PACIFIC JOURNAL OF MATHEMATICS
Vol 123, No. 1, 1986

ON A PROBLEM OF GAUSS-KUZMIN TYPE FOR CONTINUED FRACTION WITH ODD PARTIAL QUOTIENTS

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Let x be a number of the unit interval. Then x may be written in a unique way as a continued fraction

$$x = 1/(\alpha_1(x) + \varepsilon_1(x)/(\alpha_2(x) + \varepsilon_2(x)/(\alpha_3(x) + \cdots)))$$

where $\varepsilon_n \in \{-1,1\}$, $\alpha_n \geq 1$, $\alpha_n \equiv 1 \pmod 2$ and $\alpha_n + \varepsilon_n > 1$. Using the ergodic behaviour of a certain homogeneous random system with complete connections we solve a variant of Gauss-Kuzmin problem for the above expansion.

1. Introduction. We define continued fraction with odd partial quotients as follows. Let us partition the unit interval into

$$\left(\frac{1}{2k}, \frac{1}{2k-1}\right], \quad k = 1,2,\ldots, \quad \text{and} \quad \left(\frac{1}{2k-1}, \frac{1}{2k-2}\right], \quad k = 2,3,\ldots$$

and define the transformation T: $[0,1] \to [0,1]$ by

$$Tx = e\left(\frac{1}{x} - (2k-1)\right)$$

where

$$e = 1 \quad \text{if } x \in \left(\frac{1}{2k}, \frac{1}{2k-1}\right],$$

and

$$e = -1 \quad \text{if } x \in \left(\frac{1}{2k-1}, \frac{1}{2k-2}\right].$$

We arrive at

$$x = \frac{1}{2k - 1 + e(Tx)}$$

and therefore the map T generates a continued fraction

$$(1.1) \qquad x = \frac{1}{\alpha_1(x) + \varepsilon_1(x)/(\alpha_2(x) + \varepsilon_2(x)/(\alpha_3(x) + \cdots))}$$

$$= \begin{bmatrix} 1, \varepsilon_1(x), \varepsilon_2(x), \ldots \\ \alpha_1(x), \alpha_2(x), \alpha_3(x), \ldots \end{bmatrix}$$

where $\varepsilon_n \in \{-1, 1\}$, $\alpha_n \geq 1$, $\alpha_n \equiv 1 \pmod 2$ and $\alpha_n + \varepsilon_n > 1$. The expression (1.1) is called the *continued fraction with odd partial quotients expansion* of x.

Let us denote

$$r_n = \alpha_n + \left[\begin{matrix}\varepsilon_{n+1}, \dots \\ \alpha_{n+1}, \dots\end{matrix}\right], \qquad n = 1, 2, \dots.$$

The purpose of this paper is to find the limit

$$\lim_{n\to\infty} \mu(r_n > t) = l$$

for a given nonatomic measure μ on the σ-algebra of the Borel sets of $[0, 1]$ and to estimate the error $\mu(r_n > t) - l$. This is the variant of Gauss-Kuzmin problem for the continued fraction with odd partial quotients expansion. For solving of the above problem we shall use the approach of the random system with complete connections.

NOTATION.
$N^* = \{1, 2, 3, \dots\}$,
$N = \{0, 1, 2, \dots\}$,
$R =$ the set of real numbers,
$[a] =$ the integral part of $a \in R$,
$I_A =$ the characteristic function of A,
$G = (\sqrt{5} + 1)/2$,
$\mathscr{B}_{[0,1]} =$ the σ-algebra of the Borel sets of $[0, 1]$,
$\mathscr{P}(X) =$ the power set of X,
$(X, \mathscr{X}^r) =$ the n-fold product measurable space of $(X, \mathscr{X})$.

2. Preliminaries.

DEFINITION 2.1. A quadruple $\{(W, \mathscr{W}), (X, \mathscr{X}), u, P\}$ is named a homogeneous random system with complete connections (RSCC) if

(i) $(W, \mathscr{W})$ and $(X, \mathscr{X})$ are arbitrary measurable spaces;
(ii) u: $W \times X \to W$ is a $(\mathscr{W} \otimes \mathscr{X}, \mathscr{W})$-measurable function;
(iii) P is a transition probability function from $(W, \mathscr{W})$ to $(X, \mathscr{X})$.

