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# ON EXTREME POINTS AND SUPPORT POINTS OF THE FAMILY OF STARLIKE FUNCTIONS OF ORDER $\alpha$

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**Let $\mathrm{St}(\alpha)$ denote the subclass of functions $f(z)$ analytic in the open unit disk $D$ which satisfy the conditions $f(0) = 0$, $f'(0) = 1$ and $\mathrm{Re}(zf'(z)/f(z)) > \alpha$ for $z$ in $D$. In this note we investigate the compact, convex family $\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha))$ which is the closed convex hull of the set of all functions analytic in $D$ that are subordinate to some function in $\mathrm{St}(\alpha)$, $\alpha < 1/2$. The principal result establishes that every support point of $\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha))$ arising from a "nontrivial" functional must also be an extreme point, hence a function of the form $f(z) = xz/(1 - yz)^{2(1-\alpha)}$, $|x| = |y| = 1$.**

**To amplify on this synopsis, let $\mathscr{A}$ denote the set of functions analytic in the open unit disk $D = \{z \in \mathbf{C} \mid |z| < 1\}$. Then $\mathscr{A}$ is a locally convex linear topological space under the topology of uniform convergence on compact subsets of $D$. A function $f$ in $\mathscr{A}$ is said to be subordinate to a function $F$ in $\mathscr{A}$ (written $f \prec F$), if there is a function $\varphi$ in $B_0$ such that $f(z) = F(\varphi(z))$, where $B_0 = \{\varphi \in \mathscr{A} \mid \varphi(0) = 0, |\varphi(z)| < 1 \text{ in } D\}$.**

**Let $\mathscr{F}$ be a compact subset of $\mathscr{A}$. A function $f$ in $\mathscr{F}$ is a support point of $\mathscr{F}$ if there is a continuous linear functional $J$ on $\mathscr{A}$ such that**

$$\mathrm{Re}\, J(f) = \max\{\mathrm{Re}\, J(g) \mid g \in \mathscr{F}\}$$

**and $\mathrm{Re}\, J$ is non-constant on $\mathscr{F}$. We use $\Sigma\mathscr{F}$ to denote the set of support points of $\mathscr{F}$ and $\overline{\mathrm{co}}\mathscr{F}$ and $\mathscr{E}\overline{\mathrm{co}}\mathscr{F}$ to denote, respectively, the closed convex hull of $\mathscr{F}$ and the set of extreme points of the closed convex hull of $\mathscr{F}$.**

Let $S(\mathrm{St}(\alpha))$ denote the set of functions in $\mathscr{A}$ that are subordinate to some function in $\mathrm{St}(\alpha)$. Then $S(\mathrm{St}(\alpha))$ is a compact subset of $\mathscr{A}$ [**11**, p. 365]. In [**3**] and [**6**] it was shown that

$$\overline{\mathrm{co}}\, \mathrm{St}(\alpha) = \left\{ \int \frac{z}{(1 - xz)^{2(1-\alpha)}}\, d\mu(x) : \mu \text{ is a probability measure the unit circle} \right\}$$

and that

$$\mathscr{E}\overline{\mathrm{co}}\, \mathrm{St}(\alpha) = \sum \mathrm{St}(\alpha) = \left\{ \frac{z}{(1 - xz)^{2(1-\alpha)}} : |x| = 1 \right\}.$$

The analogous questions for $S(\mathrm{St}(\alpha))$ have not been so readily answered and only recently has a reasonably complete description been presented. Hallenbeck [8] and Hallenbeck and MacGregor [9] obtained $\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha))$ for $\alpha \le 0$ and $\alpha = 1/2$ in 1974. The missing link, $0 < \alpha < 1/2$, was completed by Perera in his doctoral dissertation [12]. Thus we now have

