows of no operator which fails to satisfy it!), and so we prove our other main theorem (2.4), which works for any T, but has a slightly weaker conclusion.

Section 2, where these results are demonstrated, constitutes the core of the paper. Section 1 summarises those parts of interpolation theory that will be needed, as well as giving a brief introduction to analytic multivalued functions. In §3 we apply what has been proved to deduce a number of consequences for the spectrum of an interpolated operator. Some of these are generalisations of theorems already known, including the 'constancy conditions' and 'spectral bounds' mentioned above, but one or two seem to be original. Thus the theory of analytic multivalued functions serves both to unify what has gone before, and also to motivate new ideas.

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1. Preliminaries. The purpose of this section is to sketch some background from two areas of analysis: analytic multivalued functions, and interpolation theory. We shall also take the opportunity to establish some notation.

We begin with analytic multivalued functions. Let X and Y be Hausdorff topological spaces and denote by $\kappa(Y)$ the collection of non-empty compact subsets of Y. A map $K\colon X \to \kappa(Y)$ is said to be *upper semicontinuous* (*u.s.c.*) if whenever U is open in Y, the set $\{x \in X; K(x) \subset U\}$ is open in X. The *graph* of K is defined as

$$\operatorname{graph}(K) = \{(x, y) \in X \times Y;\ y \in K(x)\}.$$

Also if $X_1 \subset X$, then $K|X_1$ denotes the restriction of K to X_1. The following fundamental result was proved by Z. Słodkowski in **[38]** (further information on pseudoconvex sets and plurisubharmonic functions may be found for example in **[23]**).

THEOREM 1.1. *Let G be an open subset of* $\mathbf{C}$, *and let* $K\colon G \to \kappa(\mathbf{C})$ *be u.s.c. The following are equivalent*:

(a) *the set* $(G \times \mathbf{C}) \setminus \operatorname{graph}(K)$ *is pseudoconvex in* $\mathbf{C}^2$;

(b) *for each* $a, b \in \mathbf{C}$, *if*

$$G_1 = \{\lambda \in G;\ a + \lambda b \notin K(\lambda)\},$$

$$\psi(\lambda, z) = -\log|z - a - \lambda b|,$$

$$\phi(\lambda) = \sup\{\psi(\lambda, z);\ z \in K(\lambda)\} \qquad (\lambda \in G_1),$$

then $\phi\colon G_1 \to [-\infty, \infty)$ *is subharmonic;*

(c) *for any* G_1 *open in* G, *and any plurisubharmonic function* ψ *defined on a neighbourhood of* graph$(K|G_1)$, *if*

$$\phi(\lambda) = \sup\{\psi(\lambda, z);\ z \in K(\lambda)\} \qquad (\lambda \in G_1),$$

then $\phi\colon G_1 \to [-\infty, \infty)$ *is subharmonic.* □

Any u.s.c. map $K\colon G \to \kappa(\mathbf{C})$ satisfying these conditions is called an *analytic multivalued function* (or *a.m.v. function* for short). Słodkowski showed that if A is a complex Banach algebra and $f\colon G \to A$ is analytic, then the spectrum of $f(\lambda)$ is an a.m.v. function. Using an idea of B. Aupetit, this breakthrough enabled him to solve the generalised Pełczyński conjecture (see **[38]**). Since then, a number of other applications have been found, both in spectral theory **[3, 4, 6, 7, 30, 31, 38, 39, 53]**, and in uniform algebras **[4, 38, 41, 42]**. Progress has also been made in investigating the abstract properties of a.m.v. functions (see **[1, 2, 4, 5, 6, 8, 30, 32, 33, 34, 38, 40, 41, 53]**), and we shall occasionally need to refer to some of these papers.

Now we turn to interpolation theory. Almost everything in this section is taken from the fundamental paper of A. P. Calderón **[12]**; details may also be found in **[9]**.

Let $B = [B_0, B_1]$ be an *interpolation pair*, that is, a pair of complex Banach spaces $(B_0, \|\cdot\|_0)$ and $(B_1, \|\cdot\|_1)$ continuously embedded in a Hausdorff topological vector space R. We shall write

$$\Delta = \Delta(B) = B_0 \cap B_1 \quad \text{and} \quad \Sigma = \Sigma(B) = B_0 + B_1.$$

Both Δ and Σ become Banach spaces when endowed with the respective norms

$$\|x\|_\Delta = \max(\|x\|_0, \|x\|_1),$$

$$\|x\|_\Sigma = \inf\{\|y\|_0 + \|z\|_1;\ y \in B_0,\ z \in B_1,\ x = y + z\}.$$

Henceforth we shall always assume that Δ *is dense in both* B_0 *and* B_1.

Let G be the open strip $\{\lambda \in \mathbf{C};\ 0 < \operatorname{Re}\lambda < 1\}$. Define $\mathscr{F}$ to be the class of all bounded continuous functions $f\colon \overline{G} \to \Sigma$ which are analytic on G, such that for $j = 0, 1$, the function $t \to f(j + it)$ takes values in B_j, is $\|\cdot\|_j$-continuous and tends to zero as $|t| \to \infty$. On the vector space $\mathscr{F}$, we introduce the norm

$$\|f\|_{\mathscr{F}} = \max_{j=0,1} \sup\{\|f(j+it)\|_j;\ -\infty < t < \infty\}$$

and this makes $\mathscr{F}$ a Banach space. For each $\lambda \in \bar{G}$, the Banach space $[B_0, B_1]_\lambda$ is defined to be the quotient of $\mathscr{F}$ by the closed subspace

$$\mathscr{N}_\lambda = \{f \in \mathscr{F};\ f(\lambda) = 0\},$$

its norm $\|\cdot\|_\lambda$ being just the quotient norm. Usually we shall abbreviate it simply to B_λ. Of course, it depends on λ only through $\operatorname{Re}\lambda$, but it will be convenient for us to define B_λ for all $\lambda \in \bar{G}$. Also it appears that we have two definitions for the spaces B_0, B_1 and their norms, but the density assumption on Δ above ensures that these definitions coincide. It also implies that Δ is dense in B_λ for every $\lambda \in \bar{G}$.