Next, denote the element $(x_1, \dots, x_n) \in X^n$ by $x^{(n)}$.

DEFINITION 2.2. The functions $u^{(n)}$: $W \times X^n \to W$, $n \in N^*$, are defined as follows:

$$u^{(n+1)}(w, x^{(n+1)}) = \begin{cases} u(w, x), & \text{if } n = 0 \\ u\big(u^{(n)}(w, x^{(n)}), x_{n+1}\big), & \text{if } n \geq 1. \end{cases}$$

Convention. We shall write $wx^{(n)}$ instead of $u^{(n)}(w, x^{(n)})$.

DEFINITION 2.3. The transition probability functions P_r, $r \in N^*$, are defined by

$$P_r(w,A)=\begin{cases} P(w,A), & \text{if } r=1\\ \sum_{x_1\in X} P(w,x_1)\sum_{x_2\in X} P(wx_1,x_2)\cdots \sum_{x_r\in X} P\big(wx^{(r-1)},x_r\big)I_A\big(x^{(r)}\big), & \text{if } r>1,\end{cases}$$

for any $w \in W$, $r \in N^*$ and $A \in \mathscr{X}^r$.

DEFINITION 2.4. Assume that $X^0 \times A = A$. Then we define

$$P_r^n(w,A) = P_{n+r-1}(w, X^{n-1} \times A),$$

for any $w \in W$, n, $r \in N^*$ and $A \in \mathscr{X}^r$.

THEOREM 2.5. (*Existence theorem.*) *Let* $\{(W,\mathscr{W}), (X,\mathscr{X}), u, P\}$ *be a homogeneous RSCC and let* $w_0 \in W$. *Then there exist a probability space* $(\Omega, \mathscr{K}, P_{w_0})$ *and two chains of random variables* $(\xi_n)_{n\in N^*}$ *and* $(\zeta)_{n\in N}$ *defined on* Ω *with values in X and W respectively, such that*

(i)(a) $P_{w_0}((\xi_n,\dots,\xi_{n+r-1}) \in A) = P_r^n(w_0, A)$,

(b) $P_{w_0}((\xi_{n+m},\dots,\xi_{n+m+r-1}) \in A|\xi^{(n)}) = P_r^m(w_0\xi^{(n)}, A)$, P_{w_0}-*a.e.*

(c) $P_{w_0}((\xi_{n+m},\dots,\xi_{n+m+r-1}) \in A|\xi^{(n)},\zeta^{(n)}) = P_r^m(\zeta_n, A)$, P_{w_0}-*a.e.*

for any n, m, $r \in N^*$ *and* $A \in \mathscr{X}^r$, *where* $\xi^{(n)}$, $\zeta^{(n)}$ *denote the random vectors* $(\xi_1,\dots,\xi_n)$ *and* $(\zeta_1,\dots,\zeta_n)$ *respectively.*

(ii) $(\zeta_n)_{n\in N}$ *is a homogeneous Markov chain with initial distribution concentrated in* w_0 *and with the transition operator U defined by*

$$Uf(w) = \sum_{x\in X} P(w,x)f(wx), \tag{2.1}$$

for any f real W-measurable and bounded function.

This theorem is proved by Iosifescu [**2**].

REMARK. (i) Letting $m = r = 1$ in (i)b we obtain

$$P_{w_0}\big(\xi_{n+1} \in A\,|\,\xi^{(n)}\big) = P\big(w_0\xi^{(n)}, A\big), \qquad P_{w_0}\text{-a.e.}$$

that is the conditioned distribution of ξ_{n+1} by the past depends actually by this, through $u^{(n)}$. This fact justifies the name of *chain of infinite order* or *chain with complete connections* used for $(\xi_n)_{n\in N}$.

(ii) On account of (2.1) we have

$$(2.2) \qquad U^n f(w) = \sum_{x^{(n)} \in X^n} P_n(w, x^{(n)}) f(wx^{(n)}), \quad n \in N^*$$

for any f real $\mathscr{W}$-measurable and bounded function.

