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EXAMPLES OF HEREDITARILY l^1 BANACH SPACES FAILING THE SCHUR PROPERTY

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A class of separable Banach sequence spaces is constructed. A member X of this class (i) is a hereditarily l^1 dual space which fails the Schur property, and (ii) is of codimension one in its first Baire class. A consequence of (ii) is that X is not isomorphic to the square of any Banach space Y.

Introduction. In this paper we introduce and study a new class of Banach sequence spaces, the X_α spaces. The definition of the norm in a particular X_α space depends on the action of special sequences of intervals of integers on a vector $x = (t_1, t_2, \ldots)$ (as in the definition of the James space J [**6**]) in conjunction with a fixed sequence of weighting factors (as in the Lorentz sequence spaces [**7**].)

Let X denote a specific X_α space, and let (e_i) denote the sequence of usual unit vectors in X (i.e. $e_i(j) = \delta_{ij}$ for integers i and j). Our main result is the following:

THEOREM 1. (1) *X is hereditarily l^1.*

(2) *The sequence (e_i) is a normalized boundedly complete basis for X. Thus, X is a dual space.*

(3) (i) *The sequence (e_i) is a weak Cauchy sequence in X with no weak limit in X. In particular, X fails the Schur property.* (ii) *There is a subspace X_0 of X which fails the Schur property, yet which is weakly sequentially complete.*

(4) *Let $B_1(X)$ denote the first Baire class of X in its second dual, i.e.,*

$$B_1(X) = \{x^{**} \varepsilon X^{**}: x^{**} \text{ is a weak* limit of a sequence } (x_n) \text{ in } X\}$$

Then $\dim B_1(X)/X = 1$.

Part (4) shows that the space X has properties analogous to those of the quasireflexive spaces of James. Since $\dim B_1(X)/X$ is an isomorphism invariant, we have the following immediate consequences of the Theorem.

COROLLARY 2. (1) *For any n and any Banach space Y, X is not isomorphic to Y^n. In particular, X is not isomorphic to its square.*

(2) *For any $n > 1$, X^n does not imbed isomorphically in X.*

(3) *Let $X = A \oplus B$. Then exactly one of A or B is weakly sequentially complete and the other is of codimension one in its first Baire class.*

The properties of the X_α spaces provide an interesting contrast to the work in the paper [**5**], where an example of a separable Banach space which has the Schur property yet fails the Radon-Nikodym property is given. The spaces presented here were designed (in part) so that the combinatorial considerations encountered in [**5**] could be avoided.

In addition to the James space and the Lorentz sequence spaces mentioned above, the X_α spaces owe their origin to the space of Maurey and Rosenthal [**8**]. A class of examples (unpublished), similar to the X_α spaces, was constructed independently by E. Odell.

The existence of hereditarily l^1 Banach spaces failing the Schur property was shown first by Bourgain [3]. However, the analysis of the X_α spaces is self contained and particularly straightforward. For example, the basic sequences which are equivalent to the usual basis of l^1 are explicitly constructed, and there is no use of Rosenthal's characterization [**9**] of Banach spaces containing l^1.

Except as indicated below, our terminology and notation are standard. The reader is referred to the books of Day [**4**] and Lindenstrausss and Tzafriri [7] for standard reference material on Banach spaces.

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Preliminaries. In this section the definition of the X_α spaces is given. First, by a block we mean an interval F (finite or infinite) of integers. For a block F and $x = (t_1, t_2, \ldots)$ a sequence of scalars such that $\sum_j t_j$ converges, define $\langle x, F \rangle = \sum_{j \in F} t_j$.

To define the norm, we consider special sequences of blocks and special sequences of nonnegative reals. Specifically, we call a sequence (finite or infinite) $F_1, F_2, \ldots, F_n, \ldots$ (where each F_i is a finite block) *admissible* if

$$\max F_i < \min F_{i+1} \quad \text{for } i = 1, 2, 3, \ldots.$$

Let us now consider a sequence α of nonnegative reals (α_i) (whose terms are used as weighting factors in the definition of the norm) which satisfies the following properties:

(1) $\alpha_1 = 1$ and $\alpha_{i+1} \le \alpha_i$ for $i = 1, 2, \ldots$.

(2) $\lim_{i\to\infty} \alpha_i = 0$.