If $(Z, \|\cdot\|)$ is a Banach space, let us write Z^* for its dual, equipped with the usual dual norm $\|\cdot\|^*$. Then $[B_0^*, B_1^*]$ $(= B^*$, say) is also an interpolation pair (take $R = \Delta(B)^*$), and it is easy to check that

$$\Delta(B^*) = \Sigma(B)^* \quad \text{and} \quad \Sigma(B^*) = \Delta(B)^*,$$

with equality of norms. Define $\mathscr{G}$ to be the collection of all continuous functions $g\colon \bar{G} \to \Sigma(B^*)$ which are analytic on G, such that

$$\|g(\lambda)\|_{\Sigma(B^*)} \leq c\cdot(1 + |\lambda|)$$

for some constant c, and such that for $j = 0, 1$,

$$g(j + it_2) - g(j + it_1)$$

takes values in B_j^* whenever $-\infty < t_1 < t_2 < \infty$, with

$$\|g\|_{\mathscr{G}} = \max_{j=0,1} \sup_{t_1 < t_2} \left\{\|g(j+it_2) - g(j+it_1)\|_j^*/(t_2 - t_1)\right\} < \infty.$$

Under this norm, the space $\mathscr{G}$ reduced modulo the constant functions becomes a Banach space. For each $\lambda \in G$, the Banach space $[B_0^*, B_1^*]^\lambda$ is defined to be the quotient of $\mathscr{G}$ by the closed subspace

$$\mathscr{M}_\lambda = \{g \in \mathscr{G};\ (dg/d\lambda)(\lambda) = 0\},$$

its norm $\|\cdot\|^\lambda$ being just the quotient norm. The whole point of introducing $\mathscr{G}$ is that it can be shown that if $\lambda \in G$, then $[B_0^*, B_1^*]^\lambda$ is isometrically isomorphic to B_λ^*. Also, if

$$(x, y) \to \langle x, y\rangle_\lambda\colon B_\lambda \times B_\lambda^* \to \mathbf{C}$$

is the naturally induced pairing, then it agrees with the pairing

$$(x, y) \to \langle x, y\rangle\colon \Delta(B) \times \Sigma(B^*) \to \mathbf{C}$$

on their common domain of definition.

Let $f \in \mathscr{F}$, $g \in \mathscr{G}$ and $\lambda \in G$; we shall denote by $[f]_\lambda$ and $[g]_\lambda$ the cosets in $B_\lambda = \mathscr{F}/\mathscr{N}_\lambda$ and $B_\lambda^* = \mathscr{G}/\mathscr{M}_\lambda$ containing f and g resepectively. We isolate the following simple lemma for future reference.

LEMMA 1.2. *Let $f \in \mathscr{F}$ and $g \in \mathscr{G}$. The function*

$$\rho(\lambda) = \langle [f]_\lambda, [g]_\lambda \rangle_\lambda$$

is bounded and analytic on G.

Proof. Boundedness is clear because

$$(1) \qquad |\langle [f]_\lambda, [g]_\lambda \rangle_\lambda| \le \|[f]_\lambda\|_\lambda \cdot \|[g]_\lambda\|^\lambda \le \|f\|_{\mathscr{F}} \cdot \|g\|_{\mathscr{G}}.$$

The inequality (1) also shows that it suffices to prove ρ is analytic whenever f lies in some dense subset of $\mathscr{F}$. Therefore, by [**12**, 7.92], we may suppose without loss of generality that f is the product of a scalar analytic function $\alpha(\lambda)$ with a constant vector $x \in \Delta$. Fixing an arbitrary $\lambda_0 \in G$, we know that g has a Taylor expansion

$$g(\lambda) = \sum_0^\infty a_n \cdot (\lambda - \lambda_0)^n \qquad (a_n \in \Sigma(B^*))$$

which converges in $\Sigma(B^*)$ uniformly on some neighbourhood of λ_0. Thus,

$$\left|\rho(\lambda) - \alpha(\lambda) \cdot \sum_0^N n(\lambda - \lambda_0)^n \cdot \langle x, a_n \rangle\right|$$

$$\le |\alpha(\lambda)| \cdot \|x\|_\Delta \cdot \left\| \sum_{N+1}^\infty n a_n (\lambda - \lambda_0)^{n-1} \right\|_{\Sigma(B^*)},$$

and since the right-hand term converges to zero uniformly on some neighbourhood of λ_0, we deduce that ρ is analytic on a neighbourhood of λ_0. □

We shall also require a rather more recent result: this is the Reiteration Theorem, which originally appeared as [**12**, 12.3], but with a hypothesis that has since been proved redundant (see [**13**]).

THEOREM 1.3. *Let $\mu, \sigma, \lambda_0, \lambda_1 \in \overline{G}$, and suppose that*

$$\mu = (1 - \sigma) \cdot \operatorname{Re} \lambda_0 + \sigma \cdot \operatorname{Re} \lambda_1.$$

Then $B_\mu = [B_{\lambda_0}, B_{\lambda_1}]_\sigma$, with equality of norms. □

Finally, suppose that $T\colon \Sigma \to \Sigma$ is a linear map such that for $j = 0, 1$, the restriction $T|B_j$ belongs to $L(B_j)$ (where $L(Z)$ denotes the algebra of bounded linear operators on a Banach space Z). In such a situation, we shall say that T is an *operator on the pair* $[B_0, B_1]$. The following standard interpolation result is an abstract version of the Riesz-Thorin Theorem.

PROPOSITION 1.4. *For every* $\lambda \in \overline{G}$, *we have* $T|B_\lambda \in L(B_\lambda)$, *and moreover if* $\|\cdot\|_\lambda$ *denotes the operator norm on* $L(B_\lambda)$, *then*

$$\|T\|_\lambda \le \|T\|_0^{1-s} \cdot \|T\|_1^s,$$

where $s = \operatorname{Re}\lambda$. □

2. The main results. We shall adopt the notation of §1: thus $[B_0, B_1]$ is an interpolation pair such that Δ is dense in both B_0 and B_1, the open strip $\{0 < \operatorname{Re}\lambda < 1\}$ is denoted by G, and $T\colon \Sigma \to \Sigma$ is an operator on the pair $[B_0, B_1]$ as in (1.4). Also, if S is an operator on a Banach space Z, we shall denote its spectrum in $L(Z)$ by $\operatorname{Sp}(S; Z)$; in the case when $Z = B_\lambda$, this will often be abbreviated simply to $\operatorname{Sp}_\lambda S$. In this section we shall investigate the properties of the set-valued map $\lambda \to \operatorname{Sp}_\lambda T\colon \overline{G} \to \kappa(\mathbf{C})$.

In carrying out such investigations one must beware of the following pitfall: the fact that T may be invertible in both $L(B_0)$ and $L(B_1)$ does *not* imply that the two inverses of T must agree on $\Delta = B_0 \cap B_1$. An example of this phenomenon will be outlined below. However, something positive can be said. Define W_T to be the set of $z \in \mathbf{C}$ such that $(T - zI)$ is invertible in both $L(B_0)$ and $L(B_1)$ and such that the two inverses agree on $\Delta = B_0 \cap B_1$; also set $E_T = \mathbf{C} \setminus W_T$. The following result is due to J. D. Stafney [**48**, Lemmas 1.7 and 1.6].

PROPOSITION 2.1. (a) *The set* E_T *consists of the union of* $\operatorname{Sp}_0 T \cup \operatorname{Sp}_1 T$ *with a subcollection of the bounded components of* $\mathbf{C} \setminus (\operatorname{Sp}_0 T \cup \operatorname{Sp}_1 T)$.