(iii) The transition probability function of the Markov chain $(\zeta_n)_{n \in N^*}$ is

$$Q(w, A) = \sum_{x \in X} P(w, x) I_A(wx) = P(w, A_w),$$

where $A_w = \{x \in X\colon\ wx \in A\}$, $w \in W$. It follows that the transition probability after n paths of the Markov chain $(\zeta_n)_{n \in N}$ is

$$Q^n(w, A) = P_n\big(w, A_w^{(n)}\big),$$

where $A_w^{(n)} = \{x^{(n)}\colon wx^{(n)} \in A\}$.

2.6. Let Q_n be the transition probability function defined by

$$Q_n(w, A) = n^{-1} \sum_{k=1}^{n} Q^k(w, A)$$

and let U_n be the Markov operator associated with Q_n. Next, denote $L(W)$ the space of all real Lipschitz functions defined on W and assume that $(L(W), \|\cdot\|)$ is a Banach space with respect to a norm $\|\cdot\|$.

(i) If there exists a linear bounded operator U^∞ from $L(W)$ to $L(W)$ such that

$$\lim_{n \to \infty} \|U_n f - U^\infty f\| = 0,$$

for any $f \in L(W)$ with $\|f\| = 1$, we say U *ordered*.

(ii) If

$$\lim_{n \to \infty} \|U^n f - U^\infty f\| = 0,$$

for any $f \in L(W)$ with $\|f\| = 1$, we say U *aperiodic*.

(iii) If U is ordered and $U^\infty(L(W))$ is one-dimensional space, it is named *ergodic* with respect to $L(W)$.

(iv) If U is ergodic and aperiodic, it is named *regular* with respect to $L(W)$ and the corresponding Markov chain has the same name.

DEFINITION 2.7. If $\{(W, \mathscr{W}), (X, \mathscr{X}), u, P\}$ is a RSCC which satisfies the properties

(i) (W, d) is a metric separable space;

(ii) $r_1 < \infty$, where

$$r_k = \sup_{w' \neq w''} \sum_{X^k} P_k(w, x^{(k)}) \frac{d(w'x^{(k)}, w''x^{(n)})}{d(w', w'')}, \qquad k \in N^*;$$

(iii) $R_1 < \infty$, where

$$R_1 = \sup_{A \in \mathscr{X}} \sup_{w' \neq w''} \frac{|P(w', A) - P(w'', A)|}{d(w', w'')};$$

(iv) there exists $k \in N^*$ such that $r_k < 1$, it is named RSCC with contraction.

This definition is due to M. F. Norman [**3**].

THEOREM 2.8. *Let (W, d) be a compact space and $\{(W, \mathscr{W}), (X, \mathscr{X}), u, P\}$ be a* RSCC *with contraction.*

The Markov chain associated to the RSCC *is regular, if and only if, there exists a point $\tilde{w} \in W$ such that*

$$\lim_{n \to \infty} d(\sigma_n(\tilde{w}), w) = 0,$$

for any $w \in W$, where $\sigma_n(w) = \operatorname{supp} Q^n(w, \cdot)$ ($\operatorname{supp} \mu$ denotes the support of the measure μ).

LEMMA 2.9. *We have*

$$\sigma_{m+n}(w) = \overline{\bigcup_{w' \in \sigma_m(w)} \sigma_n(w')},$$

for any m, $n \in N$, $w \in W$ (the line designates the topological aderence).

Theorem 2.8 and Lemma 2.9 are due to Iosifescu [**1**].

DEFINITION 2.10. Let $\{(W, \mathscr{W}), (X, \mathscr{X}) u, P\}$ be a RSCC. The RSCC is called uniformly ergodic if for any $r \in N^*$ there exists a probability P_r^∞ on $\mathscr{X}^r$ such that $\lim \varepsilon_n = 0$, as $n \to \infty$, where

$$\varepsilon_n = \sup_{\substack{w \in W, r \subseteq N^* \\ A \in \mathscr{X}^r}} |P_r^n(w, A) - P_r^\infty(A)|.$$

THEOREM 2.11. *Let (W, d) be a compact space. If the* RSCC $\{(W, \mathscr{W}), (X, \mathscr{X}), u, P\}$ *with contraction has regular associated Markov chain, then it is uniform ergodic.*

This result one can find in [**1**].

