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MINIMAL NON-PERMUTATIVE PSEUDOVARIETIES OF SEMIGROUPS. I

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A semigroup is permutative if it satisfies an identity of the form $x_1x_2 \cdots x_n = x_{\sigma 1}x_{\sigma 2} \cdots x_{\sigma n}$ where σ is a non-identical permutation of $\{1, 2, \ldots, n\}$. The finite permutative semigroups form a pseudovariety and permutative pseudovarieties enjoy many properties first obtained for commutative pseudovarieties. Several types of permutation identities are considered, and all pseudovarieties minimal with respect to the property of containing a finite semigroup which fails an identity of a given type are determined. This includes the cases of the commutativity identity, the general permutation identities, and the "strong (left) permutation identities". As a preliminary, all minimal non-commutative pseudovarieties of groups and monoids are also determined.

1. Introduction. As a natural generalization of commutativity, permutation identities have been considered by several authors and shown to play an important role in various contexts (see Yamada [**19, 20**], Perkins [**12**], Pollák [**14, 15**], Higgins [**7, 8**], Almeida [**1**]). The non-collapse of this extended concept is peculiar to the class of semigroups, as a permutative monoid is necessarily commutative.

The problem of determining the minimal non-commutative pseudovarieties of groups was first considered by S. Oates (see Neumann [**11**], p. 42) although this author did not exhibit all such pseudovarieties, rather just established that they must contain a non-abelian metabelian group. As a step towards the solution of a problem in language theory, Margolis and Pin [**10**] then extended this result by showing that a non-commutative pseudovariety of monoids all of whose group members are abelian must contain one of three monoids which they describe. They also claim that their methods can be adjusted to yield a list of generators (up to the group case) of the minimal non-commutative pseudovarieties of semigroups. However, as we show here, their list is incomplete.

In the first part of this work, we determine explicitly all minimal non-commutative pseudovarieties of groups and semigroups. We also consider some special types of permutation identities and produce a complete list of minimal non-strongly permutative and non-strongly left permutative pseudovarieties of semigroups. This is based on the solution of the monoid case, together with the reduction of the statement that a semigroup S satisfies a certain kind of permutation identity to a much simpler statement of pseudo-identical type involving idempotents (cf. Reiterman [**18**]). In this way, we eliminate the bothersome presence of existential quantifiers on natural numbers.

All minimal pseudovarieties with respect to failing one of the above properties are characterized here both by a generator of minimum size and by a finite basis of identities.

Part II of this work presents a complete list of minimal non-permutative pseudovarieties of semigroups (with no restriction on the nature of the excluded permutation identities).

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2. Preliminaries. For general undefined terms and background, we refer the reader to Clifford and Preston [**3**] and Lallement [**9**]. For the notion of "pseudovariety" and its relationship with varieties in Birkhoff's sense, see Ash [**2**]. Finally, the definition of "implicit operation" on a class of algebras and its connection with pseudovarieties can be found in Reiterman [**18**].

Here, we will call *pseudo-identity* an expression of the form $\pi = \rho$, where π and ρ are implicit operations on the class of all finite semigroups. For a set Σ of pseudo-identities, $[\![\Sigma]\!]$ denotes the class of all finite semigroups satisfying Σ. By Theorem 3.1 [**18**], every pseudovariety is of the form $[\![\Sigma]\!]$ for some set Σ of pseudo-identities, while by Propositions 1.2 and 2.1 [**18**], every pseudovariety which is generated by a single semigroup is of the form $[\![\Sigma]\!]$ for some set Σ of identities.

For a finite semigroup S, $\underline{V}(S)$ denotes the pseudovariety generated by S. Thus $\underline{V}(S)$ consists of all homomorphic images of finite subdirect powers of S. As usual, S^1 stands for the smallest monoid containing S, i.e., $S^1 = S$ if S is a monoid, otherwise $S^1 = S \cup \{1\}$ where 1 acts as an identity. Next, S^r denotes the set S under the reverse operation $a * b = ba$, and $E(S)$ represents the set of all idempotents in S. If $A, B \subseteq S$, we let $AB = \{ab\colon a \in A,\ b \in B\}$.

For a set P, we let $|P|$ denote its cardinality.

Our first result is a simple sharpening of Proposition III. 9.2 of Eilenberg [**6**] and will be of crucial importance later.

PROPOSITION 2.1. *Let S be a finite semigroup and $E = E(S)$. Let $n = |S|$, $k = |E|$. Then, either $S^{n-k} = SES$ or E is a band ideal of S with S/E cyclic (and so $S^{n-k+1} = E$). In particular, $S^{n-k+1} = SES$.*

Proof. We first show that $S^{n-k+1} = SES$. Let $s_1, s_2, \ldots, s_{n-k+1} \in S$ and let $t_i = s_1 s_2 \cdots s_i$. If the elements t_i $(i = 1, \ldots, n-k+1)$ are all distinct, then $t_i = e \in E$ for some i and so $s_1 s_2 \cdots s_{n-k+1} = s_1 \cdots s_i \cdot e \cdot e s_{i+1} \cdots s_{n-k+1} \in SES$. If $t_i = t_j$ with $i < j$, let $(s_{i+1} s_{i+2} \cdots s_j)^l = e \in E$, to obtain $t_i = t_i e$ and so $s_1 s_2 \cdots s_{n-k+1} = s_1 \cdots s_i \cdot e \cdot s_{i+1} \cdots s_{n-k+1} \in SES$. Hence $S^{n-k+1} \subseteq SES$. Since $SES \subseteq S^r$ for all $r \geq 1$, we conclude that $S^{n-k+1} = SES$.

