REALIZING TRANSFER MAPS FOR
RAMIFIED COVERINGS

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In this paper we study transfer maps for a general class of ramified covering spaces. This class includes finite covering spaces, as well as more general orbit maps $Y/H \to Y/G$, where H is a subgroup of a finite group G, and where Y is any G-space. This class also includes classical branched coverings of manifolds. The goal of this paper is to identify and to analyze the obstructions to realizing such a homology transfer homomorphism by a stable map between spaces.

The class of ramified coverings that we study was defined by L. Smith. By the use of symmetric product spaces, Smith gave an elegant, combinatorial description of a transfer homomorphism

$$\tau_* \colon H_*(X; Z) \to H_*(\tilde{X}; Z)$$

for any d-fold ramified covering $\pi \colon \tilde{X} \to X$. In particular this homomorphism has the usual property that the composition

$$\pi_* \tau_* \colon H_*(X; Z) \to H_0(X; Z)$$

is multiplication by d.

In the case of a regular covering map, or equivalently the orbit map of a free group action $X/H \to X/G$, it is well known that there exists a stable map (i.e. a map of suspension spectra)

$$\tau \colon \Sigma^\infty X/G \to \Sigma^\infty X/H$$

that realizes $\pi_* \colon H_*(X/G) \to H_*(X/H)$. These maps were discovered independently by Roush [8] and by Kahn and Priddy [4], and they have had several important applications in stable homotopy theory. In particular this was one of the main tools used in Kahn and Priddy's proof of their famous theorem that any stable map $\lambda \colon RP^\infty \to S^0$ that is nonzero on the first stable homotopy group, $\pi_1^s(\_)$, in fact induces a split surjection in 2-local homotopy groups in positive dimensions [4].

This naturally leads us to the following question. When does there exist stable realizations $\tau \colon \Sigma^\infty X \to \Sigma^\infty \tilde{X}$ of the transfer homomorphism of more general ramified covers $\pi \colon \tilde{X} \to X$? In his thesis [5], R. Kubelka showed that in general the induced transfer in mod p cohomology, τ^*:

$H^*(\tilde{X}; \mathbf{Z}/p) \to H^*(X; \mathbf{Z}/p)$ does not commute with the action of the Steenrod algebra, so such realizations do not in general exist.

In ths note we address the following local version of this question. If n is an integer let $X[1/n]$ denote the localizations of X obtained by inverting all primes dividing n. So in particular $H_*(X[1/n], \mathbf{Z}) \cong H_*(X; \mathbf{Z}) \otimes \mathbf{Z}[1/n]$. In this paper we use Smith's homology transfer to give easy proofs of the following results, the first of which is a well known folk theorem.

THEOREM 1. *Let Y be an arbitrary G-space where G is a finite group of order n. Let $\pi\colon Y \to Y/G$ be the orbit map. Then there exists a stable map of local spectra*

$$\tau\colon \Sigma^\infty Y/G\left[\frac{1}{n}\right] \to \Sigma^\infty Y\left[\frac{1}{n}\right]$$

that realizes the local transfer homomorphism

$$\tau_* \otimes 1\colon H_*(Y/G; \mathbf{Z}) \otimes \mathbf{Z}\left[\frac{1}{n}\right] \to H_*(Y; \mathbf{Z}) \otimes \mathbf{Z}\left[\frac{1}{n}\right].$$

THEOREM 2. *Let $\pi\colon \tilde{X} \to X$ be any d-fold ramified cover in the sense of Smith [9]. Then there exists a stable map*

$$\tau\colon \Sigma^\infty X\left[\frac{1}{d!}\right] \to \Sigma^\infty \tilde{X}\left[\frac{1}{d!}\right]$$

that realizes $\tau_* \otimes 1\colon H_*(X; \mathbf{Z}) \otimes \mathbf{Z}[1/d!] \to H_*(\tilde{X}; \mathbf{Z}) \otimes \mathbf{Z}[1/d!]$.