3. The Gauss-Kuzmin type equation. Let μ be a nonatomic measure on $\mathscr{B}_{[0,1]}$ and define

$$F_n(w) = F_n(w, \mu) = \mu\left(r_{n+1}^{-1} < w\right), \qquad n \in N,\ w \in [0,1].$$

Clearly $F_0(w) = \mu([0, w])$.

PROPOSITION 3.2. (*The Gauss-Kuzmin type equation*) F_n, $n \in N$, *satisfy the relation*

$$F_{n+1}(w) = \sum_{\substack{(k,\varepsilon) \\ k \equiv 1 \,(\mathrm{mod}\, 2) \\ |\varepsilon| = 1,\ k + \varepsilon > 1}} \varepsilon\left(F_n\left(\frac{1}{k}\right) - F_n\left(\frac{1}{k + \varepsilon w}\right)\right), \qquad w \in [0,1].$$

Proof. We start from the relation

$$r_{n+1} = \alpha_{n+1} + \frac{\varepsilon_{n+1}}{r_{n+2}}.$$

Thus

$$\begin{aligned} F_{n+1}(w) &= \mu\left(r_{n+2}^{-1} < w, \varepsilon_{n+1} = 1\right) + \mu\left(r_{n+1}^{-1} < w, \varepsilon_{n+1} = -1\right) \\ &= \sum_{k = 1\,(\mathrm{mod}\, 2)} \mu\left((k + w)^{-1} < r_{n+1}^{-1} < k^{-1}\right) \\ &\quad + \sum_{\substack{k \equiv 1\,(\mathrm{mod}\, 2) \\ k \neq 1}} \mu\left(k^{-1} < r_{n+1}^{-1} < (k - w)^{-1}\right) \\ &= \sum_{(k,\varepsilon)} \varepsilon\left(F_n\left(\frac{1}{k}\right) - F_n\left(\frac{1}{k + \varepsilon w}\right)\right). \end{aligned}$$

and this completes the proof.

Further, suppose that F_0' exists and it is bounded (μ has bounded density). By induction we obtain that F_n' exists and it is bounded too for any $n \in N^*$. Deriving the Gauss-Kuzmin type equation we arrive at

$$F_{n+1}'(w) = \sum_{(k,\varepsilon)} \frac{1}{(k + \varepsilon w)^2} F_n'\left(\frac{1}{k + \varepsilon w}\right). \tag{3.1}$$

Let us denote for $\rho(w) = (G - 1 + w)^{-1} - (-G - 1 + w)^{-1}$, $w \in [0,1]$ and $n \in N$

$$f_n(w) = F_n'(w)/\rho(w).$$

Then (3.1) becomes

$$f_{n+1}(w) = \left(G^2 - (1-w)^2\right)$$
$$\times \sum_{(k,\varepsilon)} \frac{1}{((G-1)(k+\varepsilon w)+1)((G+1)(k+\varepsilon w)-1)} f_n\left(\frac{1}{k+\varepsilon w}\right).$$

Now, we prove

PROPOSITION 3.3. *The function*

$$P(w,(k,\varepsilon)) = \frac{G^2-(1-w)^2}{((G-1)(k+\varepsilon w)+1)((G+1)(k+\varepsilon w)-1)}$$

defines a transition probability function from $([0,1], \mathscr{B}_{[0,1]})$ *to* $(X, \mathscr{P}(X))$ *where*

$$X = \left\{(k,\varepsilon)\colon k \geq 1,\ k \equiv 1 \ (\mathrm{mod}\, 2),\ |\varepsilon| = 1,\ k+\varepsilon > 1\right\}.$$