THEOREM (*Hallenbeck, MacGregor, Perera*). *Let* $\alpha \le 1/2$. *Then*

$$\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha)) = \left\{ \int \frac{xz}{(1-zy)^{2(1-\alpha)}}\, d\mu(x,y) : \mu \text{ is a probability measure on the torus} \right\},$$

$$\mathscr{E}\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha)) = \left\{ \frac{xz}{(1-yz)^{2(1-\alpha)}} : |x| = |y| = 1 \right\}.$$

If $1/2 < \alpha < 1$ and $p = 2(1-\alpha)$, then $0 < p < 1$ and the usual arguments break down. One encounters difficulties analogous to those for the families $V_k$ of functions with bounded boundary rotation, when $2 < k < 4$, or the families $C(\beta)$ of close-to-convex functions of order $\beta$, when $0 < \beta < 1$.

Also in [3] the following sharp inequalities were obtained (for $\alpha = 0$ see [13]): If $f$ is in $S(\mathrm{St}(\alpha))$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n$, then, for

$$\alpha \le 0, \quad |a_n| \le \frac{(2-2\alpha)(3-2\alpha)\cdots(n-2\alpha)}{(n-1)!} \qquad (n = 1, 2, \ldots)$$

and, for $1/2 \le \alpha < 1$, $|a_n| \le 1$ $(n = 1, 2, \ldots)$.

In [12] Perera also obtains, for $\alpha \le 1/2$, the support points of $\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha))$ as a consequence of a somewhat more general result. In this note we show that the first inequality above for the coefficients also obtains in the range $0 < \alpha < 1/2$, and examine the support points of $S(\mathrm{St}(\alpha))$ for $\alpha < 1/2$. In [10] Hallenbeck and MacGregor discussed the case $\alpha = 0$ and we extend this by showing, for $\alpha < 1/2$, that if $f$ is a support point of $S(\mathrm{St}(\alpha))$ corresponding to a continuous linear functional $J$ on $\mathscr{A}$ not of the form $J(f) = af(0) + bf'(0)$ $(f \in \mathscr{A}, a, b \in \mathbf{C})$, then $f$ is an extreme point of $\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha))$.

## 1. Extreme points of the closed convex hull of $S(\mathrm{St}(\alpha))$ ($\alpha \leq 1/2$).

LEMMA 1.1. *Let $U$ denote the unit circle $\{z \in \mathbf{C} \mid |z| = 1\}$ and let $\mu$ and $\nu$ be two probability measures on $U$. If $p$ and $q$ are two non-negative real numbers with $p + q \geq 1$, then there exists a probability measure $\lambda$ on $U \times U$ such that*

$$\left\{\int_U \frac{xz}{(1-xz)^p}\,d\mu(x)\right\}\left\{\int_U \frac{1}{(1-yz)}\,d\nu(y)\right\}^q = \int_{U\times U} \frac{xz}{(1-yz)^{p+q}}\,d\lambda(x,y).$$

*Proof.* It is well known that $\log(1-z)$ is univalent and convex. It follows that, if $f(z) \prec 1/(1-z)^p$ and $g(z) \prec 1/(1-z)^q$, then

$$f(z)\cdot g(z) \prec \frac{1}{(1-z)^{p+q}}.$$

This fact together with a trivial modification of the Herglotz formula yields

$$\frac{1}{(1-xz)^p}\cdot\left\{\int_U \frac{1}{(1-yz)}\,d\nu(y)\right\}^q \prec \frac{1}{(1-z)^{p+q}}.$$

Since $p + q \geq 1$ a result of Brannan, Clunie and Kirwan ([**2**], p. 5) yields

$$\frac{1}{(1-xz)^p}\cdot\left\{\int_U \frac{1}{(1-yz)}\,d\nu(y)\right\}^q = \int_U \frac{1}{(1-wz)^{p+q}}\,d\alpha(w),$$

for some probability measure $\alpha$ on $U$. Hence we have

$$\left\{\int_U \frac{xz}{(1-xz)^p}\,d\mu(x)\right\}\left\{\int_U \frac{1}{(1-yz)}\,d\nu(y)\right\}^q = \int_{U\times U} \frac{xz}{(1-wz)^{p+q}}\,d\alpha(w)\,d\mu(x).$$

Now it is easy to see that the right hand side of the above equation belongs to the set

$$\left\{\int_{U\times U} \frac{xz}{(1-yz)^{p+q}}\,d\lambda(x,y)\,\middle|\,\lambda \text{ is a probability measure on } U\times U\right\}$$

and the lemma follows.