(b) *If* $z_0 \notin E_T$ *then* $T - z_0 I$ *is invertible on each* B_λ, *and all the inverses agree on* Δ. *Hence, for all* $\lambda \in \overline{G}$ *we have* $\operatorname{Sp}_\lambda T \subset E_T$. □

Consider the following example. For $1 < p < \infty$ define $T \in L(L_p(0, \infty))$ to be the Cesaro operator

$$Tf(x) = \frac{1}{x}\int_0^x f(t) \cdot dt \qquad (f \in L_p).$$

Fix any p_0 and p_1 with $1 < p_0 < p_1 < \infty$, and take $B_0 = L_{p_0}$ and $B_1 = L_{p_1}$. Then for $\lambda \in \overline{G}$ we have $B_\lambda = L_p$, where

$$1/p = \operatorname{Re}\bigl((1 - \lambda)/p_0 + \lambda/p_1\bigr)$$

(see [**9**, Chapter 5]). Now it is shown in [**10**] that $\mathrm{Sp}(T; L_p)$ is just the circle with centre $2(p-1)/p$ and the same radius. Consequently, if E_T were equal to $\mathrm{Sp}_0 T \cup \mathrm{Sp}_1 T$, then this example would violate (2.1)(b). This justifies the assertion made prior to (2.1).

We are therefore led to make the following definitions.

(I) T satisfies the *uniqueness-of-resolvent* (*U.R.*) *condition* if whenever $0 \leq s \leq t \leq 1$ and $z \notin \mathrm{Sp}_s(T) \cup \mathrm{Sp}_t(T)$, then the inverses of $(T - zI)$ in $L(B_s)$ and $L(B_t)$ agree on Δ.

(II) T satisfies the *local uniqueness-of-resolvent* (*U.R.*) *condition* if whenever $0 < s < 1$ and $z \notin \mathrm{Sp}_t(T)$ for all t in some neighbourhood of s, then the inverses of $(T - zI)$ in $L(B_t)$ agree on Δ for all t in some (possibly smaller) neighbourhood of s.

The first definition was introduced in [**52**]. The second definition is new: it is much milder than the first and may possibly be vacuous. Certainly the author knows of no T which fails to satisfy it, though there is a hint of a counterexample in [**44**]. However, it will be useful to assume it later on.

Combining (1.3), (2.1) and the definition (I) above immediately yields the following crude but useful corollary. (If $Q \in \kappa(\mathbf{C})$, then we shall write $Q\hat{}$ for its polynomial hull, which equals the union of Q with all the bounded components of $\mathbf{C} \setminus Q$.)

COROLLARY 2.2. *Let* $0 \leq s \leq \operatorname{Re}\lambda \leq t \leq 1$.

(a) $\mathrm{Sp}_\lambda T \subset (\mathrm{Sp}_s T \cup \mathrm{Sp}_t T)\hat{}$.

(b) *If T satisfies the U.R. condition, then* $\mathrm{Sp}_\lambda T \subset \mathrm{Sp}_s T \cup \mathrm{Sp}_t T$. □

Now we give some simple criteria which are sufficient to ensure that T satisfies the U.R. conditions. First we need another piece of terminology. Let (X, μ) be a σ-finite measure space; a complex Banach space $(Z, \|\cdot\|)$ of measurable functions on X (with two functions being identified if they agree μ-almost everywhere) is a *Banach lattice* if whenever $f \in Z$ and g is measurable with $|g| \leq |f|$, then $g \in Z$ and $\|g\| \leq \|f\|$. Also Z is said to satisfy the *dominated convergence condition* if whenever f_n is a sequence of elements of Z converging pointwise to zero on X in such a way that $|f_n| \leq f$ for some $f \in Z$, then $\|f_n\| \to 0$.

PROPOSITION 2.3. (I) *Each of the following two conditions ensures that T satisfies the U.R. condition*:

(a) *one of the spaces B_0, B_1 is contained inside the other*;

(b) *the set* $\mathrm{Sp}_0 T \cup \mathrm{Sp}_1 T$ *has empty interior and connected complement.*

(II) *Each of the following two conditions ensures that T satisfies the local U.R. condition*:

(c) *T satisfies the U.R. condition*;

(d) *the spaces B_0 and B_1 are Banach lattices of measurable functions on a σ-finite measure space (X, μ) which both satisfy the dominated convergence condition.*

Proof. (a) This becomes clear upon remarking that if say $B_0 \subset B_1$, then $B_s \subset B_t$ whenever $s \leq t$.

(b) Let $0 \leq s \leq t \leq 1$, and set $F = \mathrm{Sp}_0 T \cup \mathrm{Sp}_1 T$. Since $\mathbf{C} \setminus F$ is connected, from (2.2)(a) we deduce that $\mathrm{Sp}_s T \cup \mathrm{Sp}_t T \subset F$. But F has empty interior, so that any subset of it must also have connected complement. The result therefore follows by applying (1.3) and (2.1)(a) to the interpolation pair $[B_s, B_t]$.

(c) This is obvious.

(d) This result is due to I. Ya. Sneiberg and lies rather deeper. We do not give a proof, but refer the reader instead to [**44**, Theorem 4]. Note in particular it implies that T will satisfy the local U.R. condition whenever B_0 and B_1 are L_p-spaces with respect to a σ-finite measure space. Thus although the Cesaro operator T mentioned earlier fails to satisfy the U.R. condition, it *does* obey the local version. □

We now state our first main theorem.

THEOREM 2.4. *For $\lambda \in \overline{G}$ define $K(\lambda)$ to be the union of $\mathrm{Sp}_\lambda T$ with those components of $\mathbf{C} \setminus \mathrm{Sp}_\lambda T$ which are contained within E_T.*

(a) *The map K: $\overline{G} \to \kappa(\mathbf{C})$ is u.s.c.*

(b) *The restriction K: $G \to \kappa(\mathbf{C})$ is a.m.v.*

In order to prove this result we require a lemma which is a variant of a classical result of E. M. Stein [**49**].

LEMMA 2.5. *Let H be an open subset of G and suppose that to each $\lambda \in H$ is associated an operator $S_\lambda \in L(B_\lambda)$ such that*:

(i) *the map $\lambda \to \|S_\lambda\|_\lambda$ is locally bounded on H*;

(ii) *for each $f \in \mathscr{F}$ and each $g \in \mathscr{G}$, the map $\lambda \to \langle S_\lambda[f]_\lambda, [g]_\lambda \rangle_\lambda$ is analytic on H.*

Then $\log\|S_\lambda\|_\lambda$: $H \to [-\infty, \infty)$ is a continuous subharmonic function.

Proof. Let

$$\mathscr{H} = \{(f, g) \in \mathscr{F} \times \mathscr{G};\ \|f\|_{\mathscr{F}} < 1,\ \|g\|_{\mathscr{G}} < 1\}.$$

From the Hahn-Banach Theorem, for each $\lambda \in H$ we have

$$(2)\quad \|S_\lambda\|_\lambda = \sup\left\{|\langle S_\lambda x, y\rangle_\lambda|;\ x \in B_\lambda,\ y \in B_\lambda^*,\ \|x\|_\lambda < 1,\ \|y\|^\lambda < 1\right\}$$

$$= \sup\left\{|h_{f,g}(\lambda)|;\ (f,g) \in \mathscr{H}\right\}$$

where for $f \in \mathscr{F}$ and $g \in \mathscr{G}$

$$h_{f,g}(\lambda) = \langle S_\lambda[f]_\lambda, [g]_\lambda\rangle_\lambda.$$

Now by assumption (ii), each $h_{f,g}$ is analytic on H, and by (2) and assumption (i), the family $\{h_{f,g};\ (f,g) \in \mathscr{H}\}$ is locally uniformly bounded on H. Using the standard Cauchy estimates, it follows that this family if equicontinuous, whence (2) shows that $\|S_\lambda\|_\lambda$ is a continuous function of λ on H. Since

$$\log\|S_\lambda\|_\lambda = \sup\left\{\log|h_{f,g}(\lambda)|;\ (f,g) \in \mathscr{H}\right\}$$

is the *pointwise supremum of a family of subharmonic functions, it is* itself subharmonic on H. $\square$

Note in particular that if S_λ is taken to be the restriction of T to B_λ for each $\lambda \in G$, then the hypothesis (2.5)(i) holds by (1.4), and (2.5)(ii) follows from (1.2) because if $f \in \mathscr{F}$ then Tf also belongs to $\mathscr{F}$. Thus $\log\|T\|_\lambda$ is a continuous subharmonic function on G.

