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UNIMODULAR APPROXIMATION IN FUNCTION ALGEBRAS

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Let A be a function algebra on the compact Hausdorff space X. The main result of this paper gives necessary and sufficient conditions for the set of quotients of inner functions in A to be dense in the set of continuous unimodular functions on X. A theorem of Douglas and Rudin concerning quotients of Blaschke products is derived. The main result is also applied in the context of the theory of compact abelian groups.

The prototype for the main result of this paper is the following theorem due to Douglas and Rudin [3].

THEOREM 1. *Suppose that u is a measurable function on the unit circle such that $|u(e^{it})| = 1$ a.e. Then u can be approximated arbitrarily closely in the essential-sup norm by quotients of Blaschke products.*

We will derive Theorem 1 from a general result concerning function algebras. Our main result will also be applied to certain algebras of functions on compact abelian groups.

Preliminaries. Consider a sub-algebra A of the algebra $C(X)$ of continuous complex-valued functions on the compact Hausdorff space X. We will assume that A contains the constant functions and is closed with respect to the sup-norm $\|\cdots\|$. We will call a function $g \in A$ *inner* if $|g| \equiv 1$. Let $I(A)$ denote the set of inner functions and let $Q(A)$ denote the set of quotients of inner functions. Of course the set $U(X) = I(C(X))$ is simply the collection of continuous unimodular functions on X. $U(X)$ is a group under pointwise multiplication. $U(X)$ has an important subgroup, namely the group $\log U(X)$ of members of $U(X)$ having continuous logarithms. We will indicate the natural quotient map from $U(X)$ to $U(X)/\log U(X)$ by π_X.

In the case where X is the n-dimensional torus T^n and A is the polydisk algebra $A(T^n)$, i.e., the closed algebra generated by the coordinate projections from T^n onto T^1, there is an abundance of inner functions. It is known that a function $f \in A(T^n)$ is inner if and only if it is of the form

$$f(z) = M(z)\hat{Q}(z)/Q(z), \qquad z = (z_1, z_2, \dots, z_n),$$

where M is a monomial, $Q(z)$ is a polynomial of degree $d(j)$ in z_j for $j = 1, 2, \ldots, n$ which does not vanish anywhere on the closed unit polydisk, and $\hat{Q}(z)$ is the unique polynomial satisfying

$$\hat{Q}(z) = z_1^{d(1)} z_2^{d(2)} \cdots z_n^{d(n)} \overline{Q(z)}$$

for every $z \in T^n$. See [**6**, p. 112]. Note in particular that every $f \in I(A(T^n))$ is a rational function.

Our reason for discussing inner functions in the polydisk algebra is that they are related to the general case via the following easily proved:

PROPOSITION 2. *Let A be the general function algebra as described above. Let $f_1, f_2, \ldots, f_n \in I(A)$ and $F \in A(T^n)$, then $F(f_1, f_2, \ldots, f_n) \in A$. In particular, if $F \in I(A(T^n))$, then $F(f_1, f_2, \ldots, f_n) \in I(A)$.*

Main Result.

THEOREM 3. *Let A be as described above. $Q(A)$ is dense in $U(X)$ if and only if the following conditions hold.*

(i) $\pi_X(Q(A)) = U(X)/\log U(X)$

(ii) *$I(A)$ separates the points of X.*

Proof. Suppose that $Q(A)$ is dense in $U(X)$. Let $u \in U(X)$. Since $\log U(X)$ is an open subgroup of $U(X)$ containing the constant function 1, it follows that there exists a $q \in Q(A)$ such that $\bar{q}u \in \log U(X)$. Thus, we have $\pi_X(u) = \pi_X(q)$. It follows that (i) is satisfied. Next, suppose $a, b \in X$ with $a \neq b$. It's easy to find a function $u_0 \in U(X)$ such that $u_0(a) \neq u_0(b)$. If $g(a)$ were equal to $g(b)$ for every g in $I(A)$, then it would follow that $q(a) = q(b)$ for every $q \in Q(A)$. But by choosing a $q_0 \in Q(A)$ sufficiently close to u_0 we can obtain $q_0(a) \neq q_0(b)$. This contradiction shows that we must have $g_0(a) \neq g_0(b)$ for some $g_0 \in I(A)$.

Now we will assume that (i) and (ii) hold. To show that $Q(A)$ is dense in $U(X)$ we observe first that the uniform closure $\overline{Q(A)}$ of $Q(A)$ is a subgroup of $U(X)$. Next we note that, because of (i), it suffices to show that $\log U(X)$ is contained in $\overline{Q(A)}$. A function u in $U(X)$ belongs to $\log U(X)$ if and only if it has the form

$$u = e^{ir}$$

where r is a continuous real valued function on X. Since

$$u = \left(e^{ir/n}\right)^n$$

for $n = 1, 2, \ldots,$ it follows that each member of $\log U(X)$ can be written as an integer power of a function in $U(X)$ which is arbitrarily close to the constant function 1. Thus, to show $\log U(X) \subseteq \overline{Q(A)}$ it is enough to show that $\overline{Q(A)}$ contains all functions of the form

$$u = v^2 \tag{1}$$

where $v \in U(X)$ and $\operatorname{Re} v > 1/2$.