We will also observe that Theorem 2 is the best possible result in the most general setting. In particular we will exhibit an explicit 3-fold ramified cover $\pi\colon \tilde{X} \to X$ with the property that if p is either 2 or 3 the induced cohomology transfer $\tau^*\colon H^*(\tilde{X}; \mathbf{Z}/p) \to H^*(X: \mathbf{Z}/p)$ does not commute with the action of the Steenrod algebra. Thus both the primes 2 and 3 must be inverted before the transfer homomorphism can be realized by a stable map. We will also see that this phenomenon is general. That is, for every d there exists a d-fold ramified cover $\pi\colon \tilde{X}_d \to X_d$ that has the property that for every prime $p \leq d$ the induced cohomology transfer $\tau^*\colon H^*(\tilde{X}; \mathbf{Z}/p) \to H^*(X; \mathbf{Z}/p)$ does not respect the action of the Steenrod algebra.

The organization of this paper is as follows. In §1 we recall the Kahn-Priddy stable transfer map for free group actions, Smith's definition of general ramified coverings, and then describe the obstruction to obtaining stable transfer maps in the general setting. In §2 we prove Theorems 1 and 2 by showing that this obstruction vanishes after inverting the requisite set of primes.

The author is grateful to G. Brumfiel for originally suggesting this problem to him and for several helpful conversations. He is also grateful to L. Smith for helpful correspondence. The author points out that Smith was aware of Theorem 2 and has an alternate proof.

1. The obstruction of realizing the transfer homomorphism. We begin by recalling the Kahn-Priddy stable transfer maps for free group actions [4].

Let X be a space of the homotopy type of a C. W. complex and suppose it is acted upon freely on the right by a finite group G. Let $H < G$ be a subgroup of index n, and consider the n-fold covering

$$p\colon X/H \to X/G$$

given by the obvious projection of orbit spaces. By the left action of G on the set of left cosets G/H we get a representation

$$\rho\colon G \to \Sigma_n$$

where Σ_n is the symmetric group on n-letters.

Let EG and $E\Sigma_n$ denote contractible spaces that are acted upon freely on the left by G and Σ_n respectively. The representation induces an equivariant map $\rho_*\colon EG \to E\Sigma_n$. Kahn and Priddy define a map

$$\Phi\colon X \times_G EG \to X/H^n \times_{\Sigma_n} E\Sigma_n$$

by the formula $\Phi(x, w) = (\overline{xg_1}, \ldots, \overline{xg_n}; \rho_*(w))$, where $\{g_1, \ldots, g_n\} \in G$ represent the n left cosets in G/H and $\overline{xg_i}$ denotes the orbit of xg_i in X/H. G acts on the left of x by $gx = xg^{-1}$, and Σ_n acts on the left of X/H^n by permuting coordinates.

Now recall from [3] that for any space Y there are Dyer-Lashof maps $\alpha_n\colon Y^n \times_{\Sigma_n} E\Sigma_n \to \Omega^\infty\Sigma^\infty Y$. Letting Y be X/H, define a stable map $T\colon \Sigma^\infty(X \times_G EG) \to \Sigma^\infty X/H$ to be the adjoint of the composition (which, by abuse of notation, we also call T)

$$T = \alpha_n \circ \Phi\colon X \times_G EG \to X/H^n \times_{\Sigma_n} E\Sigma_n \to \Omega^\infty\Sigma^\infty(X/H).$$

Now since the action of G on X is free, the projection map $X \times_G EG \to X/G$ is a homotopy equivalence, so T can be viewed as a stable map

$$T\colon \Sigma^\infty X/G \to \Sigma^\infty X/H.$$

In [4] Kahn and Priddy showed that $T_*\colon H_*(X/G) \to H_*(X/H)$ is the usual transfer homomorphism for covers as defined, for example in [10].

As we shall see, if $\pi\colon \tilde{X} \to X$ is a general ramified cover, the obstruction to obtaining a stable transfer map $T\colon \Sigma^\infty X \to \Sigma^\infty \tilde{X}$ is measured by the inability in general to describe X in the form $\tilde{X} \times_G EG$. In

order to make this precise, we now recall Smith's definition of a ramified covering and its corresponding transfer homomorphism [9].