Proof of (2.4). (a) Let us begin by noting that $\lambda \to \|T\|_\lambda$ is an u.s.c. function on $\overline{G}$: upper semicontinuity on ∂G follows from (1.4), and on G itself we even know that the function is continuous, by the remark immediately above. Hence, if $r_\lambda(T)$ denotes the spectral radius of T in $L(B_\lambda)$, then $\lambda \to r_\lambda(T)$ is u.s.c. on $\overline{G}$, since it is the pointwise infimum of the u.s.c functions

$$\lambda \to \|T^n\|_\lambda^{1/n} \qquad (n \geq 1).$$

Fix $\lambda_0 \in \overline{G}$, and let $z_0 \notin K(\lambda_0)$. We have to prove that there exist neighbourhoods M of z_0 and N of λ_0 such that

$$\lambda \in N \cap \overline{G} \Rightarrow K(\lambda) \cap M = \varnothing.$$

Let D be the component of $\mathbf{C} \setminus K(\lambda_0)$ which contains z_0. By the definition of K, the set $D \setminus E_T$ is non-empty and open, and therefore infinite. Pick $z_1 \in D \setminus E_T$ with $z_1 \neq z_0$ and let M be a compact disc with centre z_0 such that $M \subset D$ and $z_1 \notin M$. By Runge's Theorem, there exists a polynomial $p(z)$ such that

$$(3)\qquad |p(1/(z - z_1))| < 1 \qquad (z \in \mathbf{C} \setminus D),$$

$$(4)\qquad |p(1/(z - z_1)) - 2| < 1 \qquad (z \in M).$$

Now $T - z_1 I$ is invertible in both $L(B_0)$ and $L(B_1)$, and its inverses agree on Δ, so $(T - z_1 I)^{-1}$ is an operator on the pair $[B_0, B_1]$, and by (1.4) it may be interpolated to define an inverse of T-$z_1 I$ on each B_λ. If we set

$$S = p\big((T - z_1 I)^{-1}\big),$$

then the Spectral Mapping Theorem gives

$$\mathrm{Sp}_\lambda S = \{\, p(1/(z - z_1));\ z \in \mathrm{Sp}_\lambda T \,\}, \tag{5}$$

so that by (3),

$$\mathrm{Sp}_{\lambda_0} S \subset \{\, z \in \mathbf{C};\ |z| < 1 \,\}.$$

By the upper semicontinuity of $r_\lambda(S)$ proved at the beginning, there is a neighbourhood N of λ_0 such that

$$\begin{aligned} \lambda \in N \cap \bar{G} &\Rightarrow \mathrm{Sp}_\lambda S \subset \{\, z \in \mathbf{C};\ |z| < 1 \} \\ &\Rightarrow \mathrm{Sp}_\lambda S \cap \{\, z \in \mathbf{C};\ |z| > 1 \} = \varnothing \\ &\Rightarrow \mathrm{Sp}_\lambda T \cap M = \varnothing, \end{aligned}$$

the last implication following from (4) and (5).

(b) We shall apply (1.1)(b). Let $a, b \in \mathbf{C}$ and set

$$G_1 = \{\lambda \in G;\ a + \lambda b \notin K(\lambda)\},$$
$$\varphi(\lambda) = \sup\{-\log|z - a - \lambda b|;\ z \in K(\lambda)\} \qquad (\lambda \in G_1).$$

Our task is to prove that φ is subharmonic on G_1. Note that for each $\lambda \in G_1$, since $z \to -\log|z - a - \lambda b|$ is subharmonic on a neighbourhood of $K(\lambda)$, it attains its maximum over $K(\lambda)$ on the boundary of $K(\lambda)$, which is contained in $\mathrm{Sp}_\lambda T$, which in turn is contained in $K(\lambda)$. Thus if $\lambda \in G_1$, then

$$\varphi(\lambda) = \sup\{-\log|z - a - \lambda b|;\ z \in \mathrm{Sp}(T; B_\lambda)\}. \tag{6}$$

Fix an arbitrary $\lambda_0 \in G_1$: we shall prove that φ is subharmonic on a neighbourhood of λ_0.

Assertion. There exists $\delta > 0$ such that $T - (a + \lambda_0 b)I$ is invertible in both $L(B_{\lambda_0 - \delta})$ and $L(B_{\lambda_0 + \delta})$, and the two inverses agree on $(B_{\lambda_0 - \delta}) \cap (B_{\lambda_0 + \delta})$.

Assume this for the time being, and set

$$B_0' = B_{\lambda_0 - \delta}, \qquad B_1' = B_{\lambda_0 + \delta},$$

so that by (1.3), if $B'_\mu = [B'_0, B'_1]_\mu$ then

$$B'_\mu = B_\lambda, \quad \text{where } \lambda = \lambda_0 + (2\mu - 1)\delta.$$

If we also set

$$\begin{aligned}
a' &= a + \lambda_0 b - \delta b, \\
b' &= 2\delta b, \\
G'_1 &= \{\mu \in G;\ \lambda_0 + (2\mu - 1)\delta \in G_1\}, \\
\varphi'(\mu) &= \sup\{-\log|z - a' - \mu b'|;\ z \in \operatorname{Sp}(T; B'_\mu)\} \qquad (\mu \in G'_1),
\end{aligned}$$

then by (6),

$$\varphi(\lambda) = \varphi'((\lambda - \lambda_0 + \delta)/2\delta) \quad \text{on } \{\lambda \in G_1;\ |\mathrm{Re}(\lambda - \lambda_0)| < \delta\}.$$

Therefore φ will be subharmonic on a neighbourhood of λ_0 if φ' is subharmonic on a neighbourhood of $1/2$. All this shows that we may "delete the primes" and assume without loss of generality that $\lambda_0 = \delta = 1/2$. Thus $T - (a + \lambda_0 b)I$ is invertible both in $L(B_0)$ and $L(B_1)$, and its inverses agree on Δ. Set

$$M = \max\left(\left\|(T - (a + \lambda_0 b)I)^{-1}\right\|_0, \left\|(T - (a + \lambda_0 b)I)^{-1}\right\|_1\right).$$

It follows from (1.4) that $T - (a + \lambda_0 b)I$ is invertible in each $L(B_\lambda)$, all the inverses agree on Δ, and

$$\left\|(T - (a + \lambda_0 b)I)^{-1}\right\|_\lambda \le M \qquad (\lambda \in \bar{G}).$$

