PACIFIC JOURNAL OF MATHEMATICS
Vol. 123, No. 2, 1986

PEAK POINTS IN BOUNDARIES NOT OF FINITE TYPE

ALAN V. NOELL

It is known that, in domains in $\mathbf{C}^2$ which are pseudoconvex and of finite type, compact subsets of peak sets for $A^\infty(D)$ are peak sets for $A^\infty(D)$. We give an example of a convex domain D (not of finite type) whose weakly pseudoconvex boundary points form a line segment K, with the property: K is a peak set for $A^\infty(D)$, but a point $p \in K$ is not a peak point for any $A^\alpha(D)$, $\alpha > 0$. We also consider briefly the case when the weakly pseudoconvex boundary points form a disc.

0. Introduction. Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$ with C^∞ boundary, and let $A^\alpha(D)$ denote the algebra of functions holomorphic in D and of class C^α in $\overline{D}$; here $0 \leq \alpha \leq \infty$. A compact set $K \subset \partial D$ is a peak set for such an algebra A if there exists $f \in A$ so that $f = 1$ on K while $|f| < 1$ on $\overline{D} \setminus K$; f is said to be a peak function for K. (If a peak set is a singleton $\{p\}$, p is called a peak point.) If D is strongly pseudoconvex, Chaumat and Chollet have proved in [3] that every compact subset of a peak set for $A^\infty(D)$ is a peak set for $A^\infty(D)$. In [5] it was shown that this also holds for domains in $\mathbf{C}^2$ of finite type. (Recall that $D \Subset \mathbf{C}^2$ is of finite type if one-dimensional complex manifolds cannot be tangent to ∂D to arbitrarily high order.) If ∂D is allowed to contain a complex manifold, it is easy to see that compact subsets of peak sets need not be peak sets (cf. Example 2.2 below). The main purpose of this paper is to show that compact subsets can fail to be peak sets even if $\partial D \Subset \mathbf{C}^2$ contains no complex manifold.

A closely related question is whether points of ∂D are peak points for some $A^\alpha(D)$. It is known ([4], [6]) that, if p is a point of strong pseudoconvexity, then p is a peak point for $A^\infty(D)$. In the case of points of weak pseudoconvexity, the following fact is an immediate consequence of a paper of Bedford and Fornæss [1]: If $D \Subset \mathbf{C}^2$ is of finite type, each point p is a peak point for some $A^\alpha(D)$, where $\alpha = \alpha(p)$ is positive, but it may approach zero as the geometry of ∂D allows complex manifolds tangent to higher and higher order at p. The main example below shows that this degeneracy of α is reasonable since, if complex manifolds can be tangent to arbitrarily high order at p, p may fail to be a peak point for $\bigcup_{\alpha>0} A^\alpha(D)$.

Our main result is the following.

EXAMPLE 1.1. There exist a convex domain $D \Subset \mathbf{C}^2$ with C^∞ boundary and a compact set $K \subset \partial D$ so that

(a) $K = w(\partial D)$ (the weakly pseudoconvex boundary points of D) is a line segment;

(b) K is a peak set for $A^\infty(D)$;

(c) $0 \in K$, but 0 is not a peak point for any $A^\alpha(D)$, $\alpha > 0$; and

(d) every point of K is a peak point for $A(D) := A^0(D)$.

In §1 we construct D and K and prove they have the desired properties. In §2 we indicate how the construction of D can be altered slightly to give that other points of K fail to be peak points for $A^\alpha(D)$, $\alpha > 0$. Section 2 also contains a brief description of the situation in which $w(\partial D)$ is a disc.

I wish to thank J. E. Fornæss for suggesting the existence of the main example and making many helpful comments on this material.