(3) $\sum_{i=1}^{\infty} \alpha_i = \infty$.

For $x = (t_1, t_2, t_3, \ldots)$ a finitely nonzero sequence of scalars, define

$$\|x\| = \max \sum_{i=1}^{n} \alpha_i |\langle x, F_i\rangle|$$

where the max is taken over all n, and admissible sequences $F_1, F_2, \ldots, F_n$. Let X $(= X_{(\alpha_i)})$ be the completion of the finitely non zero sequences of scalars $x = (t_1, t_2, \ldots)$ in this norm. An X_α space is a Banach space constructed in this fashion from some sequence α satisfying (1)–(3) above.

REMARK. Property (3) of the sequence (α_i) is introduced to insure a new class of spaces. Indeed, if we consider sequences (α_i) which satisfy (1) and

(2′) there is a $\delta > 0$ such that $\alpha_i > \delta$ for all i, then the spaces X we obtain are all isomorphic to l^1. If we require (1), (2) and

(3′) $\sum_{i=1}^{\infty} \alpha_i < \infty$,

then the spaces X are all isomorphic to c_0.

Proofs of the results. For the remainder of the paper let us pick and fix a sequence (α_i) satisfying (1)–(3) above, and let $X = X_{(\alpha_i)}$. This section contains the analysis of the stucture of the space X.

What we will show in the proof of Theorem 1 is that an l^1 subspace of X is obtained by considring block basic subsequences (u_i) of (e_i) which have the property (roughly) that the number of sets m in an admissible sequence $F_1, F_2, \ldots, F_m$ needed to norm u_n goes to ∞ as $n \to \infty$.

Before beginning our detailed analysis, we collect some basic facts about the space X into the following lemma:

LEMMA 3. (a) *The sequence (e_i) forms a monotone, subsymmetric basis for the space X. (Recall that a basic sequence is subsymmetric if it is equivalent to each of its subsequences.)* (b) *For each integer n,*

$$\left\| \sum_{i=1}^{n} (e_{2i-1} - e_{2i}) \right\| = \sum_{i=1}^{2n} \alpha_i.$$

The proof of part (a) of the lemma follows immediately from the definition of the norm in X. Part (b) follows from the obvious selection of the admissible sequence $F_i = \{i\}$ for $i = 1, 2, \dots, 2n$.

This next simple lemma provides the key to the analysis of the space X.

LEMMA 4. *Let the sequence (α_i) be as above, let $n_0 > 0$ be an integer and let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that, if $b_1, b_2, \dots, b_n$ are ≥ 0, $b_i < \delta$ for all i, and $\sum_{i=1}^n \alpha_i b_i = 1$, then $\sum_{i=1}^n \alpha_{i+n_0} b_i \geq 1 - \varepsilon$.*

Proof. The series of nonnegative reals $\sum_{i=1}^\infty [\alpha_i - \alpha_{i+n_0}]$ converges, say to c. So, for any n, $\sum_{i=1}^n [\alpha_i - \alpha_{i+n_0}] \leq c$. Thus,

$$\sum_{i=1}^{n} \left[\alpha_i - \alpha_{i+n_0}\right] b_i \leq \left[\max b_i\right] \cdot c < \varepsilon$$

if $\max b_i$ is small enough.

Lemma 4 provides us with a tool for calculating the norm of linear combinations of vectors in terms of the norms of the individual components. We apply this to obtain a criterion for a sequence of vectors to have a subsequence which is equivalent to the usual basis of l^1.

For $x \in X$, put $s(x) = \max |\langle x, G \rangle|$ where the max is taken over all blocks G.

LEMMA 5. *Let (u_i) be a sequence of norm one vectors in X and (G_i) an admissible sequence of blocks such that $\{j\colon u_i(j) \neq 0\} \subset G_i$. For each i, put $s_i = s(u_i)$. If $\lim_{i\to\infty} s_i = 0$, then a subsequence (v_k) of (u_k) is equivalent to the usual basis of l^1.*

Proof. We select the sequence (v_k) by induction. Let $v_1 = u_1$. Pick n_1 and admissible blocks $F_1, F_2, \dots, F_{n_1}$ satisfying $\max F_{n_1} = \max G_1$ and $\sum_{i=1}^{n_1} \alpha_i |\langle v_1, F_i \rangle| = \|v_1\| = 1$. Let δ_1 be any δ guaranteed by Lemma 4 for the integer n_1 and $\varepsilon = 1/2$. (To simplify notation in the remainder of the proof, let $n_0 = 0$.)