Proof. We must verify that

$$\sum_{(k,\varepsilon)} P(w,(k,\varepsilon)) = 1.$$

Indeed, noting that $(G-1)^{-1} = G$ and $(G+1)^{-1} = -G+2$, we have

$$\sum_{(k,\varepsilon)} \frac{G^2-(1-w)^2}{((G-1)(k+\varepsilon w)+1)((G+1)(k+\varepsilon w)-1)}$$
$$= \frac{G^2-(1-w)^2}{G^2-1}\left(\sum_{k=1,3,\dots} \frac{1}{(k+w+G)(k+w+G-2)}\right.$$
$$\left. + \sum_{k=3,5,\dots} \frac{1}{(k-w+G)(k-w+G-w)}\right)$$
$$= \frac{G^2-(1-w)^2}{2G}\left(\sum_{k=1,3,\dots}\left(\frac{1}{k-2+w+G} - \frac{1}{k+w+G}\right)\right.$$
$$\left. + \sum_{k=3,5,\dots}\left(\frac{1}{k-2-w+G} - \frac{1}{k-w+G}\right)\right)$$
$$= \frac{G^2-(1-w)^2}{2G}\left(\frac{1}{G-(1-w)} + \frac{1}{G+(1-w)}\right) = 1$$

that is the desired result.

Now, we can define a random system with complete connections as follows.

$$\{(W, \mathscr{W}), (X, \mathscr{X}), u, P\} \tag{3.2}$$

where

$$W = [0,1], \quad \mathscr{W} = \mathscr{B},$$

$$X = \{(k, \varepsilon) \colon k \geq 1,\ k \equiv 1 \ (\mathrm{mod}\, 2),\ |\varepsilon| = 1,\ k + \varepsilon > 1\},$$

$$\mathscr{X} = \mathscr{P}(X), \quad u(w, (k, \varepsilon)) = \frac{1}{k + \varepsilon w},$$

$$P(w, (k, \varepsilon)) = \frac{G^2 - (1 - w)^2}{((G - 1)(k + \varepsilon w) + 1)((G + 1)(k + \varepsilon w) - 1)}.$$

4. The ergodic behaviour of the RSCC. In this section we study the ergodic behaviour of RSCC (3.2) in order to solve a Gauss-Kuzmin type theorem.

In what follows we shall introduce the norm $\|\cdot\|_L$ defined by

$$\|f\|_L = \sup_{w \in W} |f(w)| + \sup_{w' \neq w''} \frac{|f(w') - f(w'')|}{|w' - w''|}, \qquad f \in L(W).$$

Then $(L(W), \|\cdot\|_L)$ is a Banach algebra.

PROPOSITION 4.1. RSCC (3.2) *is a* RSCC *with contraction and its associated Markov operator U is regular with respect to* $L(W)$.

Proof. We have

$$\frac{dP}{dw} = \frac{2(1 - w)((G - 1)(x + \varepsilon w) + 1)((G + 1)(x + \varepsilon w) - 1)}{((G - 1)(k + \varepsilon w) + 1)^2((G + 1)(k + \varepsilon w) - 1)^2}$$
$$- \frac{2\varepsilon(G^2 - (1 - w)^2)((G^2 - 1)(x + \varepsilon w) + 1)}{((G - 1)(k + \varepsilon w) + 1)^2((G + 1)(k + \varepsilon w) - 1)^2},$$

$$\frac{du}{dw} = \frac{\varepsilon}{(k + \varepsilon w)^2}.$$

Therefore

$$\sup_{w \in W, (x, \varepsilon) \in X} \left| \frac{d}{dw} P(w, (x, \varepsilon)) \right| < \infty,$$

$$\sup_{w \in W} \left| \frac{d}{dw} u(w, (x, \varepsilon)) \right| < \frac{1}{(x - 1)^2}, \qquad k = 3, 5, \ldots$$

It follows that $R_1 < \infty$ and $r_1 < 1$, that is, (3.2) is a SALC with contraction.

To prove the regularity of U with respect to $L(W)$, define the recurrence relation $w_{n+1} = (w_n + 1)^{-1}$, $n \in N$, with $w_0 = w$. Clearly $w_{n+1} \in \sigma(w_n)$. Then using Lemma 2.9 and by induction we obtain $w_n \in \sigma_n(w)$, $n \in N^*$. Because w_n tends to $G - 1$ as $n \to \infty$, for every $w \in [0,1]$, then

$$|\sigma_n(w), G - 1| \leq |w_n - G + 1| \to 0$$

as $n \to \infty$. The regularity of U with respect to $L(W)$ follows from Theorem 2.8. and the proof is completed.