THEOREM 1.2. *Let $U$ be the unit circle $\{z \in \mathbf{C} \mid |z| = 1\}$ and $\alpha \leq 1/2$. Also let $\mathscr{F}$ consist of the functions*

$$f_\lambda(z) = \int_{U\times U} \frac{xz}{(1 - yz)^{2(1-\alpha)}}\, d\lambda(x, y),$$

*where $\lambda$ varies over the probability measures on $U \times U$. Then $\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha)) = \mathscr{F}$ and*

$$\mathscr{E}\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha)) = \left\{ \frac{xz}{(1 - yz)^{2(1-\alpha)}} \,\middle|\, |x| = |y| = 1 \right\}.$$

*Proof.* This theorem was known for $\alpha \leq 0$ and $\alpha = 1/2$ ([**9**], [**8**]). Our aim here is to prove it for $0 \leq \alpha < 1/2$. The main tool is Lemma 1.1.

Suppose that $f$ is in $\mathscr{E}\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha))$. Then a result in [**11**, p. 366] implies that $f \prec g$ for some $g \in \mathscr{E}\overline{\mathrm{co}}\,\mathrm{St}(\alpha)$. $\mathscr{E}\overline{\mathrm{co}}\,\mathrm{St}(\alpha))$ was found in [**3**, p. 417] to be the set of all functions

$$\frac{z}{(1 - xz)^{2(1-\alpha)}} \quad \text{with } |x| = 1.$$

Hence we have

$$f(z) = \frac{\varphi(z)}{(1 - c\varphi(z))^{2(1-\alpha)}}$$

for some $|c| = 1$ and $\varphi$ in $B_0$. Write $f(z)$ in the form

$$f(z) = \bar{c}\left\{\frac{c\varphi(z)}{1 - c\varphi(z)}\right\} \cdot \left\{\frac{1}{1 - c\varphi(z)}\right\}^{(1-2\alpha)}$$

First using trivial modifications of the Herglotz formula and then applying the Lemma 1.1 with $p = 1$ and $q = 1 - 2\alpha (q \geq 0$ if $\alpha \leq 1/2)$ we obtain

$$f(z) = \bar{c}\int_{U\times U} \frac{xz}{(1 - yz)^{2(1-\alpha)}}\, d\lambda(x, y)$$

for some probability measure $\lambda$ on $U \times U$. Since

$$\frac{\bar{c}xz}{(1 - yz)^{2(1-\alpha)}} \in \mathscr{F}, \quad \text{for all } |c| = |x| = |y| = 1,$$

and $\mathscr{F}$ is compact and convex, it is clear that $f \in \mathscr{F}$. Hence $\mathscr{E}\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha)) \subseteq \mathscr{F}$ and $\overline{\mathrm{co}}\, S(\mathrm{St}(\alpha)) \subseteq \mathscr{F}$. On the other hand

$$\frac{xz}{(1 - yz)^{2(1-\alpha)}} \in S(\mathrm{St}(\alpha)),$$

which implies that $\mathscr{F} \subseteq \overline{\text{co}}\, S(\text{St}(\alpha))$ and $\overline{\text{co}}\, S(\text{St}(\alpha)) = \mathscr{F}$. Now Theorem 1.1 in [4] yields

$$\mathscr{E}\,\overline{\text{co}}\, S(\text{St}(\alpha)) \subseteq \left\{ \frac{xz}{(1-yz)^{2(1-\alpha)}} \,\middle|\, |x| = |y| = 1 \right\}.$$

These sets are actually equal. For, if

$$\frac{x_0 z}{(1-y_0 z)^{2(1-\alpha)}} = \int_{U\times U} \frac{xz}{(1-yz)^{2(1-\alpha)}}\, d\lambda(x,y),$$

then by now standard methods we obtain $x_0 = \int_{U\times\{y_0\}} x\, d\lambda(x,y)$ and $\lambda(\{x_0, y_0\}) = 1$. Hence the theorem.

