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ONE-DIMENSIONAL ALGEBRAIC FORMAL GROUPS

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Let K be an algebraically closed field of characteristic zero. We shall call an element of $K[[x_1,\dots,x_n]]$ algebraic if it is algebraic over $K(x_1,\dots,x_n)$. Thus a one-dimensional algebraic formal group is an element $F \in K[[x_1, x_2]]$ such that F is a formal group and F is algebraic. As is well known, such formal groups arise from one-dimensional algebraic groups. Our intention is to show that this is the only way they arise. All formal groups mentioned in this note shall be one-parameter formal groups.

DEFINITION. Two algebraic formal groups $F, F' \in K[[x_1, x_2]]$ are said to be algebraically isomorphic if there exists an algebraic element $f \in xK[[x]]$ such that $f \neq 0$ and

$$f(F(x_1, x_2)) = F'(f(x_1), f(x_2)).$$

It is easy to see that there exists a unique element $f^* \in xK[[x]]$ such that $f \circ f^* = x$. It then follows that

$$f^*F'(x_1, x_2) = F(f^*(x_1), f^*(x_2))$$

and that f^* is algebraic.

Now suppose $(X, e, [+])$ is a one-dimensional algebraic group over K. Let $z \in K(X)$ be a local parameter at e. Let ρ_1, ρ_2: $X \times X \to X$ be the natural projections. Then $\{z \circ \rho_1, z \circ \rho_2\}$ is a set of local parameters at $e \times e$ in $X \times X$, and so there exists a unique power series $H(x, y) \in K[[x, y]]$ such that

$$H(z \circ \rho_1, z \circ \rho_2) = z(\rho_1 [+] \rho_2)$$

as elements of the complete local ring at $e \times e$ on $X \times X$. It is easy to see that H is an algebraic formal group. We shall call such a formal group a *formal algebraic group*.

PROPOSITION A. *Every algebraic formal group is algebraically isomorphic to a formal algebraic group.*

We will prove a stronger statement than Proposition A. We call a differential $\omega \in K[[x]]\,dx$ algebraic if ω/dx is an algebraic element of $K[[x]]$. If $H(x, y)$ is a formal group and

$$g(x) = \frac{d}{dy}H(x, y)\Big|_{y=0},$$

then $g(0) = 1$, and

$$\omega = g\,dx$$

is the invariant differential of H. If H is an algebraic, then so is ω. We will prove

PROPOSITION B. *Let ω be an algebraic differential. Suppose that there exist nonzero algebraic elements f_1, f_2 of $xK[[x]]$ such that*

$$f_1^*(\omega) = af_2^*(\omega)$$

where $a \in \mathbf{C}^*$, a is not a root of unity. Then there exist a formal algebraic group with invariant differential ω' and an algebraic element u of $K[[x]]$ such that

$$eu^*(\omega') = \omega$$

where $e = \mathrm{Res}_0(\omega/x)$.

To deduce Proposition A from Proposition B, let F be an algebraic formal group, ω its invariant differential, $f_2(x) = x$, $f_1(x) = F(x, x)$. Then

$$f_1^*(\omega) = 2\omega = 2f_2^*(\omega). \tag{0}$$

It follows that there exists a formal algebraic group H with invariant differential ω' and an algebraic element $g \in xK[[x]]$ such that

$$g^*(\omega') = \omega. \tag{1}$$

We claim

$$g(F(x, y)) = H(g(x), g(y)).$$

Indeed, if $\lambda, \lambda' \in xK[[x]]$, $d\lambda = \omega$, $d\lambda' = \omega'$, then (1) implies $\lambda' \circ g = \lambda$. On the other hand,

$$\lambda F(x, y) = \lambda(x) + \lambda(y)$$

$$\lambda' H(x, y) = \lambda'(x) + \lambda'(y),$$

so that

$$\begin{aligned} gF(x, y) &= \lambda'^{-1} \circ \lambda F(x, y) = \lambda'^{-1}(\lambda(x) + \lambda(y)) \\ &= H(\lambda'^{-1} \circ \lambda(x) + \lambda'^{-1} \circ \lambda(y)) = H(g(x), g(y)) \end{aligned}$$

as required.