1. The main example.

1.1. Let (z, w) be coordinates in $\mathbf{C}^2$ with $z = x + iy$ and $w = u + iv$. D will be defined near $[-2, 2] \times \{0\}$ by

$$D := \{(z, w) \colon u + \chi(|x|) + \phi(|y|) + v^2(1 + |z|^2/100) < 0\},$$

where χ and ϕ have the following properties:

(1.1.1) Each is C^∞, non-negative, and strictly convex off its zero set.

(1.1.2) $\{x; \chi(x) = 0\} = [-2, 2]$, and $\{y; \phi(y) = 0\} = \{0\}$.

(1.1.3) ϕ vanishes to infinite order at 0.

The precise form of χ is irrelevant for this section. We will choose ϕ later so that (c) above holds. We extend the definition of D so that it is convex with C^∞ boundary and strongly pseudoconvex away from $K := [-2, 2] \times \{0\}$. Note that (a) and (b) above are obvious—a peak function for K is e^w. We defer the proof of (d) above to 1.5 below.

The idea of the construction is as follows. For $0 < \varepsilon \ll 1$, the plane $\{w = -\varepsilon\}$ intersects the set $D \cap \{(x + iy, -\varepsilon); |x| < 1\}$ in a set R_ε which is the interior of a rectangle. If we had a peak function f for 0, the function $g := 1 - \operatorname{Re} f$ would be a harmonic function on R_ε, so $g(0, -\varepsilon)$ would be given as a Poisson integral over ∂R_ε. We could multiply g by a large positive constant so that it would be larger than 1 on a significant portion of ∂R_ε; furthermore, if we chose ϕ properly, the height of R_ε would decrease slowly as $\varepsilon \to 0$. These two facts would give a lower bound for $g(0, -\varepsilon)$. Any Lipschitz regularity of g would give a contradictory upper bound.

In practice it is much easier to make the above estimates if we first replace the family $\{R_\varepsilon\}$ by a one-parameter family of convex lenses whose Poisson kernels are analyzed more simply.

1.2. *A one-parameter family of lenses.* Let $D_t^j \subset \mathbf{C}(z)$, $j = 1, 2$, be the open disc with center $(-1)^j i/t$ and radius $\sqrt{1 + t^2}/t$ for $0 < t \ll 1$. Let L_t be the lens $D_t^1 \cap D_t^2$. ∂L_t intersects the $\operatorname{Re} z$ axis at ± 1 and the $\operatorname{Im} z$ axis at $\pm\delta(t)$, where $\delta(t) := t/(\sqrt{t^2 + 1} + 1)$. The interior angle ∂L_t makes at ± 1 is $\alpha(t) := 2\tan^{-1} t$. Note that

$$\alpha(t) \geq t \geq \delta(t) \quad \text{for small } t. \tag{1.2.1}$$

We map L_t to the unit disc U by a biholomorphism G_t: Map L_t to the wedge $\{z;\ |\operatorname{Arg} z| < \alpha(t)/2\}$ by $z \mapsto -(z + 1)/(z - 1)$; the wedge to the right half-plane by $z \mapsto z^{n(t)}$, where $n(t) := \pi/\alpha(t)$; and, the right half-plane to U by $z \mapsto (z - 1)/(z + 1)$. We extend G_t, the composition of these maps, to a homeomorphism between $\overline{L}_t$ and $\overline{U}$, and we put $H_t := (G_t)^{-1}$. G_t fixes the points -1, 0, and 1. Also, the first and last maps of which G_t is the composition are biholomorphic near -1 and the inverse image of -1, respectively, while the map $z \mapsto z^{n(t)}$ is Lipschitz of order $n(t)$ at 0. Thus, H_t is Lipschitz of order $1/n(t)$ at -1, and there exists a constant c_1 independent of t so that

$$|G_t(z) + 1| \geq \left[c_1|z + 1|\right]^{n(t)} \quad \text{if } |z + 1| < \tfrac{1}{2} \text{ and } z \in \overline{L}_t. \tag{1.2.2}$$

We use these facts to get the following estimate.