Roughly speaking, a ramified covering is a surjective, finite to one map $\pi\colon \tilde{X} \to X$ together with a notion of multiplicity, so that if at each point $x \in X$ one writes down the unordered list of preimages $\{\pi^{-1}(x)\}$ according to their multiplicity, then this list varies continuously with $x \in X$. Here is how Smith makes this idea precise.

For every positive integer $d \in \mathbf{Z}^+$, let $\mathrm{SP}^d(Y)$ be the d-fold symmetric product $\mathrm{SP}^d(Y) = Y^d/\Sigma_d$, where again Σ_d acts on Y^d by permuting coordinates.

DEFINITION 1.1 [9]. A surjective, finite to one map $\pi\colon \tilde{X} \to X$ is called a *d-fold ramified covering* iff there is a map

$$\mu\colon X \to \mathbf{Z}^+$$

called the multiplicity map such that

(1) For all $x \in X$, $\Sigma_{\tilde{x} \in \pi^{-1}(x)}\mu(\tilde{x}) = d$,

(2) The function $\tau_\pi\colon X \to \mathrm{SP}^d(\tilde{X})$ defined by sending x into $\pi^{-1}(x)$, where each $\tilde{x} \in \pi^{-1}(x)$ occurs $\mu(x)$ times, is continuous.

As observed by Smith, this definition together with the Dold-Thom theorem [2] allows for an elegant description of the transfer homomorphism $\pi_*\colon H_*(X) \to H_*(\tilde{X})$ in the following manner.

For every integer d, one can include Y^d into Y^{d+1} as the subspace of $(d+1) -$ tuples whose last coordinate is a given fixed basepoint $y_0 \in Y$. This respects the symmetric group actions, and so defines inclusions $\mathrm{SP}^d(Y) \to \mathrm{SP}^{d+1}(Y)$. Let

$$\mathrm{SP}^\infty(Y) = \varinjlim_d \mathrm{SP}^d(Y).$$

THEOREM 1.2 (*Dold and Thom* [2]). $\mathrm{SP}^\infty(Y)$ *is homotopy equivalent to the infinite loop space*

$$\varinjlim_n \Omega^n(K(\mathbf{Z}, n) \wedge Y),$$

where $K(\mathbf{Z}; n)$ is an Eilenberg-MacLane space of type $(\mathbf{Z}, n)$. Thus $\pi_*\mathrm{SP}^\infty(Y) \cong \tilde{H}_*(Y; \mathbf{Z})$.

Now let $\pi\colon \tilde{X} \to X$ be a d-fold ramified covering and consider the composition

$$\tau\colon X \xrightarrow{\tau_\pi} \mathrm{SP}^d(\tilde{X}) \to \mathrm{SP}^\infty(\tilde{X}).$$

τ extends to a map, which by abuse we also call τ: $SP^\infty X \to SP^\infty \tilde{X}$ defined by the maps

$$(\tau_\pi)^k : SP^k(X) \to SP^{kd}(\tilde{X}).$$

Passing to a homotopy group, we get a transfer homomorphism

$$\tau_* : \tilde{H}_*(X) \cong \pi_* SP^\infty(X) \to \pi_* SP^\infty(\tilde{X}) \cong \tilde{H}_*(\tilde{X}).$$

This homomorphism was studied in detail in [9] and was shown to have the usual properties of transfer homomorphisms. For example the following are easy corollaries of the definition (see [9]).

PROPOSITION 1.3. *Let G be a finite group that acts on a space Y with orbit map π: $Y \to Y/G$, structured as a $d = |G|$-fold ramified cover. Then the composition $\tau_* \circ \pi_*$: $\tilde{H}_* Y \to \tilde{H}_* Y$ is given by the formula*

$$\tau_* \pi_*(u) = \sum_{g \in G} g_*(u).$$

PROPOSITION 1.4. *Let π: $\tilde{X} \to X$ be a d-fold ramified cover. Then the composite homomorphism*

$$\pi_* \circ \tau_* : \tilde{H}_*(X) \to \tilde{H}_*(X)$$

is multiplication by d.