Set $r = \min(\delta, 1/2M|b|)$. Then if $|\lambda - \lambda_0| < r$, the operator $T - (a + \lambda b)I$ is invertible in $L(B_\lambda)$. Since by (6)

$$\varphi(\lambda) = \log r_\lambda\left((T - (a + \lambda b)I)^{-1}\right) \qquad (|\lambda - \lambda_0| < r),$$

it follows that φ is the limit of the decreasing sequence of functions

$$(1/2^n) \cdot \log\left\|(T - (a + \lambda b)I)^{-2^n}\right\|_\lambda.$$

It is therefore sufficient to prove that for each integer $k \ge 1$, the map

$$\lambda \to \log\left\|(T - (a + \lambda b)I)^{-k}\right\|_\lambda$$

is subharmonic on the set $H = \{\lambda;\ |\lambda - \lambda_0| < r\}$, and we shall do this by verifying that the family of operators $S_\lambda \in L(B_\lambda)$ defined by

$$S_\lambda = (T - (a + \lambda b)I)^{-k} \qquad (\lambda \in H)$$

satisfies the hypotheses (i) and (ii) of (2.5).

(i) If $|\lambda - \lambda_0| < r$, then

$$\begin{aligned}
&\left\|(T-(a+\lambda b)I)^{-1}\right\|_\lambda \\
&= \left\|(T-(a+\lambda_0 b)I)^{-1} \cdot \left(I-(T-(a+\lambda_0 b)I)^{-1}b(\lambda-\lambda_0)\right)^{-1}\right\|_\lambda \\
&\leq \left\|(T-(a+\lambda_0 b)I)^{-1}\right\|_\lambda \cdot \sum_{j=0}^{\infty} \left\|(T-(a+\lambda_0 b)I)^{-j} b^j (\lambda-\lambda_0)^j\right\|_\lambda \\
&\leq 2M,
\end{aligned}$$

and hence

$$\lambda \in H \Rightarrow \|S_\lambda\|_\lambda \leq (2M)^k.$$

(ii) Suppose $|\lambda - \lambda_0| < r$. In the space $L(B_\lambda)$, we then have

$$\begin{aligned}
S_\lambda &= (T-(a+\lambda_0 b)I)^{-k} \cdot \left(I-(T-(a+\lambda_0 b)I)^{-1}b(\lambda-\lambda_0)\right)^{-k} \\
&= \sum_{j=0}^{\infty} \binom{k+j-1}{j} (T-(a+\lambda_0 b)I)^{-(k+j)} b^j \cdot (\lambda-\lambda_0)^j \\
&= \sum_{j=0}^{\infty} A_j \cdot (\lambda-\lambda_0)^j,
\end{aligned}$$

say, where each A_j is an operator on the pair $[B_0, B_1]$, and

$$\|A_j\|_\lambda \leq \binom{k+j-1}{j} (M|b|)^j M^k,$$

so that the series really does converge in $L(B_\lambda)$. Therefore if $f \in \mathscr{F}$ and $g \in \mathscr{G}$, then

$$(7) \qquad \langle S_\lambda [f]_\lambda, [g]_\lambda \rangle_\lambda = \sum_{j=0}^{\infty} \langle A_j [f]_\lambda, [g]_\lambda \rangle_\lambda \cdot (\lambda-\lambda_0)^j,$$

the convergence of the right-hand side being uniform on H. For each $j \geq 0$, the function

$$\lambda \to \langle A_j [f]_\lambda, [g]_\lambda \rangle_\lambda$$

is analytic on G, by (1.2). Hence the right-hand side of (7) is analytic on H, whence so also is the left-hand side, as was to be proved.

It remains to justify the Assertion. Since $\lambda_0 \in G_1$, the point $a + \lambda_0 b$ lies in some component D of $\mathbf{C} \setminus K(\lambda_0)$ which is not contained in E_T. Choose $z_0 \in D \setminus E_T$ and join $a + \lambda_0 b$ to it by a continuous path σ in D.

Now $K(\lambda)$ depends on λ only through $\operatorname{Re}\lambda$, and by part (a) it is u.s.c; thus since $[\sigma]$ (the image of σ) is a compact set disjoint from $K(\lambda_0)$, there exists $\delta > 0$ such that

$$(8) \qquad |\operatorname{Re}(\lambda - \lambda_0)| \leq \delta \Rightarrow K(\lambda) \cap [\sigma] = \varnothing.$$

Define V_T to be the set of $z \in \mathbf{C}$ such that $T - zI$ is invertible in both $L(B_{\lambda_0-\delta})$ and $L(B_{\lambda_0+\delta})$, and such that the two inverses agree on $(B_{\lambda_0-\delta}) \cap (B_{\lambda_0+\delta})$. By (1.3) and (2.1), the set V_T is a union of a subcollection of the components of $\mathbf{C} \setminus (\operatorname{Sp}_{\lambda_0-\delta}(T) \cup \operatorname{Sp}_{\lambda_0+\delta}(T))$. By (8), the points $a + \lambda_0 b$ and z_0 lie in the same component of the latter set. Also $z_0 \notin E_T$, which implies that $z_0 \in V_T$. Therefore $a + \lambda_0 b \in V_T$, as desired. □

COROLLARY 2.6. *The map* $\lambda \to (\operatorname{Sp}_\lambda T)\hat{}$ *is u.s.c. on* $\overline{G}$ *and a.m.v. on* G.

Proof. Since $(\operatorname{Sp}_\lambda T)\hat{} = K(\lambda)\hat{}$, the result is an immediate consequence of (2.4) plus the easily proved fact that the polynomial hull of an u.s.c. (respectively a.m.v.) function is u.s.c. (respectively a.m.v.). □

Unfortunately (2.4) asserts 'a.m.v.-ness' only for $K(\lambda)$ rather than for $\operatorname{Sp}_\lambda T$, which is the more natural object to consider. Our second main result rectifies this situation, though at the cost of making an extra assumption about the operator T.

THEOREM 2.7. *For* $\lambda \in G$, *define* $L(\lambda) = \operatorname{Sp}_\lambda T$.

(a) *The map* $L\colon G \to \kappa(\mathbf{C})$ *is u.s.c.*

(b) *If* T *satisfies the local U.R. condition, then* $L\colon G \to \kappa(\mathbf{C})$ *is a.m.v.*

Proof. (a) This part is due to I. Ya. Sneiberg [**44**]; his ingenious argument is repeated in the less obscure reference [**52**].

(b) The proof is almost identical to that of (2.4)(b) with K replaced by L. The only difference is that the Assertion made in the middle of the proof now follows directly from the local U.R. assumption on T. □

REMARKS. (1) As already pointed out, it may well be the case that every T satisfies the local U.R. condition: if this is true, then (2.7)(b) is stronger than (2.4)(b) because $\partial K(\lambda) \subset L(\lambda) \subset K(\lambda)$ for every $\lambda \in G$.