Let u and v be as in (1) and let $\varepsilon \in (0, \frac{1}{4})$. By (ii) and the Stone-Weierstrass theorem we can find inner functions $g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_n$ and constants $c_1, c_2, \ldots, c_n$ such that

$$\left\| v - \sum_{j=1}^{n} c_j \bar{g}_j h_j \right\| < \varepsilon. \tag{2}$$

Clearly, (2) can be re-written in the form

$$\left\| v - \bar{g} \sum_{j=1}^{n} c_j f_j \right\| < \varepsilon, \tag{3}$$

where $g = \prod_{j=1}^{n} g_j$ and $f_1, f_2, \ldots, f_n \in I(A)$. It follows easily from (3) that

$$\operatorname{Re} \bar{g} \sum_{j=1}^{n} c_j f_j > \frac{1}{4}. \tag{4}$$

Since $\bar{v} = 1/v$, it follows again from (3) that

$$\left\| v - \bar{g} / \sum_{j=1}^{n} \bar{c}_j \bar{f}_j \right\| < 4\varepsilon.$$

Thus, we have

$$\left\| u - \bar{g}^2 \sum_{j=1}^{n} c_j f_n / \sum_{j=1}^{n} \bar{c}_j \bar{f}_j \right\| < 8\varepsilon. \tag{5}$$

We now define a mapping $\Psi\colon X \to T^{n+1}$ via

$$\Psi(x) = \big(g(x), f_1(x), \ldots, f_n(x)\big).$$

Consider the function as defined on $\Psi(X)$ by

$$r(z) = r(z_0, z_1, \ldots, z_n) = \bar{z}_0 \sum_{j=1}^{n} c_j z_j.$$

We observe that, by (4),

$$\operatorname{Re} r(z) > 1/4 \tag{6}$$

whenever $z \in \Psi(X)$. It follows easily from (6) and the Tietze extension theorem that r has an extension r^* to all of T^{n+1} which satisfies

$$\operatorname{Re} r^*(z) > 1/4$$

for all z in T^{n+1}. In particular, r^* never vanishes on T^{n+1}. Thus $r^*/\bar{r}^* \in U(T^{n+1})$. Using the main result of [5], we can find F_1, $F_2 \in I(A(T^{n+1}))$ such that

$$\|r^*/\bar{r}^* - F_1\bar{F}_2\| < \varepsilon, \tag{7}$$

where the norm indicated is of course the sup-norm over T^{n+1}. Now (5) can be re-written

$$\|u - r^* \circ \Psi/\bar{r}^* \circ \Psi\| < 8\varepsilon. \tag{8}$$

Thus, using (7) and (8), we have

$$\|u - F_1 \circ \Psi \overline{F_2 \circ \Psi}\| < 9\varepsilon.$$

By Proposition 2 it follows that $F_1 \circ \Psi$ and $F_2 \circ \Psi$ belong to $I(A)$. The proof of Theorem 3 is now complete.

Applications and examples. (I) Let $d\mu$ be a measure on some set and let $H^\infty(d\mu)$ be a sub-algebra of $L^\infty(d\mu)$ which is closed in the essential-sup norm and contains the functions which are constant a.e. We will call a function $h \in H^\infty(d\mu)$ *inner* if $|h| = 1$ a.e. The Gelfand transform $f \to \hat{f}$ is an isometric isomorphism from $L^\infty(d\mu)$ onto $C(M_\mu)$, where M_μ denotes the maximal ideal space of $L^\infty(d\mu)$. The algebra $H^\infty(d\mu)$ is carried by the Gelfand transform onto a closed sub-algebra $\hat{H}^\infty)(d\mu)$. Furthermore, h is inner in $H^\infty(d\mu)$ iff $\hat{h} \in \hat{I}(H^\infty(d\mu)$. For our purposes it is important to note that every $u \in L^\infty(d\mu)$, such that $|u| = 1$ a.e., can be written in the form $u = e^{ir}$, where $r \in L^\infty(d\mu)$. It follows that $U(M_\mu)/\log U(M_\mu)$ is trivial.

By the remarks above and by Theorem 2 we have the following:

COROLLARY 4. *Every member of $L^\infty(d\mu)$ which is unimodular a.e. can be approximated in the essential sup-norm by quotients of inner functions from $H^\infty(d\mu)$ iff $I(\hat{H}^\infty(d\mu)$ separates the points of M_μ.*

In [1] Bernard el al show that if $H^\infty(d\mu)$ is strongly logmodular, then $I(\hat{H}^\infty(d\mu))$ separates the points of M_μ. Strong logmodularity holds in the classical case where μ is arc-length measure σ on T^1 and $H^\infty(d\sigma)$ is the usual Hardy space. Thus, Corollary 4 and the fact that every inner function in $H^\infty(d\mu)$ can be approximated by Blaschke products yield Theorem 1. (See [4, Chapter 10] for details.)