Now, by virtue of Theorem 2.11, RSCC (3.2) is uniformly ergodic. Moreover, Theorem 2.1.57 of Iosifescu-Theodorescu [2], implies that $Q^n(\cdot, \cdot)$ converges uniformly to a probability Q^∞ and there exist two positive constants $q < 1$ and c such that

$$\|U^n f - U^\infty f\|_L \leq cq^n \tag{4.1}$$

for all $n \in N^*$, $f \in L(W)$, where

$$U^\infty f = \int_W f(w) Q^\infty(dw). \tag{4.2}$$

Further, by virtue of Lemma 2.1.58 of Iosifescu-Theodorescu [**2**], U has no eigenvalues of modulus 1 other than 1. Then, taking into account Proposition 2.1.6 of [**2**] the adjoint of the operator U with the transition probability function Q has the only eigenvector the measure Q^∞, that is

$$\int_0^1 Q(w, B) Q^\infty(dw) = Q^\infty(B), \tag{4.3}$$

for all the Borel sets B of $[0,1]$.

Generally, the form of Q^∞ cannot be identified but in our case this is possible as we shall show below

PROPOSITION 4.2. *The probability Q^∞ has the density*

$$\rho(w) = \frac{1}{w + G - 1} - \frac{1}{w - G - 1}, \qquad w \in [0,1]$$

and the normalizing constant $1/(3 \log G)$.

Proof. By virtue of uniqueness of Q^∞ we have to prove the equality (4.3) where

$$Q(w, B) = \sum_{\substack{(x,\varepsilon)\in X \\ (x+\varepsilon w)^{-1}\in B}} P(w,(x,\varepsilon)), \qquad w \in [0,1],\ B \in \mathscr{B}_{[0,1]}.$$

Since the intervals $[0, u] \subset [0,1]$ generate $\mathscr{B}_{[0,1]}$, it suffices to verify the equality (4.3) only for $B = [0, u]$, $0 < u \leq 1$. First, we consider that $[u^{-1}]$ is even. Then

$$\int_0^1 Q(w, [0, u))\rho(w)\, dw$$

$$= \int_0^1 \left(\sum_{\substack{k=1,3,\dots \\ k>[u^{-1}-w]}} P(w,(k,1)) + \sum_{\substack{k=3,5,\dots \\ k>[u^{-1}]}} P(w,(k,-1)) \right) \rho(w)\, dw$$

$$= \int_0^1 \frac{G^2 - (1-w)^2}{2(G^2-1)}$$

$$\times \left(\sum_{k=[u^{-1}]+1}^{\infty} \left(\frac{1}{k-2+w+G} - \frac{1}{k+w+G} \right) \right) \rho(w)\, dw$$

$$+ \int_0^{1+[u^{-1}]-u^{-1}} \frac{G^2-(1-w)^2}{2G}$$

$$\times \left(\sum_{k=[u^{-1}]+1}^{\infty} \left(\frac{1}{k-2+w+G} - \frac{1}{k+w+G} \right) \right) \rho(w)\, dw$$

$$+ \int_{1+[u^{-1}]-u^{-1}}^{1} \frac{G^2-(1-w)^2}{2G}$$

$$\times \left(\sum_{k=[u^{-1}]+3}^{\infty} \left(\frac{1}{k-2-w+G} - \frac{1}{k-w+G} \right) \right) \rho(w)\, dw$$

$$= \log \frac{G+[u^{-1}]}{G+[u^{-1}]-1} \cdot \frac{G+[u^{-1}]-1}{G+u^{-1}-2} \cdot \frac{G+u^{-1}}{G+[u^{-1}]}$$