Corollary 1.3. *Let* $f(z) \in S(\text{St}(\alpha))$ *and* $f(z) = \sum_{n=1}^{\infty} a_n z^n$. *If* $\alpha \leq 1/2$, *then*

$$|a_n| \leq \frac{(2-2\alpha)(3-2\alpha)\cdots(n-2\alpha)}{(n-1)!} \qquad (n = 1, 2, \ldots)$$

*and the inequality is sharp.*

*Proof.* This follows immediately from Theorem 1.2 and the argument given in [11, p. 366].

Remarks. (1) Corollary 1.3 was known for $\alpha = 0$, a result of W. Rogosinski [13, p. 72] and for $\alpha \leq 0$ and for $\alpha = 1/2$ [8, p. 61]. Since the sharp bounds for the Taylor coefficients were also known for $1/2 \leq \alpha < 1$ [3, p. 423], we have now completed the determination of sharp bounds for the Taylor coefficients of the functions in $S(\text{St}(\alpha))$.

(2) It was noted in [8] that Theorem 1.2 is not true for $1/2 < \alpha < 1$. We note that if $1/2 < \alpha < 1$ then $\overline{\text{co}}\, S(\text{St}(\alpha))$ has a large number of extreme points. We claim that if $1/2 < \alpha < 1$, then

$$\psi(z)/(1 - x\psi(z))^{2(1-\alpha)}$$

belongs to $\mathscr{E}\,\overline{\text{co}}\, S(\text{St}(\alpha))$ where $\psi(z)$ is an inner function with $\psi(0) = 0$ and $|x| = 1$. For, if

$$\frac{\psi(z)}{(1 - x\psi(z))^{2(1-\alpha)}} = tf_1(z) + (1-t)f_2(z),$$

where $0 < t < 1$ and $f_1(z), f_2(z) \in \mathscr{E}\,\overline{\text{co}}\, S(\text{St}(\alpha))$, then

$$\frac{z}{(1-xz)^{2(1-\alpha)}} \in H^q$$

for some $q > 1$ (since $1/2 < \alpha < 1$) and

$$\|f_1\|_q, \|f_2\|_q \leq \left\| \frac{z}{(1-z)^{2(1-\alpha)}} \right\|_q .$$

The conclusion that $f_1(z) = f_2(z)$ can be drawn exactly the same way as in [**9**, p. 466]. Hence the claim.

**2. Support points of a family related to** $S(\mathrm{St}(\alpha))$. Let $U$ be the unit circle and

$$\mathscr{G}_p = \left\{ \int_{U\times U} \frac{xz}{(1-yz)^p}\, d\mu(x, y) \,\middle|\, \mu \text{ is a probability measure on } U \times U \right\} \quad (p > 0).$$

In §1 we showed that, if $\alpha \leq 1/2$, then $\mathscr{G}_{2(1-\alpha)} = \overline{\mathrm{co}}\, S(\mathrm{St}(\alpha))$. In this section we are interested in determining the support points of the compact convex family $\mathscr{G}_p$. In §3 we use this result when we consider the problem of support points of $S(\mathrm{St}(\alpha))$. We first need a theorem from the first named author's doctoral dissertation and a lemma. We reproduce the proof of the theorem for completeness.