Next consider a plane domain Ω bounded by a finite number of analytic Jordan curves. Let $H^\infty(\Omega)$ denote the algebra of bounded analytic functions on Ω. by Fatou's theorem $H^\infty(\Omega)$ can be identified with the algebra $H^\infty(ds)$ of functions which are non-tangential limits ds— a.e. of functions in $H^\infty(\Omega)$. Here of course ds denotes arc length measure on the boundary of Ω. It is shown in [**1**, Lemma 4.8] that the inner functions in $\hat{H}^\infty(ds)$ separate the points of the maximal ideal space of $L^\infty(ds)$. Thus, Corollary 4 can be applied to $H^\infty(ds)$.

(II) Another application of Theorem 3 occurs in the theory of compact abelian groups. In the discussion which follows G will denote a compact abelian group and $G\#$ will denote its dual group, i.e., the group of continuous characters on G.

We will need the following.

LEMMA 5. *Let $S \subseteq G^\#$ and let $\langle S \rangle$ denote the subgroup of $G^\#$ generated by S. If S separates the points of G, then $\langle S \rangle = G^\#$.*

Proof. Suppose that S separates the points of G and that $\langle S \rangle$ is a proper sub-group of $G^\#$. Since $G^\#$ is discrete, it follows that there is a non-constant character q on $G^\#$ such that $q(c) = 1$ for all $c \in \langle S \rangle$ (see [**7**, pp. 35–36]). By the Pontryagin duality theorem there is a point $g_0 \in G$ such that $q(b) = b(g_0)$ for all $b \in G^\#$. In particular, $c(g_0) = 1$ for every $c \in S$. Since S separates points it follows that g_0 must be the identity in G. But then q cannot be non-constant, since $q(b) = b(g_0) = 1$ for every $b \in G^\#$.

THEOREM 6. *Suppose that G is either totally disconnected or connected. Let S be a point separating subset of $G^\#$ and let A_S denote the closed sub-algebra of $C(G)$ generated by S. Then $Q(A_S)$ is dense in $U(G)$.*

Proof. Suppose G is totally disconnected. Then $U(G)/\log U(G)$ is trivial. Thus, the hypotheses of Theorem 3 are satisfied by A_S. It follows that $Q(A_S)$ is dense in $U(G)$.

In the case where G is connected we bring in a result due to Taylor [**8**, p. 80] which asserts that for each $u \in U(G)$ there is a continuous character c such that $cu \in \log U(G)$. By Lemma 5 it follows that $c \in Q(A_S)$. Thus, condition (i) of Theorem 3 is satisfied by the algebra A_S. Since $S \subseteq A_S$, the condition (ii) is also satisfied. Hence, $Q(A_S)$ is dense in $U(G)$.

(III) It is not hard to give examples where condition (ii) of Theorem 3 fails. If X is a compact subset of the plane and $R(X)$ is the closure of the rational functions in $C(X)$, then the inner functions in $R(X)$ separate the points of X iff $R(X) = C(X)$. A more striking example is provided by an algebra constructed by Browder and Wermer in [**2**, Corollary 1]. The Browder and Wermer algebra is of the form

$$A(\Psi) = \{f | f, f \circ \Psi \in A(T^1)\},$$

where Ψ is a homeomorphism of T^1 onto itself which is singular i.e. Ψ maps a set of Lebesgue measure 0 onto a set of full Lebesgue measure. The algebra $A(\Psi)$ is a Dirichlet sub-algebra of $A(T^1)$. If $h \in I(A(\Psi))$, then both h and $h \circ \Psi$ must be finite Blaschke products. It follows easily that h must be a constant.

Final Remark. Condition (i) of Theorem 2 remains something of a mystery to us. We have been unable to find an example of an algebra of functions which satisfies (ii) but not (i).

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References

[1] A. Bernard, J. B. Garnett and D. E. Marshall, *Algebra generated by inner functions*, J. Funct. Anal., **25** (1977), 275–285.

[2] A. Browder and J. Wermer, *A method for constructing Dirichlet algebras*, Proc. Amer. Math. Soc., **15** (1964), 546–552.

[3] R. G. Douglas and W. Rudin, *Approximation by inner functions*, Pacific J. Math., **31** (1969), 313–320.

[4] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall 1962.

[5] J. N. Mc Donald, *Approximation by quotients of rational inner functions*, Proc. Amer. Math. Soc., **74**, No. 2, (1979), 299–300.

[6] W. Rudin, *Function Theory on Polydisks*, W. A. Benjamin 1969.

[7] ______, *Fourier Analysis on Groups*, Interscience, 1960.

[8] J. L. Taylor *Measure Algebras*, C.B.M.S. Regional Conference series #16.

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