Next, suppose $S \neq E$, i.e., $k < n$, and further suppose that $S^{n-k} \neq SES$. Let $s_1, s_2, \ldots, s_{n-k} \in S$ be such that $s_1 s_2 \cdots s_{n-k} \notin SES$. By the same argument as above, it follows that the elements $t_i = s_1 s_2 \cdots s_i$ $(i = 1, \ldots, n-k)$ are all distinct and non-idempotent. Thus, $S = \{t_1, t_2, \ldots, t_{n-k}\} \cup E$ and E is an ideal of S. In particular, E is a band.

With the previous choice of elements $s_1, s_2, \ldots, s_{n-k} \in S$, if $s_i \neq s_1$ for some i, then, since $s_i \notin E$, we must have $s_i = t_j$ for some $j \in \{2, \ldots, n-k\}$. Hence, $s_1 s_2 \cdots s_{n-k} \in S^{n-k+1} = SES$, a contradiction. Therefore, $s_1 = s_2 = \cdots = s_{n-k}$ and so s_1 generates the Rees quotient S/E.

A *permutation identity* is an identity of the form

$$x_1 x_2 \cdots x_n = x_{\sigma 1} x_{\sigma 2} \cdots x_{\sigma n} \tag{1}$$

where σ is a permutation of $\{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_n$ are distinct variables. We say that a semigroup S is *permutative* if it satisfies some nontrivial (i.e., $\sigma \neq \iota$) permutation identity. S is *strongly left* (resp. *right*) *permutative* if it satisfies an identity (1) with $\sigma 1 \neq 1$ (resp. $\sigma n \neq n$). S is *strongly permutative* if it is both strongly left and right permutative. Note that a permutative monoid is commutative.

We denote by $\underline{\mathrm{Perm}}_{(k,m,l)}$ the pseudovariety defined by the identities of the form (1) with $n = k + m + l$ and $\sigma i = i$ for $1 \leq i \leq k$ and for $n - l + 1 \leq i \leq n$, where $k, l \geq 0$ and $m \geq 2$. Then, $\underline{\mathrm{Perm}}_{(0,2,0)} = \underline{\mathrm{Com}}$ is the class of all finite commutative semigroups, and we let

$$\underline{\mathrm{Perm}}_{(0,\infty,0)} = \bigcup_{m \geq 2} \underline{\mathrm{Perm}}_{(0,m,0)},$$

$$\underline{\mathrm{Perm}}_{(0,\infty,\infty)} = \bigcup_{m \geq 2,\, l \geq 0} \underline{\mathrm{Perm}}_{(0,m,l)}, \quad \text{and}$$

$$\underline{\mathrm{Perm}} = \bigcup_{k,\, l \geq 0,\, m \geq 2} \underline{\mathrm{Perm}}_{(k,m,l)}.$$

These classes of finite semigroups are all pseudovarieties and are identified by other means in the next section.

3. Simplification of permutativity conditions using idempotents. In this section, we take advantage of the abundance of idempotents in finite semigroups as expressed in Proposition 2.1 to obtain equivalent conditions to various types of permutativity.

LEMMA 3.1. *Let S be a strongly left permutative semigroup. Then, for any $e \in E(S)$, $s \in eSe$ and $t \in S$, we have $ts = ste$.*

Proof. Let S satisfy the identity (1) with $\sigma 1 \neq 1$. Let $\alpha = \sigma^{-1}(1)$ and let r, s, t be as in the statement of the lemma. Then

$$ts = te^{\alpha-2}se^{n-\alpha} = se^*te^* = ste.$$

LEMMA 3.2. *Let S be a finite semigroup with n elements and let $E = E(S)$. Suppose that all submonoids of S are commutative.*

(i) *If, for any $e \in E$ and $t \in S$, $te = ete$, then $S \in \underline{\mathrm{Perm}}_{(0,2,n-1)}$.*

(ii) *If every idempotent in S is central, then $S \in \underline{\mathrm{Perm}}_{(0,m,0)}$ where $m = \max\{2, n-1\}$.*

Proof. By Proposition 2.1, we have either $S^{n-1} = SES$, or $|E| = 1$ and S is commutative. Thus, we may assume that $S^{n-1} = SES$.