LEMMA 2.1. (*D. Cantor, R. R. Phelps* [**5**].) *Let $a_1, \ldots, a_n$ be complex numbers with $|a_k| = 1$ $(k = 1, 2, \ldots, n)$ and $b_1, \ldots, b_n$ be distinct complex numbers with $|b_k| = 1$ $(k = 1, 2, \ldots, n)$. Then there exists a finite Blaschke product $B(z)$ such that $B(b_k) = a_k$ $(k = 1, 2, \ldots, n)$.*

THEOREM 2.2.

$$\sum \mathscr{G}_p = \left\{ \int_U \frac{\overline{B(y)}z}{(1-yz)^p}\, d\mu(y) \,\middle|\, B \text{ is a finite Blaschke product and } \nu \text{ is a probability measure on } U \right\}.$$

*Proof.* First note that

$$\mathscr{E}\mathscr{G}_p = \left\{ \frac{xz}{(1-yz)^p} \,\middle|\, |x| = |y| = 1 \right\}.$$

We begin as in [7]. Suppose that $f$ is a support point of $\mathcal{G}_p$. Then there is a continuous linear functional $J$ on $\mathcal{A}$ such that $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) | g \in \mathcal{G}_p\}$ and $\operatorname{Re} J$ is non constant on $\mathcal{G}_p$. If we let $M = \max\{\operatorname{Re} J(g) | g \in \mathcal{E}\mathcal{G}_p\}$, then the above equation becomes $\operatorname{Re} J(f) = M$, and

$$f(z) = \int_{U\times U} \frac{xz}{(1-yz)^p}\, d\mu(x, y),$$

for some probability measure $\mu$ on $U \times U$. Hence we have

$$\operatorname{Re} J\left\{\frac{xz}{(1-yz)^p}\right\} = M,$$

$\mu$ a.e. on $U \times U$, i.e. $\operatorname{Re} xF(y) = M$, $\mu$ a.e. on $U \times U$, where $F(y) = J\{z/(1-yz)^p\}$ is analytic in $\overline{D}$. If $\operatorname{Re} xf(y) = M$ holds at $(x_1, y_1)$ then $F(y_1) \neq 0$, for otherwise $M = 0$ and it follows that $J$ is constant on $\mathcal{G}_p$. Thus $|f(y)| = M$, $\mu$ a.e. on $U \times U$, and $x$ is uniquely determined by $xF(y) = |F(y)|$.

*Case* (i). $|F(y)| = M$ holds only for finitely many values of $y$. Then

$$f(z) = \sum_{k=1}^{n} \lambda_k \frac{x_k z}{(1-y_k z)^p} \quad \text{where } |x_k| = 1 = |y_k|, \quad \lambda_k > 0, \quad (k = 1, 2, \ldots, n)$$

and $\sum_{k=1}^{n} \lambda_k = 1$.

*Case* (ii). $|F(y)| = M$ holds for infinitely many values of $y$.

Then, as in [**10**, p. 539], $F(y) = MB(y)$ for some finite Blaschke product $B(z)$, $x$ is determined by $xB(y) = 1$ and the support of $\mu$ is the set $T = \{(x, y) \in U \times U | xB(y) = 1\}$. Then

$$f(z) = \int_T \frac{\overline{B}(y)z}{(1-yz)^p}\, d\mu(x, y).$$

Now for any Borel set $A$ of $U$ define $\nu(A) = \mu(C)$ where $C(\subseteq T)$ is the image of $A$ under the homeomorphism $y \to (\overline{B(y)}, y)$ of $U$ onto $T$. Clearly $\nu$ is a probability measure and $f(z)$ takes the form

$$f(z) = \int_U \frac{\overline{B(y)}z}{(1-yz)^p}\, d\nu(y).$$

The form for $f(z)$, obtained in case (i), can also be written in the above form. For we can use Lemma 2.1 with $b_k = y_k$ and $a_k = \overline{x}_k$.