Proof of Proposition B. Let $\mathbf{P}^1$ denote the projective line over K and regard x as the standard parameter on $\mathbf{P}^1$. In doing this we will identify $K[[x]]$ with the formal completion of the ring of functions on $\mathbf{P}^1$ regular at 0, $\widehat{\mathcal{O}_{\mathbf{P}^1,0}}$.

Let $f_0 = \omega/dx$. Then for $i = 0, 1, 2$ there exist complete pointed curves (X_i, e_i) over K together with morphisms

$$x_i, \tilde{f}_i \colon Y_i \to \mathbf{P}^1$$

such that x_i is a local uniformizing parameter at e_i and $x_i^* f_i$ is the formal expansion of $\tilde{f}_i$ in x_i at e_i. In other words, $x_i^* f_i$ is the image of f_i in $\mathcal{O}_{Y_i, e_i}$.

Now set $\tilde{\omega} = \tilde{f}_0\, dx_0 \in \Omega^1_{Y_0/k}$. Also note that $f_i(e_i) = 0$ as $f_i(0) = 0$, $i = 1, 2$. Let (Z_i, e_i) denote the fiber product of (Y_0, e_0) and (Y_i, e_i) over $(\mathbf{P}^1, 0)$ with respect to the morphisms x_0 and $\tilde{f}_i$, $i = 1, 2$. Thus (Z_i, e_i') fits into a commutative diagram

$$\begin{array}{ccc} (Z_i, e_i') & \xrightarrow{y_i} & (Y_i, e) \\ \tilde{\tilde{f}}_i \downarrow & & \downarrow \tilde{f}_i \\ (Y_0, e_0) & \xrightarrow{x_0} & (\mathbf{P}^1, 0). \end{array}$$

Moreover, $(x_i \circ y_i)^* f_i^* \omega$ is the formal expansion of $\tilde{\tilde{f}}_i^* \tilde{\omega}$ at e_i' in $x_i \circ y_i$. Now let (W, e) denote the fiber product of (Z_1, e_1') and (Z_2, e_2') with respect to the morphisms $x_1 \circ y_1$ and $x_2 \circ y_2$. Thus we have a commutative diagram

$$\begin{array}{ccc} (W, e) & \xrightarrow{z_2} & (Z_2, e_2') \\ z_1 \downarrow & & \downarrow x_2 \circ y_2 \\ (Z_1, e_1) & \xrightarrow{x_1 \circ y_1} & (\mathbf{P}^1, 0). \end{array}$$

Let (W^c, e) denote the connected component of (W, e) passing through e. Let

$$\bar{f}_i \colon (W^c, e) \to (Y_0, e_0)$$

denote the restriction of $\tilde{\tilde{f}}_i \circ z_i$ to W^c. Then

$$(x_i \circ y_i \circ z_i)^* f_i^* \omega$$

is the formal expansion of $\bar{f}_i^*\tilde{\omega}$ at e in $x_i \circ y_i \circ z_i$. Since $x_1 \circ y_1 \circ z_1 = x_2 \circ y_2 \circ z_2$, it follows from the hypothesis that

$$\bar{f}_1^*\tilde{\omega} = a\bar{f}_2^*\tilde{\omega}.$$

Taking $X_1 = X_0$, $X_2 = W^c$ and $\omega_1 = \tilde{\omega}$ we see that Proposition B follows from:

PROPOSITION C. *Let X_1, X_2 be two curves. Let ω_1 be a nonzero differential on X_1 and f_1, f_2 two nonconstant morphisms from X_2 to X_1 such that*

$$f_1^*(\omega_1) = af_2^*(\omega_1) \tag{2}$$

for some $a \in K^*$, a not a root of unity. Then there exists a one-dimensional algebraic group G with invariant differential ω, and a morphism $f\colon X_1 \to G$ such that

$$f^*(\omega) = \omega_1.$$

Proof. For a curve C let $\overline{C}$ denote its complete nonsingular model. Let $\omega_2 = f_2^*(\omega_1)$. Let S_i denote the set of poles of ω_i on $\overline{X}_i$. Clearly, $|S_1| \le |S_2|$, $|S_i|$ denotes the order of S_i. We also claim:

$$g(X_1) < g(X_2) \quad \text{or} \quad g(X_2) \le 1$$

where $g(X_i)$ denotes the genus of X_i. Indeed, if this is not the case, then by the Hurwitz genus formula we see that $g(X_1) = g(X_2) > 1$ and $1 = \deg(f_1) = \deg(f_2)$. but then $\bar{f}_i\colon \overline{X}_2 \to \overline{X}_1$ is biregular ($\bar{f}_i$ is the "lifting" of f_i), so that $\alpha = \bar{f}_2^{-1} \circ \bar{f}_1$ is an automorphism of X_2. But α is of finite order since $g(X_2) > 1$. On the other hand, the hypotheses of the lemma imply

$$\alpha^*(\omega_2) = a\omega_2.$$

Since a is not a root of unity, we obtain a contradiction, so we have our claim.

We also claim that there exists a curve X_0 with a differential ω_0 and two morphisms $g_1, g_2\colon X_1 \to X_0$ such that $g_2^*(\omega_0) = \omega_1$ and $g_1^*(\omega_0) = ag_2^*(\omega_0)$. Thus (X_0, ω_0) satisfies the same hypotheses as (X_1, ω_1), so once we establish this claim, we will be able to use induction to suppose that $|S_1| = |S_2|$ and $g(X_2) \le 1$.

For the results on generalized Jacobians used below, see [S].

Proof of Claim. Without loss of generality X_i is nonsingular, ω_i has no poles on X_i, and $f_i X_2 = X_1$, for $i = 1, 2$.

Let $i = 1$ or 2 in the following: Let M_i denote the polar divisor of ω_i. Let J_i denote the generalized Jacobian of X_i corresponding to M_i. There exists a unique invariant differential ν_i on J_i and an embedding of X_i in J_i (as $\omega_i \neq 0$) well defined up to translation such that ω_i is the pullback of ν_i to X_i. Henceforth we will view X_i as a subvariety of J_i. From the functoriality of generalized Jacobians there exists a canonical affine transformation

$$f_i'\colon J_2 \to J_1$$

whose restriction to X_2 is f_i. Let T_i denote translation on J_2 by $[-]f_i'(0)$ where $[-]$ denotes inversion on J_1. Set $f_i'' = T_i \circ f_i'$. Then f_i'' is a homomorphism from J_2 to J_1. It follows that

$$\left(f_1''\right)^*\nu_1 = a\left(f_2''\right)^*\nu_1 = a\nu_2.$$

There also exists a homomorphism $h\colon J_1 \to J_2$ such that

$$f_2'' \circ h = [d]$$

where d denotes the degree of f_2 and $[d]$ denotes multiplication by d on J_1. Let

$$e = \left(f_1'' \circ h \circ f_2'' - [d] \circ f_1''\right)\colon J_2 \to J_1.$$

Then e is a homomorphism and

$$\begin{aligned} e^*\nu_1 &= \left(f_2''\right)^*h^*\left(f_1''\right)^*\nu_1 - g_1^*[d]^*\nu_1 = a\left(f_2''\right)^*h^*\nu_2 - dg_1^*\nu_1 \\ &= a\left(f_2''\right)^*h^*f_2^*\nu_1 - da\nu_2 = a\left(f_2''\right)^*[d]^*\nu_1 - da\nu_2 = 0. \end{aligned}$$

Let A denote the quotient of J_1 by $e(J_2)$ and $\rho\colon J_1 \to A$ the quotient morphism. Since $e^*\nu_1 = 0$, it follows that there exists an invariant differential ν_0 on A such that $\rho^*\nu_0 = \nu_1$. Let

$$X_0 = \left(\rho \circ [d] \circ T_1\right)(X_1) \subseteq A.$$

As $\rho \circ e = 0$ we have $\rho \circ [d] \circ f_1'' = \rho \circ f_1'' \circ h \circ f_2''$. Hence as $f_1'(X_2) = f_2'(X_2) = X_1$,

$$\begin{aligned} X_0 &= \left(\rho \circ [d] \circ T_1 \circ f_1'\right)(X_2) \\ &= \left(\rho \circ [d] \circ f_1''\right)(X_2) = \left(\rho \circ f_1'' \circ h \circ f_2''\right)(X_2) \\ &= \left(\rho \circ f_1'' \circ h \circ T_2\right)(X_1). \end{aligned}$$