$$= \log \frac{(G+1)(G-1+u)}{(G-1)(G+1-u)} = \int_0^u \rho(w)\, dw.$$

Analogously if $[u^{-1}]$ is odd we have

$$\int_0^1 Q(w,[0,u))\rho(w)\,dw$$

$$= \int_0^{u^{-1}-[u^{-1}]} \frac{G^2-(1-w)^2}{2G}\cdot\frac{\rho(w)}{[u^{-1}]+w+G}\,dw$$

$$+\int_{u^{-1}-[u^{-1}]}^1 \frac{G^2-(1-w)^2}{2G}\cdot\frac{\rho(w)}{[u^{-1}]-2+w+G}\,dw$$

$$+\int_0^1 \frac{G^2-(1-w)^2}{2G}\cdot\frac{\rho(w)}{[u^{-1}]-w+G}\,dw$$

$$= \log\frac{u^{-1}+G}{[u^{-1}]+G}\cdot\frac{[u^{-1}]+G-1}{u^{-1}+G-1}\cdot\frac{[u^{-1}]+G}{[u^{-1}]+G-1}$$

$$= \log\frac{(G+1)(G-1+u)}{(G-1)(G+1-u)} = \int_0^u \rho(w)\,dw.$$

5. The Gauss-Kuzmin type theorem. Now, we may determine where $\mu(r_n > t)$ tends as $n\to\infty$ and give the rate of this convergence.

PROPOSITION 5.1. (*The solution of Gauss-Kuzmin type problem.*) *If the density F_0' of μ is a Riemann integrable function, then*

$$\lim_{n\to\infty}\mu(r_n>t) = \frac{1}{3\log G}\cdot\log\frac{(G+1)(t(G-1)+1)}{(G-1)(t(G+1)-1)},\qquad t\ge 1.$$

If the density F_0' of μ is a Lipschitz function, then there exist two positive constants c and $q<1$ such that for all $t\ge 1$, $n\in N^*$

$$\mu(r_n>t) = \frac{1}{3\log G}(1+\theta q^n)\log\frac{(G+1)(t(G-1)+1)}{(G-1)(t(G+1)-1)}$$

where $\theta=\theta(\mu,n,t)$ with $|\theta|\le c$.

Proof. Let F_0' be a Lipschitz function. Then $f_0\in L(W)$ and by virtue of (4.2)

$$U^\infty f_0 = \int_0^1 f_0(w)Q^\infty(dw) = \frac{1}{3\log G}\int_0^1 F_0'(w)\,dw = \frac{1}{3\log G}.$$

According to (4.1) there exist two constants c and $q<1$ such that

$$U^n f_0 = U^\infty f_0 + T^n f_0, \qquad n \in N^*,$$

with $\|T^n f_0\|_L \leq cq^n$.

Further, consider $C[0,1]$ the metric space of real continuous functions defined on $[0,1]$ with the norm $|\cdot| = \sup|\cdot|$. Since $L([0,1])$ is a dense subset of $C([0,1])$ we have

$$\lim_{n\to\infty} |T^n f_0| = 0 \tag{5.1}$$

for $f_0 \in C([0,1])$. Therefore (5.1) is valid for measurable f_0 which is Q^∞-almost surely continuous, that is for Riemann integrable f_0. Thus

$$\begin{aligned}\lim_{n\to\infty} \mu(r_n > t) &= \lim_{n\to\infty} F_{n-1}\left(\frac{1}{t}\right)\\ &= \lim_{n\to\infty} \int_0^{1/t} U^{n-1} f_0(u)\rho(u)\,dw\\ &= \frac{1}{3\log G}\int_0^{1/t} \rho(w)\,dw\end{aligned}$$

and the desired result follows.

References

[1] S. Grigorescu and M. Iosifescu, *Dependence with Complete Connections and Applications*, Ed. St. Encicl., Bucharest, 1982.

[2] M. Iosifescu and R. Theodorescu, *Random Processes and Learning*, Springer-Verlag, Berlin, 1969.

[3] M. F. Norman, *Markov Processes and Learning Models*, Academic Press, New York, 1972.

[4] F. Schweiger, *Continued fractions with odd and even partial quotients*, Arbeitsbericht Math. Instit. der Univ. Salzburg, 4 (1982), 59–70.

Received October 5, 1984.

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