(i) Since $SE = ESE \subseteq ES$, we deduce $S^{n-1} = ES$. Let s_1, s_2, $t_1, \ldots, t_{n-1} \in S$. Then $t_1 t_2 \cdots t_{n-1} = et'$ for some $e \in E$ and $t' \in S$. Hence, $t_1 t_2 \cdots t_{n-1} = et_1 t_2 \cdots t_{n-1}$, and so

$$\begin{aligned} s_1 s_2 t_1 \cdots t_{n-1} &= s_1 s_2 e t_1 \cdots t_{n-1} \\ &= e s_1 e s_2 e t_1 \cdots t_{n-1} \quad \text{since } te = ete \\ &= e s_2 e s_1 e t_1 \cdots t_{n-1} \quad \text{since } eSe \text{ is commutative} \\ &= s_2 s_1 e t_1 \cdots t_{n-1} = s_2 s_1 t_1 \cdots t_{n-1}. \end{aligned}$$

This establishes $S \in \underline{\mathrm{Perm}}_{(0,2,n-1)}$.

(ii) If $n - 1 < 2$, the claim can be easily verified. So, suppose $m = n - 1$ and let $s_1, s_2, \ldots, s_m \in S$. Since $S^{n-1} = SES$ and idempotents are central, for each permutation σ of $\{1, 2, \ldots, m\}$, there exists $e_\sigma \in E$ such that $s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma m} = e_\sigma s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma m}$. Then, for every such σ, we have

$$\begin{aligned} s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma m} &= (e_\sigma s_{\sigma 1})(e_\sigma s_{\sigma 2}) \cdots (e_\sigma s_{\sigma m}) \quad \text{since } e_\sigma \text{ is central} \\ &= (e_\sigma s_1)(e_\sigma s_2) \cdots (e_\sigma s_m) \quad \text{since } e_\sigma S = e_\sigma S e_\sigma \text{ is commutative} \\ &= e_\sigma s_1 s_2 \cdots s_m, \end{aligned}$$

so that, if ι denotes the identity permutation of $\{1, 2, \ldots, m\}$, then

$$\begin{aligned} s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma m} &= e_\sigma s_1 \cdots s_m = e_\sigma e_\iota s_1 \cdots s_m = e_\iota e_\sigma s_1 \cdots s_m \\ &= e_\iota s_{\sigma 1} \cdots s_{\sigma m} = e_\iota s_1 \cdots s_m = s_1 \cdots s_m, \end{aligned}$$

as desired.

THEOREM 3.3. *The following conditions are equivalent for a finite semigroup S.*

(i) $S \in \underline{\mathrm{Perm}}_{(0,\infty,0)}$.

(ii) $S \in \underline{\mathrm{Perm}}_{(0,m,0)}$ *where* $m = \max\{2, |S| - 1\}$.

(iii) *S is strongly permutative.*

(iv) *All submonoids of S are commutative and every idempotent of S is central.*

(v) *Every element of S which lies in some submonoid of S is central in S.*

Proof. (ii) ⇒ (i) ⇒ (iii) follow directly from the definitions, while (v) ⇒ (iv) is obvious. (iii) ⇒ (v) follows from Lemma 3.1 and its left-right dual. Finally, (iv) ⇒ (ii) is Lemma 3.2(ii).

NOTATION. x^ω denotes the implicit unary operation on finite semigroups which associates with an element s of a finite semigroup S the unique idempotent in the subsemigroup of S generated by s.

COROLLARY 3.4. $\underline{\mathrm{Perm}}_{(0,\infty,0)} = [\![x^\omega yz = zyx^\omega]\!]$.

THEOREM 3.5. *The following conditions are equivalent for a finite semigroup S.*

(i) $S \in \underline{\mathrm{Perm}}_{(0,\infty,\infty)}$.

(ii) $S \in \underline{\mathrm{Perm}}_{(0,2,n-1)}$ *where* $n = |S|$.

(iii) *S is strongly left permutative.*

(iv) *All submonoids of S are commutative and, for any $e \in E(S)$ and $t \in S$, $te = ete$.*

(v) *For any $e \in E(S)$ and $s, t \in S$, $este = tse$.*

Proof. The proof proceeds along the same lines as the proof of Theorem 3.3 with the difference that in establishing (iii) ⇒ (v), one first uses Lemma 3.1 to obtain $te = ete$ and then deduces that

$$este = esete = etese = tse.$$

COROLLARY 3.6. $\underline{\mathrm{Perm}}_{(0,\infty,\infty)} = [\![x^\omega yzx^\omega = zyx^\omega]\!]$.