(2) Sneiberg's proof of (2.7)(a) is completely different from that of (2.4)(a). Though at first sight his result appears to be stronger, this is not quite the case, for (2.4)(a) asserts the upper semicontinuity of K on the

whole of $\bar{G}$, whereas (2.7)(a) does not guarantee that L is u.s.c. on ∂G. Indeed, this can sometimes fail to be true, even if T satisfies the U.R. condition. The following example is an adaptation of one due to Sneiberg [**44**].

For each integer n and each $s \in [0,1]$ define

$$E_{n,s} = \begin{cases} l_p(\mathbf{Z}), & \text{if } n \le 0, \text{ where } 1/p = 1 - s/2, \\ L_q[0,1], & \text{if } n \ge 1, \text{ where } 1/q = s/2. \end{cases}$$

Define B_λ to be the l_2-direct sum of the spaces $(E_{n,\mathrm{Re}\,\lambda})$ $(n \in \mathbf{Z})$. Then B_0 is dense in B_1, so by (2.3)(a) the spaces $[B_0, B_1]$ form an interpolation pair on which any operator must satisfy the U.R. condition. Also, by standard results in interpolation theory (see [**9**, Chapter 5]), we have $B_\lambda = [B_0, B_1]_\lambda$. For $j = 0, 1$, define $T \in L(B_j)$ by

$$(9) \qquad (Tx)_n = \begin{cases} x_{n-1}, & n \ne 1, \\ Sx_0, & n = 1, \end{cases}$$

where S is the operator given by

$$S(\ldots, a_{-1}, a_0, a_1, \ldots) = \sum_{-\infty}^{\infty} a_k e^{2\pi i k t},$$

mapping l_1 to L_∞, and l_2 to L_2. By (1.4), T interpolates to an operator on each B_λ which is also of the form (9). Now $0 \notin \mathrm{Sp}_\lambda T$ if and only if $S\colon E_{0,\mathrm{Re}\,\lambda} \to E_{1,\mathrm{Re}\,\lambda}$ is invertible. For $\mathrm{Re}\,\lambda = 1$ this is indeed the case, since $S\colon l_2 \to L_2$ is well known to be an isomorphism (even an isometry), but if $\mathrm{Re}\,\lambda < 1$ then this cannot happen, because if $p < 2 < q$ then l_p and L_q are not even isomorphic Banach spaces. Thus $\mathrm{Sp}_\lambda T$ fails to be u.s.c. at $\lambda = 1$. In fact, it is easy to show that

$$\mathrm{Sp}_\lambda T = \begin{cases} \{z;\ |z| \le 1\}, & \text{if } \mathrm{Re}\,\lambda < 1, \\ \{z;\ |z| = 1\}, & \text{if } \mathrm{Re}\,\lambda = 1. \end{cases}$$

Before going on to consider applications, we state one more general result due to C. J. A. Halberg and A. E. Taylor. A partial proof, which uses the ideas developed in this paper, will be given in the next section. For the full proof the reader is referred to [**17**].

THEOREM 2.8. *Suppose that T satisfies the U.R. condition. Let $\lambda_0, \lambda_1 \in \bar{G}$ and let C be a component of $\mathrm{Sp}_{\lambda_0} T$. Then $C \cap \mathrm{Sp}_{\lambda_1} T \ne \varnothing$.* □

COROLLARY 2.9. *Suppose that T satisfies the U.R. condition. If $\mathrm{Sp}_{\lambda_0} T$ is totally disconnected for some $\lambda_0 \in \bar{G}$, then $\mathrm{Sp}_{\lambda_0} T \subset \mathrm{Sp}_\lambda T$ for all $\lambda \in \bar{G}$.* □

3. Applications. Throughout this section we shall maintain the notation developed in §§1 and 2.

We begin with a spectral analogue of the Riesz-Thorin Theorem. It is a generalisation of a result of J. D. Stafney [**46**, Theorem 1.9, and **48**, Theorem 5.5].

THEOREM 3.1. *Let u be a subharmonic function defined on an open neighbourhood U of E_T, and for $\alpha \in [-\infty, \infty)$ set*

$$E_\alpha = \{z \in E_T;\ u(z) \leq \alpha\}.$$

If $\mathrm{Sp}_0 T \subset E_\alpha$ *and* $\mathrm{Sp}_1 T \subset E_\beta$, *then for any* $\lambda \in \overline{G}$ *we have*

$$\mathrm{Sp}_\lambda T \subset E_{(1-s)\alpha + s\beta},$$

where $s = \mathrm{Re}\,\lambda$.

Proof. Suppose initially that both $\alpha, \beta > -\infty$. For $\lambda \in \overline{G}$ and $z \in U$, define

$$\psi(\lambda, z) = u(z) - \mathrm{Re}((1-\lambda)\alpha + \lambda\beta),$$

and for $\lambda \in \overline{G}$, set

$$\varphi(\lambda) = \sup\{\psi(\lambda, z);\ z \in \mathrm{Sp}_\lambda T\}.$$

If $K\colon \overline{G} \to \kappa(\mathbf{C})$ is defined as in (2.4), then since

$$\partial K(\lambda) \subset \mathrm{Sp}_\lambda T \subset K(\lambda) \subset E_T \subset U \qquad (\lambda \in \overline{G}),$$

it follows from the maximum principle for subharmonic functions that

$$\sup\{u(z);\ z \in \mathrm{Sp}_\lambda T\} = \sup\{u(z);\ z \in K(\lambda)\} \qquad (\lambda \in \overline{G}),$$

whence

$$\varphi(\lambda) = \sup\{\psi(\lambda, z);\ z \in K(\lambda)\} \qquad (\lambda \in \overline{G}).$$

Now ψ is u.s.c. on $\overline{G} \times U$ and plurisubharmonic on $G \times U$, a neighbourhood of graph$(K|G)$, so by (2.4)(b) and (1.1)(c), the last equation implies that φ is u.s.c. and bounded on $\overline{G}$, and subharmonic on G. Now by assumption, if $\mathrm{Re}\,\lambda = 0$ or 1 then $\varphi(\lambda) \leq 0$. Therefore, by the extended maximum principle [**19**, Theorem 5.16], we have $\varphi \leq 0$ on the whole of $\overline{G}$, which yields the conclusion of the theorem.

Finally, if either α or β happens to be $-\infty$, then the result follows easily by applying what has already been proved to sequences α_n and β_n decreasing to α and β respectively. □

As a corollary, we deduce a simple 'constancy condition' for $\mathrm{Sp}_\lambda T$. It is a generalisation of a theorem of P. Sarnak [**36**], who stated it for the special case when T is a convolution operator on the L_p-spaces of a locally compact Abelian group.

First we recall that a subset P of $\mathbf{C}$ is said to be *polar* if there exists a subharmonic function u: $\mathbf{C} \to [-\infty, \infty)$ which identically $-\infty$ on P but not on $\mathbf{C}$. A compact polar set is totally disconnected and so automatically has empty interior and connected complement. A countable subset of $\mathbf{C}$ is always polar. Details on polar sets may be found in [**20**].

COROLLARY 3.2. *Suppose that* $\mathrm{Sp}_{\lambda_0} T$ *is a polar set for some* $\lambda_0 \in \overline{G}$. *Then* $\mathrm{Sp}_\lambda T = \mathrm{Sp}_{\lambda_0} T$ *for all* $\lambda \in G$.