REMARK. 1.4 is immediate after observing that the composite

$$X \xrightarrow[\tau_\pi]{} SP^d(\tilde{X}) \xrightarrow[\pi_*]{} SP^d(X)$$

is given by the d-fold diagonal map.

Now if Y is any space consider the projection h_n: $Y^n \times_{\Sigma_n} E\Sigma_n \to Y^n/\Sigma_n = SP^n(Y)$. Observe that if p: $X/H \to X/G$ is the projection of orbit spaces coming from a G-action on X, then the following diagram commutes:

$$
\begin{array}{ccc}
X \times_G EG & \xrightarrow{\Phi} & X/H^n \times_{\Sigma_n} E\Sigma_n \\
\downarrow f & & \downarrow h_n \\
X/G & \xrightarrow[\tau_p]{} & SP^n(X/H)
\end{array}
$$

where f is the projection map and n is the index of H in G. Thus if the action is free and hence f is a homotopy equivalence, the Kahn-Priddy map Φ: $X/G \to X/H^n \times_{\Sigma_n} E\Sigma_n$ that was used to define the stable transfer map, homotopy lifts the Smith map τ_p: $X/G \to SP^n(X/H)$ that was used to define the homology transfer homomorphism.

For a general ramified cover $\pi\colon \tilde{X} \to X$, the obstruction to the existence of a such a homotopy lifting of τ_π will measure our obstruction to the existence of a stable transfer map $\tau\colon \Sigma^\infty X \to \Sigma^\infty \tilde{X}$. We end this section by identifying this obstruction more precisely.

Let $\pi\colon \tilde{X} \to X$ be a d-fold ramified cover and let $C(h)$ be the homotopy cofiber (i.e. mapping cover) of $h_d\colon \tilde{X}^d \times_{\Sigma_d} E\Sigma_d \to \mathrm{SP}^d(\tilde{X})$. For n a positive integer let $\{X, C(h)[1/n]\}$ be the abelian group of stable maps (i.e. maps of suspension spectra) from X to the localization of $C(h)$. We then define a class

$$\theta_{\bar{\pi}}\left[\frac{1}{n}\right] \in \left\{X, C(h)\left[\frac{1}{n}\right]\right\}$$

to be represented by the localization of the composition

$$X \underset{\tau_\pi}{\to} \mathrm{SP}^d(\tilde{X}) \underset{j}{\to} C(h)$$

where j is the inclusion of $\mathrm{SP}^d(\tilde{X})$ into the mapping cone of h_d.

PROPOSITION 1.5. *If* $\theta_\pi[1/n] \in \{X, C(h)[1/n]\}$ *is zero then there exists a map of localized suspension spectra*

$$T\colon \Sigma^\infty X\left[\frac{1}{n}\right] \to \Sigma^\infty \tilde{X}\left[\frac{1}{n}\right],$$

so that in homology,

$$T_* = \tau_* \otimes 1\colon \tilde{H}_*(X; \mathbf{Z}) \otimes \mathbf{Z}\left[\frac{1}{n}\right] \to \tilde{H}_*(\tilde{X}, \mathbf{Z}) \otimes \mathbf{Z}\left[\frac{1}{n}\right].$$

Proof. By definition, it is immediate that $\theta_\pi[1/n]$ is the obstruction to the existence of a stable homotopy lifting

$$\tilde{T}\colon \Sigma^\infty X\left[\frac{1}{n}\right] \to \Sigma^\infty\left(\tilde{X}^d \times_{\Sigma_d} E\Sigma_d\right)\left[\frac{1}{n}\right]$$

of the localization of $\tau\colon X \to \mathrm{SP}^d(\tilde{X})$. It is therefore sufficient to show how to construct the required map $T\colon \Sigma^\infty X[1/n] \to \Sigma^\infty \tilde{X}[1/n]$ out of such a lifting $\tilde{T}$.