Assume now that we have selected for $k = 1, \dots, p-1$

(1) an integer m_k $(> m_{k-1})$ so that $v_k = u_{m_k}$.

(2) an integer n_k ($> n_{k-1}$), blocks $F_{n_{k-1}+1}, \dots, F_{n_k}$ and $\delta_k > 0$ such that

(a) $\max F_{n_k} = \max G_{m_k}$.

(b) The sequence $F_1, F_2, \dots, F_{n_1}, \dots, F_{n_2}, \dots, F_{n_k}$ is admissible.

(c) $\sum_{i=1}^{n_k - n_{k-1}} \alpha_i |\langle v_k, F_i \rangle| = \|v_k\| = 1$.

(d) δ_k is any δ guaranteed by Lemma 4 for the integer n_{k-1} and $\varepsilon = 1/2$.

Now let $\delta_p > 0$ be any δ guaranteed by Lemma 4 for the integer n_{p-1} and $\varepsilon = 1/2$. Pick m_p ($> m_{p-1}$) so that $s_{m_p} < \delta_p$ and let $v_p = u_{m_p}$. Finally, pick blocks $F_{n_{p-1}+1}, \dots, F_{n_p}$ such that (a), (b) and (c) above are satisfied for v_p and G_{m_p}. This completes the induction process.

Observe that $|\langle v_k, F_{i+n_{k-1}} \rangle| < s_{n_k} < \delta_k$ for $i = 1, \dots, n_k - n_{k-1}$. By Lemma 4,

$$\sum_{i=1}^{n_k - n_{k-1}} \alpha_{i+n_{k-1}} |\langle v_k, F_{i+n_{k-1}} \rangle| > \frac{1}{2}.$$

This inequality can be rewritten as

$$\sum_{i=n_{k-1}+1}^{n_k} \alpha_i |\langle v_k, F_i \rangle| > \frac{1}{2}.$$

Now, let scalars $t_1, t_2, \dots, t_k$ be given. Since the sequence $F_1, \dots, F_{n_k}$ is admissible, it follows from the observation above that

$$\left\| \sum_{j=1}^{n} t_j v_j \right\| \geq \sum_{j=1}^{n_n} \alpha_i \left| \left\langle \sum_{j=1}^{k} t_j v_j, F_i \right\rangle \right|$$

$$= \sum_{j=1}^{k} |t_j| \sum_{i=n_{j-1}+1}^{n_j} \alpha_i |\langle v_j, F_i \rangle| > \frac{1}{2} \sum_{j=1}^{k} |t_j|.$$

Thus, the sequence (v_k) is equivalent to the usual basis of l^1.

Proof of Theorem 1 (1). By standard perturbation arguments, we need only establish the result for norm one vectors (u_i) and blocks (G_i) with $\max G_i < \min G_{i+1}$ such that $\{j\colon u_i(j) \neq 0\} \subset G_i$.

Let (s_i) be as in the statement of Lemma 5. If some subsequence of $(s_i) \to 0$, then we're done. If not, then there is a $\delta > 0$ such that, for each i, there is a block F_i with $F_i \subset G_i$ and $|\langle u_i, F_i \rangle| > \delta$.

Select a sequence of (u_i) (which we don't rename) so that $\lim_{i \to \infty} \langle u_i, N \rangle$ exists. Put $v_i = u_{2i-1} - u_{2i}$. Then $\|v_i\| \leq 2$ and $\lim_{i \to \infty} \langle v_i, N \rangle = 0$. By passing to a subsequence of (v_i) and again not renaming, we may assume that

$$\sum_{j=1}^{\infty} |\langle v_j, N\rangle| \le 1.$$

Thus, if F is any block, and $m \le n$, it follows that

$$\left|\sum_{j=m}^{n} \langle v_j, F\rangle\right| \le 5.$$

To see this, suppose that $H_1, H_2, \ldots$ is an admissible sequence of blocks, so that each v_i is supported in H_i (i.e. $\{j\colon v_i(j) \neq 0\} \subset H_i$.) Pick i_0 and j_0 so that $\inf F \in H_{i_0}$ and $\sup F \in H_{j_0}$. Then (since $|\langle x, F\rangle| \le \|x\|$ for any block F) it follows that