Proof. Using (1.3), we may assume without loss of generality that $\lambda_0 = 0$. By the Evans-Selberg Theorem [**19**, Theorem 5.11] there is a subharmonic function u: $\mathbf{C} \to [-\infty, \infty)$ such that

$$\{z \in \mathbf{C};\ u(z) = -\infty\} = \mathrm{Sp}_0 T.$$

Applying (3.1) we deduce that for each $\lambda \in G$

$$\mathrm{Sp}_\lambda T \subset \mathrm{Sp}_0 T. \tag{10}$$

To prove the reverse inclusion, note first that if $\lambda \in G$ then from (10) the set $\mathrm{Sp}_0 T \cup \mathrm{Sp}_\lambda T$ is polar, so has empty interior and connected complement. Therefore by (2.3)(b), the operator T on the pair $[B_0, B_\lambda]$ satisfies the U.R. condition. Also, as $\mathrm{Sp}_0 T$ is polar it must be totally disconnected, whence from (2.9) applied to T on $[B_0, B_\lambda]$, and then (1.3), we have

$$\mathrm{Sp}_0 T \subset \mathrm{Sp}_\lambda T,$$

which completes the proof. □

REMARKS. (1) The constancy of $\mathrm{Sp}_\lambda T$ need not extend to $\lambda \in \partial G$. Indeed from [**36, 11**], if T is the operator on the L_p-spaces of the circle group defined as convolution with the Cantor-Lebesgue measure, then $\mathrm{Sp}(T; L_2)$ is a countable set of real numbers, whereas $\mathrm{Sp}(T; L_1)$ is not entirely real.

(2) An operator S is said to be *quasi-algebraic* if there exist monic polynomials p_n of degree n such that

$$\|p_n(S)\|^{1/n} \to 0 \quad \text{as } n \to \infty.$$

By a result of P. Halmos [**18**], an operator S is quasi-algebraic if and only if its spectrum is a polar set. Thus (3.2) shows that if $T|B_{\lambda_0}$ is quasi-algebraic for some $\lambda_0 \in \overline{G}$, then $T|B_\lambda$ is quasi-algebraic for all $\lambda \in G$. This result is an analogue of a theorem of M. Krasnoselskii **[24]** which shows that the same holds if 'quasi-algebraic' is replaced by 'compact', at least provided that B_0 and B_1 satisfy certain conditions.

The proof of (3.2) made use of Halberg and Taylor's Theorem (2.8), or rather, its corollary (2.9). It is interesting to note that the methods of this paper can be used to prove at least the following partial form of (2.8) (the only difference being that λ_0 is assumed to be in G rather than just in $\overline{G}$).

THEOREM 3.3. *Suppose that T satisfies the U.R. condition. Let $\lambda_0 \in G$ and $\lambda_1 \in \overline{G}$, and let C be a component of $\mathrm{Sp}_{\lambda_0}T$. Then $C \cap \mathrm{Sp}_{\lambda_1}T \neq \varnothing$.*

Proof. Suppose that the result is false. By (1.3), we may assume without loss of generality that $\lambda_1 = 1$. Choose disjoint open sets U_0 and U_1 such that

$$C \subset U_1, \tag{11}$$

$$\mathrm{Sp}_1 T \subset U_0, \tag{12}$$

$$\mathrm{Sp}_{\lambda_0} T \subset U_0 \cup U_1 = U, \quad \text{say}. \tag{13}$$

By (13) and (2.7)(a), there exists t with $0 < t < \operatorname{Re}\lambda_0$ such that

$$t \le \operatorname{Re}\lambda \le \operatorname{Re}\lambda_0 \Rightarrow \mathrm{Sp}_\lambda T \subset U,$$

and since T satisfies the U.R. condition, (12) and (2.2)(b) imply that

$$t \le \operatorname{Re}\lambda \le 1 \Rightarrow \mathrm{Sp}_\lambda T \subset U.$$

Define u: $U \to [-\infty, \infty)$ by

$$u(z) = \begin{cases} 0, & z \in U_0, \\ 1, & z \in U_1. \end{cases}$$

Applying (3.1) to T on the pair $[B_t, B_1]$ and then using (1.3), we deduce that

$$\mathrm{Sp}_{\lambda_0} T \subset \{z \in U;\, u(z) \le (1 - \operatorname{Re}\lambda_0)/(1 - t)\} = U_0,$$

which is a contradiction. □

It is perhaps worth noting (as was done in **[15]** for the special case of l_p-spaces) that with almost no extra work we can deduce the following strengthened form of (3.3).

COROLLARY 3.4. *Suppose that T satisfies the U.R. condition. Let $\lambda_0 \in G$ and $\lambda_1, \lambda_2 \in \bar{G}$ be such that* $\operatorname{Re}\lambda_1 \le \operatorname{Re}\lambda_0 \le \operatorname{Re}\lambda_2$, *and let C be a component of* $\mathrm{Sp}_{\lambda_0}T$. *Then*

$$C \cap \mathrm{Sp}_{\lambda_1}T \cap \mathrm{Sp}_{\lambda_2}T \neq \varnothing .$$

Proof. From (3.3) we know $C \cap \mathrm{Sp}_{\lambda_1}T$ and $C \cap \mathrm{Sp}_{\lambda_2}T$ are non-empty closed subsets of C. By (2.2)(b) their union equals C, so if they were disjoint then they would disconnect C, which is impossible. □

Theorems (2.4)(a) and (2.7)(a) were results about upper semicontinuity. We also have a form of lower semicontinuity.

THEOREM 3.5. *Let $\lambda_0 \in G$ and let $z_0 \in \partial\,\mathrm{Sp}_{\lambda_0}T$. Suppose that either*
(a) *T satisfies the local U.R. condition, or*
(b) *z_0 belongs to the boundary of* $(\mathrm{Sp}_{\lambda_0}T)\hat{}$.
Then $\operatorname{dist}(z_0, \mathrm{Sp}_\lambda T) \to 0$ *as* $\lambda \to \lambda_0$.

Proof. (a) Suppose the result is false, so there exist $\varepsilon > 0$ and $\lambda_n \to \lambda_0$ such that

$$\operatorname{dist}\big(z_0, \mathrm{Sp}_{\lambda_n}T\big) \ge 3\varepsilon \qquad (n \ge 1). \tag{14}$$

Since $z_0 \in \partial\mathrm{Sp}_{\lambda_0}T$, we may pick $z_1 \notin \mathrm{Sp}_{\lambda_0}T$ with $|z_1 - z_0| < \varepsilon$. By upper semicontinuity, there exists $\delta > 0$ such that

$$|\operatorname{Re}(\lambda - \lambda_0)| < \delta \Rightarrow z_1 \notin \mathrm{Sp}_\lambda T.$$

Define φ: $\{\lambda;\ |\operatorname{Re}(\lambda - \lambda_0)| < \delta\} \to [-\infty, \infty)$ by

$$\phi(\lambda) = \sup\{1/|z - z_1|;\ z \in \mathrm{Sp}_\lambda T\}.$$

From (2.7)(b) and (1.1)(c), the function φ is subharmonic. Since $\varphi(\lambda)$ depends on λ only through $\operatorname{Re}\lambda$, this means that $\varphi(t)$ is a convex function of t on $\{t;\ |t - \operatorname{Re}\lambda_0| < \delta\}$, and is therefore continuous at $t = \operatorname{Re}\lambda_0$. However,

$$\varphi(\operatorname{Re}\lambda_0) \ge 1/|z_0 - z_1| \ge 1/\varepsilon,$$

whereas from (14) we have

$$\varphi(\operatorname{Re}\lambda_n) \le 1/2\varepsilon \qquad (n \ge 1),$$

which is a contradiction.