For any space Y let

$$H\colon \Omega^\infty \Sigma^\infty Y \to \mathrm{SP}^\infty(Y) \simeq \varinjlim_q \Omega^q(K(\mathbf{Z}, q) \wedge Y)$$

be the infinite loop map induced by the inclusions

$$\Sigma^q Y = S^q \wedge Y \underset{i \wedge 1}{\to} K(\mathbf{Z}, q) \wedge Y$$

where $i \in \pi_q K(\mathbf{Z}, q) = \mathbf{Z}$ is a generator. It is clear that $h_*: \pi_*(\Omega^\infty \Sigma^\infty Y) \to \pi_*(\mathrm{SP}^\infty(Y))$ is the stable Hurewicz homomorphism $h_*: \pi_*^s(Y) \to \tilde{H}_*(Y)$. Moreover, it is also well known (see [6] for example) that the following diagram homotopy commutes:

$$
\begin{array}{ccc}
Y^k \times_{\Sigma_k} E\Sigma_k & \overset{\alpha_k}{\to} & \Omega^\infty \Sigma^\infty Y \\[2mm]
\downarrow h_k & & \downarrow h \\[2mm]
\mathrm{SP}^k(Y) & \to & \mathrm{SP}^\infty(Y)
\end{array}
$$

where α_k is the Dyer-Lashof map mentioned above.

Thus a lifting

$$
\tilde{T}: \Sigma^\infty X\left[\frac{1}{n}\right] \to \Sigma^\infty\left(\tilde{X}^d \times_{\Sigma_d} E\Sigma_d\right)\left[\frac{1}{n}\right]
$$

of the localization of $\tau: \Sigma^\infty X \to \Sigma^\infty \mathrm{SP}^d \tilde{X}$ allows us to define a stable map

$$
\overline{T}: \Sigma^\infty X\left[\frac{1}{n}\right] \underset{\tilde{T}}{\to} \Sigma^\infty\left(\tilde{X}^d \times_{\Sigma_d} E\Sigma_d\right)\left[\frac{1}{n}\right] \underset{\alpha_d}{\to} \Sigma^\infty(\Omega^\infty \Sigma^\infty \tilde{X})\left[\frac{1}{n}\right]
$$

or equivalently, its adjoint

$$
\overline{T}: X\left[\frac{1}{n}\right] \to \Omega^\infty \Sigma^\infty(\Omega^\infty \Sigma^\infty \tilde{X})\left[\frac{1}{n}\right],
$$

that homotopy lifts the localization of the composition

$$
\overline{\tau}: X \underset{\tau_\pi}{\to} \mathrm{SP}^d(\tilde{X}) \to \mathrm{SP}^\infty(\tilde{X}) \to \Omega^\infty \Sigma^\infty(\mathrm{SP}^\infty(\tilde{X})).
$$

Now since $\Omega^\infty \Sigma^\infty \tilde{X}$ and $\mathrm{SP}^\infty \tilde{X}$ are infinite loop spaces and $h: \Omega^\infty \Sigma^\infty X \to \mathrm{SP}^\infty X$ is an infinite loop map, there exist infinite loop retractions $a: \Omega^\infty \Sigma^\infty(\Omega^\infty \Sigma^\infty \tilde{X}) \to \Omega^\infty \Sigma^\infty \tilde{X}$ and $b: \Omega^\infty \Sigma^\infty(\mathrm{SP}^\infty(\tilde{X})) \to \mathrm{SP}^\infty(\tilde{X})$ that make the following diagram homotopy commute:

$$
\begin{array}{ccc}
\Omega^\infty \Sigma^\infty(\Omega^\infty \Sigma^\infty \tilde{X}) & \overset{a}{\to} & \Omega^\infty \Sigma^\infty \tilde{X} \\[2mm]
\downarrow \Omega^\infty \Sigma^\infty h & & \downarrow h \\[2mm]
\Omega^\infty \Sigma^\infty(\mathrm{SP}^\infty(\tilde{X})) & \overset{b}{\to} & \mathrm{SP}^\infty(\tilde{X}).
\end{array}
$$