$$\left|\sum_{j=m}^{n} \langle v_j, F\rangle\right| \le |\langle v_{i_0}, F\rangle| + \sum_{j=i_0+1}^{j_0-1} |\langle v_j, F\rangle| + |\langle v_{j_0}, F\rangle|$$

$$\le \|v_{i_0}\| + 1 + \|v_{j_0}\| \le 5.$$

Finally, we show that for any subsequence (z_i) of (v_i), $\|z_1 + \cdots + z_n\| \to \infty$ as $n \to \infty$. For each i pick a block $F_i \subset H_i$ such that $|\langle z_i, F_i\rangle| > \delta$ and $\langle z_j, F_i\rangle = 0$ if $j \neq i$. Clearly, the sequence $F_1, F_2, \ldots$ is admissible. So, if $z^n = z_1 + \cdots + z_n$,

$$\|z^n\| \ge \sum_{i=1}^{n} \alpha_i |\langle z^n, F_i\rangle| \ge \sum_{i=1}^{n} \alpha_i |\langle z_i, F_i\rangle| \ge \delta \sum_{i=1}^{n} \alpha_i.$$

Thus, $\|z^n\| \to \infty$ as $n \to \infty$.

Now, observe that if F is any block,

$$\left|\left\langle \frac{z^n}{\|z^n\|}, F\right\rangle\right| = \frac{1}{\|z^n\|} |\langle z^n, F\rangle| \le \frac{5}{\|z^n\|} \to 0$$

as $n \to \infty$.

At last we are ready to select a sequence (x_i) equivalent to the usual basis of l^1. Let $n_1 = 1$. Inductively pick n_{k+1} so that $\|v_{n_k+1} + \cdots + v_{n_{k+1}}\| \ge 5 \cdot 2^k$.

Let $x_1 = v_1/\|v_1\|$ and, for $k > 1$, let

$$x_{k+1} = \frac{v_{n_k+1} + \cdots + v_{n_{k+1}}}{\|v_{n_k+1} + \cdots + v_{n_{k+1}}\|}$$

Then $\|x_k\| = 1$, and the sequence (x_k) satisfies the hypotheses of Lemma 5 for some admissible sequence $G_1, G_2, \ldots,$ so a subsequence of (x_k) is equivalent to the usual basis of l^1.

Proof of Theorem 1 (2). Suppose that (t_j) is a sequence of scalars such that, for each integer n, $\|\sum_{j=1}^{n} t_j e_j\| \leq 1$, yet $\sum_{j=1}^{\infty} t_j e_j$ does not converge.

Without loss of generality, we may assume that

(i) $\sup\|\sum_{j=1}^{n} t_j e_j\| = 1$.

(ii) There exists an $\varepsilon > 0$, such that if m is any integer, there is a $k > m$ with $\|\sum_{j=m}^{k} t_j e_j\| > \varepsilon$.

We claim that for every $\delta > 0$, there is an integer n such that, if F is a block with $\min F > n$, then $|\langle \sum_{j=1}^{\infty} t_j e_j, F\rangle| < \delta$. Let us assume for the moment that the claim has been established and finish the proof of (2).

Using property (i), we first find an integer p_0 such that, if $x = \sum_{j=1}^{p_0} t_j e_j$, then $\|x\| > 1 - \varepsilon/4$. Now pick an admissible sequence $F_1, F_2, \ldots, F_{n_0}$ such that

$$\|x\| = \sum_{i=1}^{n_0} \alpha_i |\langle x, F_i\rangle|.$$

Let $\delta > 0$ be any δ guaranteed by Lemma 4 for $\varepsilon = 1/2$ and the integer n_0. Using the claim, pick $p_1 > p_0$ so that if F is any block with $\min F \geq p_1$, then $|\langle \sum_{j=1}^{\infty} t_j e_j, F\rangle| < \delta$.