LEMMA 3.7. *Let S be a permutative semigroup. Then, for any e, $f \in E(S)$ and s, $t \in S$, we have $estf = etsf$.*

Proof. Let S satisfy the identity (1) with $k + 1 = \min\{i\colon \sigma i \neq i\}$ and $n - l = \max\{j\colon \sigma j \neq j\}$. Then

$$\begin{aligned} estf = e^{k-1}stf^{n-k-1} &= e^{k-1}sff^*tf^*f^l \quad \text{using (1)} \\ &= esftf. \end{aligned}$$

Hence,

$$\begin{aligned} estf = esftf &= (e \cdot es \cdot f)tf \\ &= e \cdot efs \cdot ftf \quad \text{by the above argument} \\ &= eftfsf \quad \text{since } fSf \text{ is commutative} \\ &= etsf \quad \text{by the above.} \end{aligned}$$

LEMMA 3.8. *Let S be a finite semigroup with n elements. Suppose that for any e, $f \in E = E(S)$ and s, $t \in S$, we have $estf = etsf$. Then $S \in \underline{\mathrm{Perm}}_{(n-1,2,n-1)}$.*

Proof. As in the proof of Lemma 3.2, we may assume that $S^{n-1} = SES$. Let $m = n - 1$ and let $s_1, \ldots, s_m, t_1, t_2, u_1, \ldots, u_m \in S$. Then, there exist e, $f \in E$ and s', s'', u', $u'' \in S$ such that $s_1 \cdots s_m = s'es''$ and $u_1 \cdots u_m = u'fu''$. Hence,

$$\begin{aligned} s_1 \cdots s_m t_1 t_2 u_1 \cdots u_m &= s'es''t_1t_2u'fu'' \\ &= s'es''t_2t_1u'fu'' \quad \text{since } estf = etsf \\ &= s_1 \cdots s_m t_2 t_1 u_1 \cdots u_m, \end{aligned}$$

as desired.

THEOREM 3.9. *The following are equivalent for a finite semigroup S.*

(i) $S \in \overline{\mathrm{Perm}}$.

(ii) $S \in \underline{\mathrm{Perm}}_{(n-1,2,n-1)}$ *where* $n = |S|$.

(iii) *S is permutative.*

(iv) *All submonoids of S are commutative and, for any e, $f \in E(S)$ and s, $t \in S$, $esf = esef$ and $este = esete$.*

(v) *For any e, $f \in E(S)$ and s, $t \in S$, $estf = etsf$.*

Proof. Here, it is now sufficient to check that (iv) $\Leftrightarrow$ (v). (v) $\Rightarrow$ (iv) is easily verified. Conversely, assume (iv) and let e, $f \in E(S)$ and s, $t \in S$. Then,

$$estf = estef = esetef = etesef = etsf,$$

as claimed.

COROLLARY 3.10.

$$\underline{\mathrm{Perm}} = [\![x^{\omega}yzt^{\omega} = x^{\omega}zyt^{\omega}]\!] = [\![x^{\omega}yzx^{\omega} = x^{\omega}zyx^{\omega}, x^{\omega}yz^{\omega} = x^{\omega}yx^{\omega}z^{\omega}]\!].$$

EXAMPLE 3.11. Let D be the semigroup with zero with presentation

$$\langle e, f;\ e = e^2,\ f = f^2,\ fe = 0\rangle.$$

Then $D = \{0, e, f, ef\}$ satisfies $xyzx = xzyx$ and so monoids in $\underline{V}(D)$ are commutative. However, $e \cdot ef \cdot f = ef \neq 0 = e \cdot fe \cdot f$, and so D is non-permutative by Theorem 3.9(v).

REMARK 3.12. If an arbitrary semigroup S satisfies an identity ε of the form (1) with $L(\varepsilon) = \min\{i\colon \sigma i \neq i\}$ and $R(\varepsilon) = n_{\varepsilon} - \max\{j\colon \sigma j \neq j\}$, then S satisfies all the identities δ of the form (1) with $L(\delta) \geq L(\varepsilon)$ and $R(\delta) \geq R(\varepsilon)$ for which the length n_{δ} is large enough (independently of S). This follows from the results of Putcha and Yaqub [**16**] (see also Pollák [**13**]).

REMARK 3.13. The bound $n - 1$ in parts (ii) of Theorems 3.3, 3.5 and 3.9 is sharp. For instance, if S is a two-element left-zero semigroup, then $S \in \underline{\mathrm{Perm}}_{(1,2,1)} \setminus \underline{\mathrm{Perm}}_{(0,2,0)}$.

4. Minimal non-commutative pseudovarieties of groups. Since a finite semigroup is a group if and only if it satisfies identities of the form $xy^n = x = y^n x$ for some n, every pseudovariety of groups can be viewed as a pseudovariety of semigroups. Moreover, as a pseudovariety which is generated by a single semigroup is defined by identities, it follows that the pseudovarieties indicated in the title of this section are essentially the same whether we work in the algebraic types of groups or semigroups. To simplify the notation, we will consider groups (rather than semigroups which are groups), thus referring to the unary operation of inversion.

A group is said to be *one-step non-commutative* if it is non-abelian but all its proper subgroups are abelian. We proceed to describe all such finite groups. In the following, p and q always denote prime numbers.