Thus the composition

$$
T: X\left[\frac{1}{n}\right] \underset{\overline{T}}{\to} \Omega^\infty \Sigma^\infty(\Omega^\infty \Sigma^\infty \tilde{X})\left[\frac{1}{n}\right] \underset{a}{\to} \Omega^\infty \Sigma^\infty \tilde{X}\left[\frac{1}{n}\right]
$$

lifts the localization of $\tau: X \to \mathrm{SP}^\infty \tilde{X}$. By abuse of notation we let T also represent the adjoint

$$
T: \Sigma^\infty X\left[\frac{1}{n}\right] \to \Sigma^\infty \tilde{X}\left[\frac{1}{n}\right].
$$

It is now clear that in homology T induces

$$\tau_* \otimes 1 \colon \tilde{H}_*(X) \otimes \mathbf{Z}\left[\frac{1}{n}\right] \to \tilde{H}_*(\tilde{X}) \otimes \mathbf{Z}\left[\frac{1}{n}\right]$$

and hence the proposition is proved.

2. Proofs of Theorems 1 and 2. We now use Proposition 1.5 to prove Theorems 1 and 2.

Proof of Theorem 1. As in the statement of the theorem we let Y be any G-space, $n = $ order of G, and $\pi\colon Y \to Y/G$ the orbit map. Consider the following homotopy commutative diagram

$$
\begin{array}{ccc}
Y \times_G EG & \overset{\Phi}{\to} & Y^n \times_{\Sigma_n} E\Sigma_n \\
\downarrow f & & \downarrow h_n \\
Y/G & \underset{\tau_\pi}{\to} & SP^n(Y).
\end{array}
$$

Now it well known (see [1] for example) that the projection map f induces an isomorphism in localized homology

$$f_*\colon \tilde{H}_*\left(Y \times_G EG; \mathbf{Z}\left[\frac{1}{n}\right]\right) \overset{\cong}{\to} \tilde{H}_*\left(Y/G; \mathbf{Z}\left[\frac{1}{n}\right]\right)$$

and hence after inverting $n = |G|$, f becomes a stable homotopy equivalence. Let

$$g\colon \Sigma^\infty Y/G\left[\frac{1}{n}\right] \to \Sigma^\infty(Y \times_G EG)\left[\frac{1}{n}\right]$$

be a stable homotopy inverse. Then by the commutativity of this diagram, the composition

$$\Sigma^\infty Y/G\left[\frac{1}{n}\right] \underset{g}{\to} \Sigma^\infty(Y \times_G EG)\left[\frac{1}{n}\right] \underset{\Phi}{\to} \Sigma^\infty\left(Y^n \times_{\Sigma_n} E\Sigma_n\right)\left[\frac{1}{n}\right]$$

is a stable homotopy lifting of the localization of $\tau\colon Y/G \to SP^\infty(Y)$. Thus $\theta_\pi[1/n] \in \{Y/G, C(h)[1/n]\}$ is trivial. Hence Theorem 1 follows from 1.5.

Proof of Theorem 2. Let $\pi\colon \tilde{X} \to X$ be an arbitrary d-fold ramified covering. Since $d! = $ the order of Σ_d, then the map

$$h_d\colon \tilde{X}^d \times_{\Sigma_d} E\Sigma_d \to SP^d(\tilde{X}) = \tilde{X}^d/\Sigma_d$$

induces an isomorphism in localized homology,

$$h_{d_*}\colon \tilde{H}_*\left(\tilde{X}^d \times_{\Sigma_d} E\Sigma_d; \mathbf{Z}\left[\frac{1}{d!}\right]\right) \underset{\cong}{\to} \tilde{H}_*\left(SP^d(\tilde{X}); \mathbf{Z}\left[\frac{1}{d!}\right]\right).$$

Thus after inverting $d!$, h_d becomes a stable homotopy equivalence. This says that the localized mapping cone, $C(h)[1/d!]$ is stably contractible. Thus $\theta_\pi[1/d!] \in \{X, C(h)[1/d!]\}$ is trivial, and so Theorem 2 follows from 1.5.