Let $y = \sum_{j=p_1}^{k} t_j e_j$ be chosen so that $\|y\| > \varepsilon$, as guaranteed by (ii). Pick blocks $G_1, G_2, \ldots, G_s$ such that $\min G_1 \geq p_1$ and

$$\|y\| = \sum_{i=1}^{s} \alpha_i |\langle x, G_i\rangle|.$$

Observe that $|\langle x, G_i\rangle| < \delta$ for all $i = 1, \ldots, s$. Thus, by the choice of δ,

$$\sum_{i=1}^{s} \alpha_{i+n_0} |\langle x, G_i\rangle| \geq \frac{\varepsilon}{2}.$$

Then the sequence $F_1, F_2, \ldots, F_{n_0}, G_1, \ldots, G_s$ is admissible, and

$$\left\| \sum_{i=1}^{k} t_i e_i \right\| \geq \sum_{i=1}^{n_0} \alpha_i |\langle x, F_i\rangle| + \sum_{i=n_0+1}^{n_0+s+1} \alpha_i |\langle y, G_{i-n_0}\rangle|$$

$$\geq 1 - \varepsilon/4 + \sum_{i=1}^{s} \alpha_{i+n_0} |\langle x, G_i\rangle| \geq 1 - \varepsilon/4 + \frac{\varepsilon}{2} > 1.$$

which is a contradiction. Thus, the basis (e_i) is boundedly complete.

It remains to prove the claim. If the claim were false, we could find blocks $G_1, G_2, \ldots$ such that $\max G_i < \min G_{i+1}$ for all i and

$$\left| \left\langle \sum_{j=1}^{\infty} t_j e_j, G_i \right\rangle \right| > \delta$$

for each i. But then, if $m > \max G_{i(m)}$, and $x^m = \sum_{j=1}^m t_j e_j$,

$$\|x^m\| > \sum_{i=1}^{i(m)} \alpha_i |\langle x^m, G_i\rangle| > \delta \sum_{i=1}^{i(m)} \alpha_i.$$

Since we can choose $i(m)$ so that $i(m) \to \infty$ as $m \to \infty$, it follows that $\|x^m\| \to \infty$ as $m \to \infty$, a contradiction. This establishes the claim and finishes the proof of part (2).

The following result is crucial to the proof of parts 3 (ii) and 4 of the Theorem:

LEMMA 6. *Let (u_i) be a bounded sequence in X and (G_i) an admissible sequence of blocks such that*

(i) $\{j\colon u_i(j) \neq 0\} \subset G_i$.

(ii) $\langle u_i, N\rangle = 0$ *for each* i.

(iii) (u_i) *is a weak Cauchy sequence in* X.

Then $(u_i) \to 0$ *weakly in* X.

Proof. First observe that (u_i) is an unconditional basic sequence in X. This follows easily from the fact that, for any scalars (t_i), and any j, $\|\sum_{i\neq j} t_i u_i\| \leq \|\sum_i t_i u_i\|$. See [7] (Proposition 1.c.6, page 18).

Now, assume that (u_i) does not converge weakly to 0. Then, there exists an $f \in X^*$, $\|f\| = 1$, and a $\delta > 0$ such that (passing to a subsequence of (u_i) and not renaming) $f(u_i) > \delta$ for all i. On the other hand, since (u_i) is unconditional and not equivalent to the usual basis of l^1, there are an N and non-negative scalars $t_1, \ldots, t_N$ such that

$$\sum_{i=1}^N t_i = 1 \quad \text{and} \quad \left\| \sum_{i=1}^N t_i v_i \right\| < \frac{\delta}{2}.$$

Thus,

$$\frac{\delta}{2} > f\left(\sum_{i=1}^N t_i v_i \right) > \sum_{i=1}^N t_i f(v_i) > \delta,$$

which contradicts the assumption that (u_i) does not converge weakly to 0. This completes the proof of Lemma 6.

Proof of Theorem 1 (3–i). If the sequence (e_i) were not weak Cauchy, we could find $n_1 < m_1 < n_2 < m_2 <, \ldots,$ a $\delta > 0$, and an $f \in X^*$ with $\|f\| = 1$ and $f(e_{n_i} - e_{m_i}) > \delta$ for all i. Thus,

$$\left\| \frac{1}{N} \sum_{i=1}^N (e_{n_i} - e_{m_i}) \right\| > \delta \quad \text{for all } N.$$

But since the basis (e_i) of X is subsymmetric, it follows from Lemma 3 that

$$\left\| \frac{1}{N} \sum_{i=1}^{N} (e_{n_i} - e_{m_i}) \right\| = \frac{1}{N} \sum_{i=1}^{2N} \alpha_i \to 0 \quad as\, N \to \infty.$$

Thus, the sequence (e_i) is weak Cauchy.