THEOREM 4.1. (*Rédei* [**17**].) (i) *The finite one-step non-commutative p-groups are the quaternion group* (*or order* 8), *the groups* (*of order* p^{m+n+1}) *defined by the relations*

$$(2) \quad a^{p^m} = b^{p^n} = 1, \quad c^p = 1, \quad ac = ca, \quad bc = cb,\ bab^{-1} = ac \qquad (m \geq n \geq 1)$$

and the group (of order p^{m+n}) defined by the relations

$$a^{p^m} = 1, \quad b^{p^m} = 1, \quad bab^{-1} = a^{1+p^{m-1}} \qquad (m \geq 2, n \geq 1). \tag{3}$$

(ii) *The remaining finite one-step non-commutative groups are obtained as follows. For two distinct primes p, q and natural number n, let m denote the multiplicative order of p modulo q and let $F = \mathrm{GF}(p^m)$. Choose $\omega \in F^*$ of order q. Take for the group the semidirect product $F * \mathbf{Z}_{q^n}$ with product given by*

$$(\alpha, a)(\beta, b) = (\alpha + \omega^a\beta, a + b) \quad (\alpha, \beta \in F,\ a, b \in \mathbf{Z}_{q^n}). \tag{4}$$

LEMMA 4.2. (i) *If G and G' are two p-groups defined by* (2) *with parameters m, n and m', n' respectively with $m \leq m'$ and $n \leq n'$, then G is a homomorphic image of G'.*

(ii) *If H and H' are two groups defined by* (4) *relative to the same pair of primes, with parameters n and n' respectively and $n \leq n'$, then H is a homomorphic image of H'.*

Proof. (i) The subgroup K of G' generated by $\{a^{p^{m'-m}}, b^{p^{n'-n}}\}$ is central and $G'/K \simeq G$.

(ii) The subgroup of H' generated by $(0, q^{n'-n})$ is central and $H'/K \simeq H$.

We let $[x, y] = x^{-1}y^{-1}xy$, $[x_1, x_2, \ldots, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}]$, $[x, (y)_1] = [x, y]$, and $[x, (y)_{n+1}] = [[x, (y)_n], y]$. The following commutator identities are easily verified (in any group):

$$[x, y] = [y, x]^{-1}$$
$$xy = yx[x, y]$$
$$[x, yz] = [x, z]z^{-1}[x, y]z = [x, z][x, y][x, y, z].$$

We denote the group given by (2) with $m = n = 1$ by G_p, while $H_{p,q}$ represents the group given by (4) when $n = 1$.

LEMMA 4.3. (i) *The pseudovariety generated by any group K in* (3) *is given by*

$$\underline{V}(K) = [\![x^{p^s} = 1, [x, y]^p = 1, [x, y, z] = 1]\!] \tag{5}$$

where $s = \max\{m, n\}$.

(ii) *The pseudovariety generated by the group G_p is given by*

$$\underline{V}(G_p) = [\![x^p = 1, [x, y, z] = 1]\!] \quad \text{in case } p > 2, \tag{6}$$

$$\underline{V}(G_2) = [\![x^4 = 1, [x, y]^2 = 1, [x, y, z] = 1]\!]. \tag{7}$$

Proof. It is easily checked that K satisfies the identities in (5). Using those identities, any word w in the variables $x_1, x_2, \dots, x_n$ can be reduced to one of the form $x_1^{u_1} \cdots x_r^{u_n} c_1^{v_1} c_2^{v_2} \cdots c_t^{v_t}$ where the c_k are distinct commutators of the form $[x_i, x_j]$ with $i < j$. Then, one checks that K satisfies $w = 1$ if and only if $p^s | u_i$ $(i = 1, \dots, r)$ and $p | v_j$ $(j = 1, \dots, t)$. This establishes (i). The proof of (ii) is similar.

PROPOSITION 4.4.

$$(8)\quad \underline{V}(H_{p,q}) = [\![x^{pq} = 1, [x^q, y^q] = [x, y]^p = 1,$$
$$[y, x, z][x, y, z] = 1, [x, y, z, t] = [x, y, t, z],$$
$$[x, y]^q [x, (y)_2]^{\binom{q}{2}} \cdots [x, (y)_{q-1}]^{\binom{q}{q-1}} [x, (y)_q] = 1]\!].$$

Proof. We first observe that $(\alpha, a)^{-1} = (-\omega^{-a}\alpha, -a)$ while, for $r \geq 1$,

$$(\alpha, a)^r = \left((1 + \omega^a + \omega^{2a} + \cdots + \omega^{(r-1)a})\alpha, ra\right).$$

Further, if $u = (\alpha, a)$, $v = (\beta, b)$, then

$$[u, v] = \left((\omega^{-b} - 1)\omega^{-a}\alpha - (\omega^{-a} - 1)\omega^{-b}\beta, 0\right);$$

in particular, $[(\alpha, 0), (\beta, b)] = ((\omega^{-b} - 1)\alpha, 0)$. Also, note that, in $F[X]$,

$$(X+1)^q - 1 = qX + \binom{q}{2}X^2 + \binom{q}{3}X^3 + \cdots + \binom{q}{q-1}X^{q-1} + X^q$$

admits the factorization $\prod_{a=0}^{q}(X - \omega^a + 1)$. Using these remarks, it is easy to check that $H_{p,q}$ satisfies the identities in (8).