We end this paper by giving examples to show that Theorem 2 is the best possible result in the general ramified cover setting. We first exhibit a 3-fold ramified cover for which, because of Steenrod algebra considerations, it is necessary to invert both the primes 2 and 3 before one can stably realize the transfer homomorphism.

Let Σ_3 act on $S^2 \times S^2 \times S^2$ be permuting coordinates and let $\Sigma_2 \subset \Sigma_3$ be the subgroup of elements that leave the first coordinate fixed. Consider the orbit space projection map

$$\pi: (S^2)^3/\Sigma_2 \to (S^2)^3/\Sigma_3 = \mathrm{SP}^3(S^2).$$

π is a 3-fold ramified cover.

Now $(S^2)^3/\Sigma_2 \cong \mathrm{SP}^2(S^2) \times S^2$ so therefore

$$\tilde{H}_q\big((S^2)^3/\Sigma_2; \mathbf{Z}\big) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \text{if } q = 2 \text{ or } 4 \\ \mathbf{Z} & \text{if } q = 6 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$\tilde{H}_q\big(\mathrm{SP}^3(S^2); \mathbf{Z}\big) = \begin{cases} \mathbf{Z}, & \text{if } q = 2, 4, \text{ or } 6 \\ 0 & \text{otherwise.} \end{cases}$$

Now let $p = 2$ or 3 and A the mod p Steenrod algebra. The dual, $A_* = \mathrm{Hom}(A, \mathbf{Z}/2)$ acts on mod p homology. The following diagrams describe the A_*-module structures of the homology of these spaces. (The nodes represent generators.)
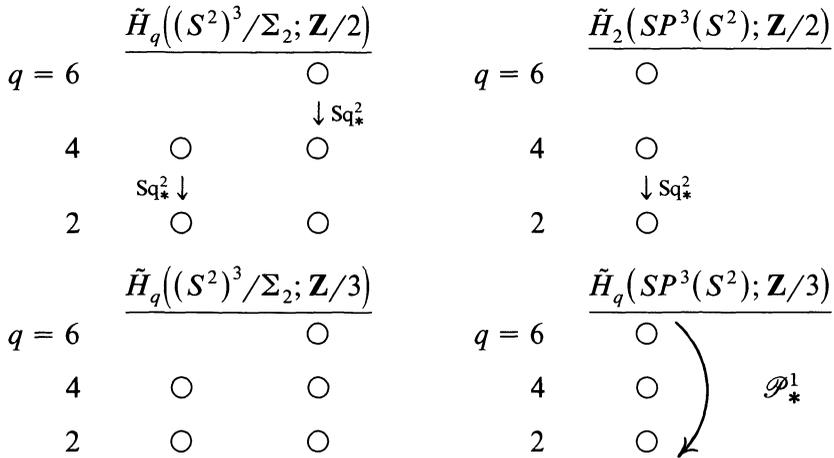
(trivial A_*-action)

The reference for these homology structures is [7]. A quick examination of these diagrams yields that the transfer homomorphisms

$$\tau_*\colon \tilde{H}_*\big(\mathrm{SP}^3(S^2); \mathbf{Z}/p\big) \to \tilde{H}_*\big((S^2)^3/\Sigma_2; \mathbf{Z}/p\big)$$

do not preserve the A_*-structures. Thus τ_* cannot be realized by a stable map unless both the primes 2 and 3 are inverted.

We remark that more general examples of d-fold ramified covers that have Steenrod algebra obstructions to constructing stable transfer maps unless $d!$ is inverted can be found in a similar manner. Namely, one studies ramified covers of the sort

$$\pi\colon X^d/\Sigma_{d-1} \to X^d/\Sigma_d = \mathrm{SP}^d(X).$$

We leave the details to the reader.

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