1.3. LEMMA. *Let L_t, $\delta(t)$, and $n(t)$ be as above. There exists a constant $c > 0$ independent of t so that, if g is a function satisfying*

(a) *g is continuous on $\overline{L}_t$ and harmonic on L_t,*

(b) *$g \geq 0$, and*

(c) *$g(z) \geq 1$ if $z \in \overline{L}_t$ and $|z + 1| < \frac{1}{2}$,*

then

$$g(0) \geq \exp[-c/\delta(t)].$$

Proof. Since $g \circ H_t$ is harmonic on U,

$$g(0) = \frac{1}{2\pi}\int_0^{2\pi} g\big(H_t(e^{i\theta})\big)\, d\theta. \tag{1.3.1}$$

Let $A_t := G_t(\{z \in \partial L_t;\ |z+1| < \frac{1}{2}\})$. By (1.2.2), A_t has length at least $(\frac{1}{2}c_1)^{n(t)}$; also, $g \circ H_t \geq 1$ on A_t. Using these two facts and (1.2.1) in (1.3.1), we get the desired inequality, if only c is large enough.

1.4. *Conclusion of the main example, part* (c). We put $\psi(t) := \exp(-c/t)$ (for $t > 0$) with c as in 1.3 above. Choose ϕ so that

$$\phi(t) \leq \exp(-1/\psi(t)).$$

Suppose f is a peak function at 0 for $A^\alpha(D)$, for some α, $0 < \alpha < 1$. Now $|\mathrm{Re} f(-1,0)| < 1$, so we may choose $M > 0$ so large that $M(1 - \mathrm{Re} f(z,w)) \geq 1$ if $(z,w) \in \overline{D}$, $|z+1| < \frac{1}{2}$, and $|w| \ll 1$. For ease of notation put $g := M(1 - \mathrm{Re} f)$. For each sufficiently small $t > 0$, the function $g(z, -\phi(\delta(t)))$ satisfies the conditions of Lemma 1.3, so

$$g(0, -\phi(\delta(t))) \geq \psi(\delta(t)). \tag{1.4.1}$$

Since $g \in C^\alpha(\overline{D})$, there is a constant $C > 0$ with

$$|g(0, -\phi(\delta(t)))| \leq C|\phi(\delta(t))|^\alpha. \tag{1.4.2}$$

Therefore, if only t is small enough,

$$\begin{aligned} C|\phi(\delta(t))|^\alpha &\geq \psi(\delta(t)) \quad \text{(by (1.4.1) and (1.4.2))} \\ &\geq [\exp(-1/\psi(\delta(t)))]^{\alpha/2} \geq |\phi(\delta(t))|^{\alpha/2}, \end{aligned}$$

so we get that $|\phi(\delta(t))|^{\alpha/2} \geq 1/C$ for all small t. This is impossible.

1.5. *Proof of* (d) *of the main example*. We show that each point of K is a peak point for $A(D)$.

Suppose μ is a representing measure for $p = (x_0, 0) \in K$. Since K is a peak set, the support of μ must be contained in K. Now the sequence $\{\exp[-n(z - x_0)^2]\}$ in $A(D)$ tends pointwise boundedly on K, as $n \to \infty$, to the characteristic function of $\{p\}$. Thus supp $\mu = \{p\}$, and by standard results from the theory of uniform algebras (e.g., 2.3.4 of [**2**]), p is a peak point for $A(D)$.

2. Two extensions of the main example.

2.1. Remark. It is reasonable to ask whether any point of the set K defined in 1.1 can be a peak point for some A^α, $\alpha > 0$. Of course, the arguments of §1 apply to each interior point of the line segment K. To show that an end-point, say $(2,0)$, can fail to be a peak point, we argue as follows. Choose the family of lenses so that the left vertex V_1 is fixed at 1 while the right vertex V_2 is at the point $2 + t$; the thickness is on the order of t as before. We map this figure to U so that $V_1 \mapsto -1$, $V_2 \mapsto 1$, and

$2 \mapsto 0$. A short computation similar to that in 1.3 above shows that we should choose χ so that $\chi(2+t) \leq \lambda(\lambda(t))$, where $\lambda(t) := t^{1/t}$, for $0 < t \ll 1$. With this choice of χ we can argue as in 1.4 that there can be no Lipschitz peak function at $(2, 0)$.