Now, let w be a word in the variables $x_1, x_2, \dots, x_n$ and suppose that $H_{p,q}$ satisfies $w = 1$. Note that, as a consequence of the identities in (8), we have that commutators commute: indeed, those identities imply that every commutator is a qth power. It follows that, using the identities in (8), w can be reduced to a word of the form $x_1^{u_1} x_2^{u_2} \cdots x_r^{u_r} c_1^{v_1} \cdots c_t^{v_t}$ where the c_i are distinct commutators of the form $[x_{i_1}, x_{i_2}, \dots, x_{i_s}]$. Since $H_{p,q}$ satisfies $w = 1$, we immediately deduce that $pq | u_i$ $(i = 1, \dots, r)$ and so $H_{p,q}$ satisfies $w' = 1$ where $w' = c_1^{v_1} \cdots c_t^{v_t}$. Using the last three identities in (8), we may assume that for any $c_j = [x_{i_1}, x_{i_2}, \dots, x_{i_s}]$ in w', the number of occurrences of any variable in c_j is less than q. Substituting $(1, 0)$ for x_{i_0} and $(0, -b_i)$ for x_i $(i \neq i_0)$, the value assumed by w' is of the form $(P(\omega^{b_1} - 1, \dots, \omega^{b_{i_0-1}} - 1, \omega^{b_{i_0+1}} - 1, \dots, \omega^{b_r} - 1), 0)$, where $P(X_1, \dots, X_{i_0-1}, X_{i_0+1}, \dots, X_r)$ is a polynomial of degree less than q on each variable. Since that value is 0 for any choice of $b_i \in \mathbf{Z}_q$ $(i \neq i_0)$, it follows that P is identically zero in $F[X_1, \dots, X_{i_0-1}, X_{i_0+1}, \dots, X_r]$ and thus, the exponents of the commutators $c_j = [x_{i_1}, x_{i_2}, \ldots, x_{i_r}]$ in w' for which $i_0 = i_1$ or $i_0 = i_2$, but $i_0 \neq i_3, \ldots, i_r$, are multiples of p. Hence, we may cancel all these c_j for each value of i_0, yielding a word w''.

At this point, we may assume that we have an identity $w'' = 1$ which holds in $H_{p,q}$, where w'' is a product of commutators of the form $[x_{i_1}, x_{i_2}, x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_r}]$ with $i_1 < i_2$. Now, if we substitute $(\omega^{-a_1}, -a_1)$ for x_1 and $(\omega^{-a_2}\lambda, -a_2)$ for x_2 we obtain

$$[x_1, x_2] = ((\omega^{a_2} - 1) - (\omega^{a_1} - 1)\lambda, 0) = (\alpha, 0).$$

Hence, any such substitution together with the assignment of the value $(0, -a_i)$ for x_i $(i > 2)$ gives the value $(\alpha P(\omega^{a_i} - 1, \ldots, \omega^{a_r} - 1), 0)$ for w'' where $P(X_1, \ldots, X_r)$ is a polynomial of degree less than q on each variable and independent of λ. With two appropriate choices for λ (such as $\lambda = 1$ and $\lambda = \omega$), and recalling that $w'' = 1$ holds in $H_{p,q}$, we deduce that $P(\omega^{a_i} - 1, \ldots, \omega^{a_r} - 1) = 0$ for all choices of $a_1, \ldots, a_r \in \mathbf{Z}_q$. Because of the degree restrictions on P, it follows that P is identically zero in $F[X_1, \ldots, X_r]$. Hence, the exponents of the commutators $[x_1, x_2, \ldots]$ in w'' are all multiples of p, and so we can cancel these commutators using the identity $[x, y]^p = 1$. Since this argument can be repeated for any pair of variables, we conclude that $w'' = 1$ is a consequence of the identities in (8).

This shows that any identity which holds in $H_{p,q}$ follows from the identities in (8) and completes the proof of the proposition.

THEOREM 4.5. *The minimal non-commutative pseudovarieties of groups are* $\underline{V}(G_p)$ *and* $\underline{V}(H_{p,q})$ *with* p, q *distinct primes. These pseudovarieties are all distinct and every non-commutative pseudovariety of groups contains one of them.*

Proof. Suppose $\underline{W}$ is a non-commutative pseudovariety of groups. Let $G \in \underline{W}$ be a non-abelian group of minimal order. Then G is one-step non-commutative and so G is isomorphic to one of the groups listed in Theorem 4.1. By Lemma 4.2 and 4.3, together with the remark that the quaternion group generates the pseudovariety in (7), we then have that $\underline{W}$ contains one of the pseudovarieties $\underline{V}(G_p)$ or $\underline{V}(H_{p,q})$.

Finally, using (2), (4) and (6)–(8), one can easily check that the particular pseudovarieties in the statement of the theorem are all distinct.

COROLLARY 4.6. G_p *(resp.* $H_{p,q}$*) is a generator of* $\underline{V}(G_p)$ *(resp.* $\underline{V}(H_{p,q})$*) of minimum size.*

5. Minimal non-commutative and minimal non-strongly left permutative pseudovarieties of semigroups. We start by introducing some semigroups which will be useful in the sequel.