Now let $g_1, g_2\colon X_1 \to X_0$ denote the restrictions of

$$\rho \circ f_1'' \circ h \circ T_2 \quad \text{and} \quad \rho \circ [d] \circ T_1$$

respectively to X_1. Also let ω_0 denote the restriction of ν_0/d to X_0. Since $(\rho \circ f_1'' \circ h)^*\nu_0 = (f_1'' \circ h)^*\nu_1 = \mathrm{ad}\,\nu_1 = a(\rho \circ [d])^*\nu_0$ it follows that

$$g_1^*\omega_0 = ag_2^*\omega_0 = a\omega_1 \tag{2}$$

and so we have our claim. Thus by induction we may suppose

$$g(X_1) = g(X_2) \leq 1 \quad \text{and} \quad |S_2| = |S_1|.$$

We also have $f_i^{-1}(S_1) = S_2$, so that f_i induces a bijection from S_2 onto S_1.

Case 1. $g(X_i) = 1$. Then $\overline{X}_i$ has a unique group structure with origin at some point P_i. It follows that f_2 and $T_R \circ f_1$ are affine transformations from X_2 to X_1. Now since $f_i|_{S_2}$: $S_2 \to S_1$ is a bijection and $f_i^{-1}(S_1) = S_2$, it follows that either

$$S_2 = S_1 = \varnothing$$

or degree $f_i = 1$, $i = 1, 2$, because f_i is étale. In the second case, f_2^{-1} exists and $\alpha = f_2^{-1} \circ f_1$ is an automorphism of X_2 such that $\alpha S_2 = S_2$. But if $S_2 \neq \varnothing$, α is of finite order. This contradicts

$$\alpha^*\omega_2 = a\omega_2.$$

Thus $S_1 = S_2 = \varnothing$, and ω_1 is an invariant differential on X_1 as required.

Case 2. $g(X_i) = 0$. Then $|S_i| \geq 1$. Let

$$A = \begin{cases} \{\infty\} & \text{if } |S_1| = 1, \\ \{\infty, 0\} & \text{if } |S_1| = 2, \\ \{\infty, 0, 1\} & \text{if } |S_1| \geq 3. \end{cases}$$

After composing with linear fractional transformations, we may suppose $A \subseteq S_2$ and $A \subseteq S_1$.

If $|S| = 1$, then $\omega_1 = b\,dx$ for some $b \in K^*$, and so is an invariant differential on $\mathbf{G}_a$. Now suppose $|S_2| \geq 1$. Let h_i be a linear fractional transformation such that

$$h_i \circ f_i(p) = p, \qquad p \in A.$$

Because $(h_i \circ f_i)^{-1}(p) = \{p\}$, $p \in A$, it follows that $h_i \circ f_i$ takes the value p with multiplicity n_i where n_i is the degree of f_i. As $\{0, \infty\} \subseteq A$ we must have

$$h_i \circ f_i = c_i x^{n_i}$$

where $c_i \in K^*$. If $|S_2| > 2$, then $1 \in A$. It follows that $c_i = 1$, and since $(h_i \circ f_i)^{-1}(1) = 1$, that $n_i = 1$. That is, $f_i = h_i^{-1}$. But then $\alpha = h_2^{-1} \circ h_1$ takes

S_2 onto itself, and $\alpha^*\omega_2 = a\omega_2$. As the group of linear fractional transformations preserving S_2 is finite this contradicts the hypothesis that a is not a root of unity. Thus $S_2 = S_1 = \{0, \infty\}$,

$$f_i = r_i x^{m_i} \quad \text{and} \quad \omega_1 = s\,dx + t\frac{dx}{x}$$

for some $r_i, t \in K^*$, $m_i \in \mathbf{Z}$, $m_i \neq 0$ and $s \in K$. So,

$$f_i^*(\omega_1) = sr_i m_i x^{m_i - 1}\,dx + tm_i\frac{dx}{x}.$$

Since $a \neq 1$, the hypothesis $f_1^*(\omega) = af_2^*(\omega_1)$ implies $s = 0$. Thus ω_1 is an invariant differential on $\mathbf{G}_m$ as required.

References

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