2.2. EXAMPLE. If we allow ∂D to contain a complex manifold, it is clear that compact subsets of peak sets for $A^\infty(D)$ may fail to be peak sets for even $A(D)$. For example, suppose $s \geq 0$ is a C^∞ increasing function so that $\{r; s(r) = 0\} = \{r; r \leq 1\}$ and s is strictly convex for $r > 1$. The domain

$$D_1 := \left\{(z, w);\ u + s(|z|) + v^2\left(1 + |z|^2/100\right) < 0\right\}$$

defined near $E := \{|z| \leq 1\} \times \{0\}$ is convex with C^∞ boundary, and E is a peak set for $A^\infty(D_1)$. However, no proper compact subset of E intersecting $U \times \{0\}$ can be a peak set for $A(D_1)$, since any such peak function would, as a holomorphic function on U, attain its maximum value at an interior point. Each point $p = (p_1, p_2)$ of $\partial U \times \{0\}$ is a peak point for $A(D_1)$ (cf. Remark 2.3 below.) One cannot guarantee greater regularity of the peak function, however. It is easy to see that no such peak function f can be in $A^1(D_1)$; if it were, $f|_U$ would have a non-zero derivative at p_1, and this would imply that f had a non-zero tangential derivative at p, a contradiction. It is also true p cannot be a peak point for $\bigcup_{\alpha > 0} A^\alpha(D_1)$; the argument is similar to that in 2.1 above. Slicing D_1 by a plane $\{w = -\varepsilon\}$ gives a disc whose radius shrinks to 1 as $\varepsilon \to 0$. Use of the explicit formula for the Poisson kernel on such a disc yields a much weaker condition on s than that imposed on ϕ and χ above; in fact, one only needs that s vanishes to infinite order at $r = 1$ to derive a contradiction to Lipschitz regularity for a peak function.

2.3. REMARK. Since $A(D)$ is a closed subalgebra of $C(\overline{D})$, standard results from the theory of uniform algebras (see, for example, 2.4.6 of **[2]**), imply that any peak set for $A(D)$ contains at least a peak point for $A(D)$. The examples of this paper illustrate that this result cannot be extended to the larger classes A^α, $\alpha > 0$.

REFERENCES

[1] E. Bedford and J. E. Fornæss, *A construction of peak functions on weakly pseudoconvex domains*, Ann. of Math., **107** (1978), 555–568.

[2] A. Browder, *Introduction to Function Algebras*, W. A. Benjamin, Inc., New York, 1969.

[3] J. Chaumat and A.-M. Chollet, *Caractérisation et propriétés des ensembles localement pics de $A^\infty(D)$*, Duke Math. J., **47** (1980), 763–787.

[4] M. Hakim and N. Sibony, *Frontière de Shilov et spectre de $A(\overline{D})$ pour des domaines faiblement pseudoconvexes*, C. R. Acad. Sci. Paris, **281** (1975), 959–962.

[5] A. Noell, *Properties of peak sets in weakly pseudoconvex boundaries in* $\mathbf{C}^2$, dissertation, Princeton University, 1983.

[6] P. Pflug, *Über polynomiale Funktionen auf Holomorphiegebieten*, Math. Z., **139** (1974), 133–139.

Received July 18, 1983 and in revised form December 20, 1984.

PRINCETON UNIVERSITY
PRINCETON, NJ 08544

Current address: Oklahoma State University
Stillwater, OK 74078