DEFINITION 5.1.

$$N = \langle a, b;\ a^2 = b^2 = ba = 0 \rangle,$$
$$T = \langle e, a;\ e = e^2, a = ea, ae = 0 \rangle,$$
$$B(m, n) = \{1, \dots, m\} \times \{1, \dots, n\} \text{ under } (i, j)(k, l) = (i, l).$$

Also, recall the semigroup D of Example 3.11. Note that D embeds both T and T^r. Further, the following equalities can be verified directly, or else encountered in the literature (cf. Edmunds [**4, 5**]):

$$\underline{V}(N) = [\![xyz = 0 = x^2]\!]$$
$$\underline{V}(B(1,2)) = [\![xy = y]\!]$$
$$\underline{V}(T) = [\![x^2y = xy,\ xy^2 = yx^2]\!]$$
$$\underline{V}(N^1) = [\![x^2 = x^3, x^2y = xyx = yx^2]\!]$$
$$\underline{V}\big(B(1,2)^1\big) = [\![x = x^2, xy = yxy]\!].$$

Moreover, the indicated semigroups are generators of minimum size of the pseudovarieties they generate.

The following result summarizes Lemmas 2.2-2.6 of Margolis and Pin [**10**].

PROPOSITION 5.2. (i) *Let $\underline{V}$ be a pseudovariety of monoids such that all groups in $\underline{V}$ are abelian and N^1, $B(1,2)^1$, $B(2,1)^1$ do not lie in $\underline{V}$. Let $M \in \underline{V}$. Then $\mathscr{H} = \mathscr{J}$ is a congruence on M and, if a, $b \in M$ and $ab \neq ba$, then $ab \not\!\mathscr{H} ba$ in the submonoid of M generated by $\{a, b\}$.*

(ii) *Part* (i) *remains valid if we replace the word "monoid" by "semigroup" and the monoids N^1, $B(1,2)^1$, $B(2,1)^1$ by the semigroups N, $B(1,2)$, $B(2,1)$.*

Here, we present an alternate way of concluding the proof of the characterization of minimal non-commutative pseudovarieties of monoids given in [**10**], thus avoiding certain language theoretic arguments.

THEOREM 5.3. *The minimal non-commutative pseudovarieties of monoids are the ones of Theorem 4.5 together with $\underline{V}(N^1)$, $\underline{V}(B(1,2)^1)$ and $\underline{V}(B(2,1)^1)$. These pseudovarieties are all distinct and every non-commutative pseudovariety of monoids contains one of them.*

Proof. In view of Theorem 4.5 and the defining identities for each of the given pseudovarieties, it suffices to show that, if a pseudovariety $\underline{W}$ does not contain any of the pseudovarieties in our list, then $\underline{W}$ is commutative.

Suppose $M \in \underline{W}$, a, $b \in M$ and $ab \neq ba$. We may assume that $\{a, b\}$ generates M and, by Proposition 5.2(i), that M is $\mathscr{J}$-trivial. We show that then necessarily $N^1 \in \underline{W}$ and thus obtain a contradiction.

If ab, $ba \in E(M)$, then $abab = ab\mathscr{R}aba\mathscr{L}ba = baba$, and so $ab = aba = ba$. Suppose then ab is not an idempotent. If ba is an idempotent, then $ba <_{\mathscr{J}} ab$. Since $ab \neq ba$ and M is $\mathscr{J}$-trivial, either $ab \nleq_{\mathscr{J}} ba$ or $ba \nleq_{\mathscr{J}} ab$. Thus, we may assume that $ab \nleq_{\mathscr{J}} ba$.

Let $I = \{x \in M\colon\ ab \nleq_{\mathscr{J}} x\}$, an ideal of M. Let $M' = M/I$ and $M'' = M' \times M' \times M'$. Let I'' be the ideal of M'' consisting of all triples with at least one zero component. Finally, let $M''' = M''/I''$ and let $x = (a, 1, b)$, $y = (b, ab, 1)$. Then $x^2 = y^2 = yx = 0$ but 1, x, y, xy, and 0 are all distinct in M'''. Hence, $N^1 \simeq \{1, x, y, xy, 0\} \subseteq M'''$ and so $N^1 \in \underline{W}$.

The next result corrects Theorem 2.8 **[10]**.

THEOREM 5.4. *The minimal non-commutative pseudovarieties of semigroups are the ones of Theorem* 4.5 *together with* $\underline{V}(N)$, $\underline{V}(B(1,2))$, $\underline{V}(B(2,1))$, $\underline{V}(T)$, *and* $\underline{V}(T^r)$. *These pseudovarieties are all distinct and every non-commutative pseudovariety of semigroups contains one of them.*

Proof. As in the proof of Theorem 5.3, we only need to consider a pseudovariety $\underline{W}$ (of semigroups) which does not contain any of the pseudovarieties in the given list and show that $\underline{W}$ must be commutative.