Suppose that this sequence has a weak limit $x \in X$. If $x = (t_j)$, then

$$t_j = \langle x, \{j\} \rangle = \lim_{i \to \infty} \langle e_i, \{j\} \rangle = 0,$$

so $x = 0$. On the other hand,

$$\langle x, N \rangle = \lim_{i \to \infty} \langle e_i, N \rangle = 1,$$

which is a contradiction.

Proof of Theorem 1. (3–ii). For each integer i, let $x_i = e_{2i} - e_{2i-1}$, and let X_0 be the closed subspace of X generated by the sequence (x_i). Since (x_i) is an unconditional basic sequence (see the proof of Lemma 6) and since X_0 contains no isomorph of c_0, it follows from **[4]** (Theorem 2, page 74) that X_0 is weakly sequentially complete. On the other hand, $\|x_i\| > 1$ for all i and, as was shown in the proof of part (3–i), $x_i \to 0$ weakly. Thus, X_0 fails the Schur property.

REMARK. Since the space X contains no isomorph of c_0 and fails to be weakly sequentially complete, it follows from a result of Bessaga and Pelczynski **[2]** that X does not imbed isomorphically in a space with an unconditional basis. (See also **[4]**, page 74.) H. Rosenthal has observed that, in fact, X does not have local unconditional structure.

Proof of Theorem 1 (4). Let $\theta_0 \in X^{**}$ be the weak* limit of the sequence (e_i) in X. We will show that if (v_i) is a weak Cauchy sequence in X, then $v_i \to x + \alpha \cdot \theta_0$, where $x \in X$ and $\alpha = \lim_{i \to \infty} \langle v_i, N \rangle$.

For each i, let $f_i \in X^*$ be defined by $f_i(e_j) = \delta_{ij}$. First, observe that if $u_i \to x^{**}$ weak*, then $x^{**} = x + \theta$, where $x \in X$ and $\theta(f_i) = 0$ for each i. (This follows from the fact that X is a dual space and the usual duality arguments.) Let $w_i = v_i - x$. Then $w_i \to \theta$ weak*. From this it follows that $f_j(w_i) \to \theta(f_j) = 0$ as $i \to \infty$. By standard pertubation arguments, we can assume that a subsequence of the (w_i) (which we don't rename) satisfies the following:

There is an admisible sequence (G_i) of blocks with $\max G_i + 1 < \min G_{i+1}$ and $\{j\colon w_i(j) \neq 0\} \subset G_i$.

Let $m_i = \max G_i + 1$, and $u_i = w_i - \langle w_i, N \rangle \cdot e_{m_i}$. By Lemma 6, $u_i \to 0$ weakly in X. On the other hand,

$$u_i = w_i - \langle w_i, N \rangle \cdot e_{m_i} \to \theta - \alpha \cdot \theta_0$$

weak* in X^{**}, where $\alpha = \lim_{i\to\infty}\langle w_i, N\rangle$. Thus, $\theta = \alpha \cdot \theta_0$. This shows that $x^{**} = x + \alpha \cdot \theta_0$ and completes the proof of part 4 and of Theorem 1.

Final remarks. There are a number of possible future directions that one might take in studying further the structure of the X_α spaces. We briefly list some of them:

(1) Determine the isomorphism types of the spaces X_α in terms of the sequence $\alpha = (\alpha_i)$.

(2) If X is isomorphic to $A \oplus B$, must one of A or B be isomorphic to X? (Corollary 2 shows that the usual decomposition techniques do not apply to the space X.)

(3) Since each X is a dual space, $X = Y^*$ for some Banach space Y. What is the subspace structure of Y? In particular, is Y hereditarily c_0?

(4) Is X hereditarily complementably l^1?

Added in proof. A. Andrew (Rocky Mountain J., to appear) has shown that X_α and X_β are isomorphic if and only if they are equal as sets, answering question (1). He also has shown that if X is isomorphic to $A \oplus B$, then one of A or B contains a complemented isomorph of X. The second named author (in preparation) has shown that the answer to question (4) is yes, and that, if $Y^* =$ X, there are many subspaces of Y isomorphic to c_0.

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