(b) The proof is just the same as for (a), but uses (2.6) in place of (2.7). □

Theorem (3.5) will now be used in the proof of a result guaranteeing the 'permanence' of certain kinds of points in $\mathrm{Sp}_\lambda T$. We have already seen an example of this phenomenon: as a consequence of (3.3), if T satisfies the U.R. condition, if $\lambda_0 \in G$ and if $\{z_0\}$ is a component of $\mathrm{Sp}_{\lambda_0} T$, then $z_0 \in \mathrm{Sp}_\lambda T$ for all $\lambda \in \bar{G}$. This will be generalised by the next result, but first we need some terminology. A subset Q of $\mathbf{C}$ is said to be *non-thin* at $z_0 \in \mathbf{C}$ if (i) z_0 belongs to the closure of Q and (ii) for every subharmonic function u defined on some neighbourhood of z_0, we have

$$\limsup_{z \to z_0,\, z \in Q \setminus \{z_0\}} u(z) = u(z_0).$$

It can be shown that any connected set is non-thin at each point of its closure (see [**20**, Theorem 10.14]).

THEOREM 3.6. *Suppose that T satisfies the U.R. condition, and let $\lambda_0 \in G$.*

(a) *If z_0 is an element of $\mathrm{Sp}_{\lambda_0} T$ such that $\mathbf{C} \setminus \mathrm{Sp}_{\lambda_0} T$ is non-thin at every point of some neighbourhood of z_0, then $z_0 \in \mathrm{Sp}_\lambda T$ for all $\lambda \in \bar{G}$.*

(b) *Suppose that $\mathbf{C} \setminus \mathrm{Sp}_{\lambda_0} T$ is non-thin at every point of $\mathrm{Sp}_{\lambda_0} T$. If $\lambda_1, \lambda_2 \in G$ and $\mathrm{Re}\,\lambda_1$ lies between $\mathrm{Re}\,\lambda_0$ and $\mathrm{Re}\,\lambda_2$, then $\mathrm{Sp}_{\lambda_1} T \subset \mathrm{Sp}_{\lambda_2} T$.*

Proof. (a) Suppose the result is false, say $z_0 \notin \mathrm{Sp}_{\lambda_1} T$. By (1.3) we may assume without loss of generality that $\lambda_1 = 1$. Choose an open disc N with centre z_0 such that $\mathrm{Sp}_1 T \cap \bar{N} = \varnothing$ and $\mathbf{C} \setminus \mathrm{Sp}_{\lambda_0} T$ is non-thin at each point of N. In particular, since z_0 cannot lie in the interior of $\mathrm{Sp}_{\lambda_0} T$, the set $\bar{N}$ is contained in a component of $\mathbf{C} \setminus \mathrm{Sp}_1 T$ which is not entirely contained within $\mathrm{Sp}_{\lambda_0} T$. Therefore, by (2.4)(a) applied to T on the pair $[B_{\lambda_0}, B_1]$, there exists $t < 1$ such that

$$t \le \mathrm{Re}\,\lambda \le 1 \Rightarrow \mathrm{Sp}_\lambda T \cap N = \varnothing. \tag{15}$$

Now

$$\mathrm{Sp}_{\lambda_0} T \cup \mathrm{Sp}_1 T \subset \mathrm{Sp}_{\lambda_0} T \cup (\mathbf{C} \setminus N) = F, \quad \text{say},$$

so if we set

$$G_1 = \{\lambda \in G;\ \mathrm{Re}\,\lambda_0 < \mathrm{Re}\,\lambda < 1\},$$

then by (2.7)(b) and (2.2)(b) the restriction of $\mathrm{Sp}_\lambda T$ to G_1 is an a.m.v. function whose range is contained within F. Now $\mathbf{C} \setminus F$ is non-thin at every point of N, so by [**32**, Corollary 2.4(a)] we deduce that for $\lambda \in G_1$, the set $\mathrm{Sp}_\lambda T \cap N$ is independent of λ. In fact, from (15) we have

$$\mathrm{Sp}_\lambda T \cap N = \varnothing \qquad (\lambda \in G_1).$$

Since $z_0 \in \mathrm{Sp}_{\lambda_0} T$, this contradicts (3.5).

(b) By part (a),

$$\mathrm{Sp}_{\lambda_0}T \subset \mathrm{Sp}_{\lambda_2}T.$$

By (2.2)(b), we therefore have

$$\mathrm{Sp}_{\lambda_1}T \subset \mathrm{Sp}_{\lambda_0}T \cup \mathrm{Sp}_{\lambda_2}T = \mathrm{Sp}_{\lambda_2}T. \qquad \square$$

For example, this result will hold whenever $\mathrm{Sp}_{\lambda_0}T$ is a subset of the real numbers. Thus in particular it will be true if B_{λ_0} is a Hilbert space and $T|B_{\lambda_0}$ is self-adjoint; this special case is a theorem of M. G. Krein and V. V. Ševcik [**25, 37**].

As a final corollary, we deduce a second 'constancy condition' for $\mathrm{Sp}_\lambda T$. Whereas (3.2) achieved this via a fairly strong condition on the spectrum at one endpoint, the next result does so using a weaker condition but at both endpoints. It is related to some theorems of G. L. Krabbe [**21, 22**].

COROLLARY 3.7. *Let* $F = \mathrm{Sp}_0T \cup \mathrm{Sp}_1T$. *Each of the following conditions ensures that for* $\lambda \in G$, *the set* $\mathrm{Sp}_\lambda T$ *is independent of* λ:

(a) *T satisfies the U.R. condition, and* $\mathbf{C} \setminus F$ *is non-thin at every point of* F;

(b) *the set F has empty interior and connected complement.*

Proof. Take $\lambda_0, \lambda_1 \in G$: then by (2.2)(b)

$$\mathrm{Sp}_{\lambda_0}T \subset F,$$

so that $\mathbf{C} \setminus \mathrm{Sp}_{\lambda_0}T$ is non-thin at every point of $\mathrm{Sp}_{\lambda_0}T$. By (3.6)(b) we have

$$\mathrm{Sp}_{\lambda_0}T \subset \mathrm{Sp}_{\lambda_1}T.$$

and as $\lambda_0, \lambda_1 \in G$ were arbitrary, this proves the result.

(b) In fact, using (2.3)(b), the hypotheses of part (b) imply those of part (a). $\square$

Note added in proof. Some of the results above have been proved independently by Z. Słodkowski in his recent paper [**54**]. In particular, his work shows that the extra hypothesis in Theorem 2.7(b) may be omitted.

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