Suppose $S \in \underline{W}$, a, $b \in S$, and $ab \neq ba$. Again, we assume that S is generated by $\{a, b\}$ and, in view of Proposition 5.2(ii), that S is $\mathscr{J}$-trivial. And, just as in the monoid case, we may here assume that ab is not an idempotent, $ab \nleq_{\mathscr{J}} ba$, and that the ideal $\{x \in S\colon\ ab \nleq_{\mathscr{J}} x\}$ of S is reduced to zero.

Let $e = a^{\omega}$, $f = b^{\omega}$ (cf. §3). If $a^n = ab$, then $a^{n+1} = aba = 0$, so that $e = 0$. Similarly, if $b^m = ab$, then $f = 0$. If $e \neq 0$, then $ab = xey$ for some x, $y \in S$; since $ba = 0$, it follows that $x = ea^i$ for some $i > 0$, and so $eab = ab$ and $ab = eb$ (as then $ab\mathscr{L}eb$). Similarly, if $f \neq 0$, then $abf = ab = af$.

There are several cases to be considered.

(i) $e \neq 0$ and $f \neq 0$. Then, by the above $ab = ef$. Thus, $\{e, f, ef, 0\} \simeq D$, so that T, $T^r \in \underline{W}$.

(ii) $e \neq 0$ and $f = 0$. Then $\{e, eb, 0\} \simeq T$ and so $T \in \underline{W}$.

(iii) $e = 0$ and $f \neq 0$. Then $\{f, af, 0\} \simeq T^r$ and so $T^r \in \underline{W}$.

(iv) $e = 0$ and $f = 0$. Consider the subsemigroup S' of $S \times S \times S$ generated by the elements $\alpha = (a, a, ab)$ and $\beta = (b, ab, b)$. Then $\alpha\beta = (ab, 0, 0) \neq 0 = (0,0,0) = \beta\alpha$. Let $I' = \{x \in S'\colon \alpha\beta \nleq_{\mathscr{J}} x\}$, an ideal of S'. Note that α^2, β^2, $\beta\alpha \in I'$, while α, β, and $\alpha\beta$ are distinct elements of $S' \setminus I'$. Hence, $S'/I' \simeq N$ and $N \in \underline{W}$.

In all cases we reach a contradiction, whence the proof is complete.

LEMMA 5.5. *Let $\underline{W}$ be a pseudovariety of semigroups and $S \in \underline{W}$. Let $e \in E(S)$ and $s \in S$.*

(i) *If $se \not\mathscr{J} ese$, then $T^r \in \underline{W}$.*

(ii) *If $se \mathscr{J} ese$, then either $se = ese$ or $B(2,1) \in \underline{W}$.*

Proof. (i) Suppose that $se \not\mathscr{J} ese$. Then $ese <_{\mathscr{J}} se$. Consider the Rees quotient S' of the subsemigroup of S generated by $\{e, s\}$ by its ideal $\{x\colon se \nleq_{\mathscr{J}} x\}$. Then $\{e, se, 0\} \simeq T^r$ and so $T^r \in \underline{W}$.

(ii) Suppose that $se \mathscr{J} ese$. Then $se \mathscr{L} ese$, say $se = tese$ for some $t \in S$. Let $f = (te)^{\omega}$. Then $se = fse$ and $fe = f$. Hence, $fef = f$ and so $ef \in E(S)$ and $ef \mathscr{L} f$. Thus, either $ef \neq f$ and $B(2,1) \simeq \{ef, f\} \in \underline{W}$, or $ef = f = fe$. In the latter case,

$$se = fse = efse = ese.$$

THEOREM 5.6. *A pseudovariety of semigroups is not strongly permutative if and only if it contains one of the pseudovarieties in Theorem 4.5 or one of $\underline{V}(N^1)$, $\underline{V}(B(1,2))$, $\underline{V}(B(2,1))$, $\underline{V}(T)$, or $\underline{V}(T^r)$. Moreover, these pseudovarieties are distinct minimal non-strongly permutative.*

Proof. It remains to show the only if part of the statement. Suppose $\underline{W}$ is a pseudovariety of semigroups which does not contain any of the pseudovarieties listed in the theorem. By Theorems 3.3(iv) and 5.3, it suffices to show that if $S \in \underline{W}$, $s \in S$ and $e \in E(S)$, then $es = se$. But, by Lemma 5.5 and its left-right dual, we have $es = ese = se$, as desired.

THEOREM 5.7. *A pseudovariety of semigroups is not strongly left permutative if and only if it contains one of the pseudovarieties in Theorem 4.5 or one of $\underline{V}(N^1)$, $\underline{V}(B(1,2)^1)$, $\underline{V}(B(2,1))$, or $\underline{V}(T^r)$. Moreover, these pseudovarieties are distinct minimal non-strongly left permutative.*

Proof. The result now follows from Theorems 3.5(iv) and 5.3 along with Lemma 5.5.

REMARK 5.8. Theorem 5.6 may also be deduced from Theorem 5.7 and its left-right dual.

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