Conversely

$$\int_U \frac{\overline{B(y)}z}{(1-yz)^p}\,d\nu(y)$$

is a support point of $\mathcal{G}_p$ for each finite Blaschke product $B(z)$ and for each probability measure $\nu$ on $U$. To see this choose a continuous linear functional $J$ on $\mathcal{A}$ such that $J\{z/(1-yz)^p\} = B(y)$. (This is easily seen to be possible.) It is immediate that $\operatorname{Re} J$ is non constant and peaks at

$$\int \frac{\overline{B(y)}z}{(1-yz)^p}\,d\nu(y).$$

**3. Support points of $S(\mathrm{St}(\alpha))$.**

LEMMA 3.1. *Let $\varphi(z)$ be a finite Blaschke product with $\varphi(0) = 0$ and let $c$ be a complex number with $|c| = 1$. If $\alpha < 1/2$ and $\varphi(z)/(1-c\varphi(z))^{2(1-\alpha)}$ is a support point of $S(\mathrm{St}(\alpha))$ then $\varphi(z) = xz$ for some $|x| = 1$.*

*Proof.* We first note that a result in [**6**, p. 83] gives

$$(*)\quad \frac{1+c\varphi(z)}{1-c\varphi(z)} = \sum_{k=1}^{n} \lambda_k \frac{1+x_k z}{1-x_k z} \quad \text{where } n \text{ is a positive integer,}$$

$$|x_k| = 1,\ \lambda_k > 0\ (k = 1, 2, \ldots, n) \text{ and } \sum_{k=1}^{n} \lambda_k = 1.$$

If we let $q = 1 - 2\alpha\ (> 0)$, then

$$\frac{\varphi(z)}{(1-c\varphi(z))^{2(1-\alpha)}} = \bar{c}\left\{\frac{c\varphi(z)}{1-c\varphi(z)}\right\}\cdot\left\{\frac{1}{1-c\varphi(z)}\right\}^q$$

$$= \sum_{k=1}^{n} \lambda_k \frac{\bar{c}x_k z}{1-x_k z}\cdot h(z) \quad \text{where } h(z) = \left\{\frac{1}{1-c\varphi(z)}\right\}^q$$

and we have used $(*)$ in the second equality. By Lemma 1.1 we have

$$\frac{x_k z}{1-x_k z}h(z) = \int \frac{xz}{(1-yz)^{2(1-\alpha)}}\,d\lambda(x, y), \quad \text{and thus}$$

$$\frac{\bar{c}x_k z}{1-x_k z}h(z) = \int \frac{xz}{(1-yz)^{2(1-\alpha)}}\,d\lambda_1(x, y).$$

By Theorem 1.2, $\bar{c}x_k z/(1 - x_k z)h(z)$ belongs to $\overline{\text{co}}\, S(\text{St}(\alpha))$. Consequently if

$$\frac{\varphi}{(1 - c\varphi)^{2(1-\alpha)}} = \left\{ \sum_{k=1}^{n} \lambda_k \frac{\bar{c}x_k z}{1 - x_k z} \right\} h(z) = \sum_{k=1}^{n} \lambda_k \left\{ \frac{\bar{c}x_k z}{1 - x_k z} h(z) \right\}$$

is a support point of $S(\text{St}(\alpha))$, hence also of $\overline{\text{co}}\, S(\text{St}(\alpha))$, then so is each term. That is, $(\bar{c}x_k z/(1 - x_k z))h(z)$ is a support point of $\overline{\text{co}}\, S(\text{St}(\alpha))$.

Now by Theorem 2.2 we must have

$$\left\{ \frac{x_k \bar{c} z}{1 - x_k z} \right\} \cdot h(z) = \int_U \frac{\overline{B_k(y)} z}{(1 - yz)^{2(1-\alpha)}} \, d\nu_k(y)$$

for some finite Blaschke product $B_k(z)$ and some probability measure $\nu_k$ on $U$ ($k = 1, 2, \ldots, n$). In view of (*) we can write this as

$$\left\{ \frac{x_k \bar{c} z}{1 - x_k z} \right\} \cdot \left\{ \sum_{j=1}^{n} \lambda_j \frac{1}{1 - x_j z} \right\}^q = \int_U \frac{\overline{B_k(y)} z}{(1 - yz)^{2(1-\alpha)}} \, d\nu_k(y).$$

Comparison of the $z$ coefficient of both sides yields

$$\int_U \overline{B_k(y)} \, d\nu_k(y) = x_k \bar{c},$$

which implies that $\nu_k$ is a point mass at some $w_k$ ($|w_k| = 1$, $k = 1, 2, \ldots, n$). Hence we have

$$\left\{ \frac{x_k \bar{c} z}{1 - x_k z} \right\} \cdot \left\{ \sum_{j=1}^{n} \lambda_j \frac{1}{1 - x_j z} \right\}^q = \frac{\overline{B_k(w_k)} z}{(1 - w_k z)^{2(1-\alpha)}} \quad \text{and}$$

$$\overline{B_k(w_k)} = x_k \bar{c}.$$

Now since $q = 1 - 2\alpha > 0$ ($\alpha < 1/2$), comparison of singularities of the above equation gives $n = 1$, as required.

THEOREM 3.2. *Let $\alpha < 1/2$ and $J$ be a continuous linear functional on $\mathscr{A}$ not of the form $J(f) = af(0) + bf'(0)$ ($a, b \in \mathbf{C}$ and $f \in \mathscr{A}$). If $f_0$ is a support point of $S(\text{St}(\alpha))$ associated with $J$, then $f_0(z) = xz/(1 - yz)^{2(1-\alpha)}$.*

*Proof.* Let $f_0 \prec g_0$ where $g_0 \in \text{St}(\alpha))$ and consider $\mathscr{G} = \{ f \in \mathscr{A} | f \prec g_0 \}$. Then $f_0$ is in $\mathscr{G}$ and $\text{Re}\, J$ peaks over $\mathscr{G}$ at $f_0$. If $\text{Re}\, J$ is constant over $\mathscr{G}$ then $\text{Re}\, J(g_0(xz^m)) =$ constant, for all $|x| = 1$, $m = 1, 2, 3, \ldots$. Hence $J\{z^m\} = 0$ ($m = 1, 2, \ldots$), which violates the assumption on the form on

$J$. Thus $\operatorname{Re} J$ is non constant over $\mathcal{G}$ and $f_0$ is a support point of $\mathcal{G}$. By a result of Abu-Muhanna [**1**], (see also [**10**]), $f_0(z) = g_0(\varphi_0(z))$ where $\varphi_0$ is a finite Blaschke product with $\varphi_0(0) = 0$. We claim $\varphi_0(z) = x_0 z$ for some $|x_0| = 1$ and $g_0(z) = z/(1 - cz)^{2(1-\alpha)}$ for some $|c| = 1$. To see this define $L$ on $\operatorname{St}(\alpha)$ by $L(g) = J\{g(\varphi_0(z))\}$. Then $L$ is a continuous linear functional on $\mathcal{A}$ and $\operatorname{Re} L$ peaks over $\operatorname{St}(\alpha)$ at $g_0$. If $\operatorname{Re} L$ is non constant over $\operatorname{St}(\alpha)$ then $g_0$ becomes a support point of $\operatorname{St}(\alpha)$, and $g_0(z) = z/(1 - cz)^{2(1-\alpha)}$ for some $|c| = 1$ [**6**, p. 89].

Hence $f_0(z) = \varphi_{0(z)}/(1 - c\varphi_0(z))^{2(1-\alpha)}$ and, by Lemma 3.1, $\varphi_0(z) = x_0 z$ with $|x_0| = 1$ as desired. If $\operatorname{Re} L$ is constant over $\operatorname{St}(\alpha)$, then $\operatorname{Re} J\{g(\varphi_0(z))\} = \operatorname{Re} J\{g_0(\varphi_0(z))\}$ for all $g$ in $\operatorname{St}(\alpha)$, and hence $g(\varphi_0(z))$ is a support point of $S(\operatorname{St}(\alpha))$ for all $g$ in $\operatorname{St}(\alpha)$. In particular this is true when $g(z) = z/(1 - cz)^{2(1-\alpha)}$ and so $\varphi_0(z)/(1 - c\varphi_0(z))^{2(1-\alpha)}$ is a support point of $S(\operatorname{St}(\alpha))$. Again, by Lemma 3.1, $\varphi_0(z) = x_0 z$ for some $|x_0| = 1$. We now have $\operatorname{Re} J\{g(x_0 z)\} =$ constant, for all $g$ in $\operatorname{St}(\alpha)$. If we take $g(z) = z/(1 - xz)^{2(1-\alpha)}$, $|x| = 1$, it follows that $J(z^n) = 0$, $n = 2, 3, \ldots$, again violating the assumed form of $J$. Consequently $\operatorname{Re} L$ is non constant over $\operatorname{St}(\alpha)$ and the theorem follows.

REMARKS. (1) It is not difficult to show that each function

$$xz/(1 - yz)^{2(1-\alpha)} \qquad (|x| = |y| = 1)$$

is a support point corresponding to a continuous linear functional $J$ not of the form $J(f) = af(0) + bf'(0)$.

(2) Theorem 3.2 is not true for $1/2 \le \alpha < 1$. For example

$$z^n/(1 - xz^n) \qquad (|x| = 1, n = 1, 2, \ldots)$$

is always a support point of $S(\operatorname{St}(\alpha))$ when $1/2 \le \alpha < 1$. Moreover, if $\alpha = 1/2$, with a trivial modification of the proof given in [**10**] for $\Sigma S(K)$, where $K$ is the usual subclass of convex functions, one can show that

$$\sum\left(S\left(\operatorname{St}\left(\tfrac{1}{2}\right)\right)\right) = \left\{f \circ \varphi \,|\, f \in \operatorname{St}\left(\tfrac{1}{2}\right) \text{ and } \varphi \text{ is a finite Blaschke product with } \varphi(0) = 0\right\}.$$

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REFERENCES

[1] Y. Abu-Muhanna, *Variability region and support points of subordinate families*, to appear.

[2] D. A. Brannan, J. G. Clunie and W. E. Kirwan, *On the coefficient problem for functions of bounded boundary rotations*, Ann. Acad. Sci. Fenn. Ser. A VI Phys., **523** (1973), 18 pp.

[3] L. Brickman, D. J. Hallenbeck, T. H. MacGregor and D. R. Wilken, *Convex hulls and extreme points of families of starlike and convex mappings*, Trans. Amer. Math. Soc., **185** (1973), 413–428.

[4] L. Brickman, T. H. MacGregor and D. R. Wilken, *Convex hulls of some classical families of univalent functions*, Trans. Amer. Math. Soc., **156** (1971), 91–107.

[5] D. G. Cantor and R. R. Phelps, *An elementary interpolation theorem*, Proc. Amer. Math. Soc., **16** (1965), 523–525.

[6] P. C. Cochrane and T. H. MacGregor, *Fréchet differentiable functionals and support points for families of analytic functions*, Trans. Amer. Math. Soc., **236** (1978), 75–92.

[7] E. Grassman, W. Hengartner and G. Schober, *Support points of the class of close to convex functions*, Canad. Math. Bull., **19**, #2 (1976), 177–179.

[8] D. J. Hallenbeck, *Extreme points of classes of functions defined through subordination*, Proc. Amer. Math. Soc., **46** (1) (1974), 59–64.

[9] D. J. Hallenbeck and T. H. MacGregor, *Subordination and extreme point theory*, Pacific J. Math., **50** (1974), 455–468.

[10] ______, *Support points of families of analytic functions described by subordinations*, Trans. Amer. Math. Soc., **278** (2) (1983), 523–546.

[11] T. H. MacGregor, *Applications of extreme point theory to univalent functions*, Michigan Math. J., **19** (1972), 361–376.

[12] A. A. S. Perera, *Support points and extreme points of some classes of analytic functions*, Doctoral Dissertation, State University of New York at Albany, 1983.

[13] W. Rogosinski, *On the coefficients of subordinate functions*, Proc. London Math. Soc., (2) **48** (1943), 48–52.


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