PACIFIC JOURNAL OF MATHEMATICS
Vol. 123, No. 2, 1986

PLANE ELLIPTIC GEOMETRY OVER RINGS

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The classical model of plane elliptic geometry is a sphere of the real affine space. The points of this model are the pairs of antipodal points of the sphere, and the lines are the great circles of the sphere. Right angles retain their ordinary meaning. This model is isomorphic to the real projective plane, where orthogonality on the set of lines is given by a symmetric bilinear form such that no line is orthogonal to itself.

In the present paper we attempt a foundation and a study of plane elliptic geometry over commutative rings.

CONTENTS

Introduction. Let R be a commutative ring with 1. The points of the projective plane $\Pi(R)$ are the sets Ra, where a is a vector of the free R-module R^3 such that a, b, c is a basis for some b, c. Replacing R^3 by the dual module R^{3*} one obtains the definition of a line Rg of $\Pi(R)$. Ra is incident with Rg if $ag = 0$. Two different points need not have a common line, and they can have more than one common line.

Let f: $R^3 \times R^3 \to R$ be a symmetric bilinear form such that $(a, a)f$ is a unit for every point Ra of $\Pi(R)$. For every $a \in R^3$ let a^* denote the linear function $R^3 \to R$, $x \mapsto (a, x)f$. Then the homomorphism $R^3 \to R^{3*}$, $a \mapsto a^*$ is a bijection; i.e. f is an inner product in the sense of [**6**]. If

Ra is a point then Ra^* is called the polar line of Ra. A line is called orthogonal to Ra^* if and only if it passes through Ra. The projective plane together with this polarity is called the elliptic plane $\Pi(R, f)$. The main purpose of Part II of our article is a synthetic approach to substructures of elliptic planes over rings.

Let Ra^* be a fixed line of $\Pi(R, f)$. Taking all points Rb such that $(a, b)f$ is a unit we obtain the point set of an affine plane $\Pi'(R)$. If R is a field then $\Pi'(R)$ contains all points except the points of the line Ra^*. But in general the affine plane will constitute a rather small part of the projective plane. This is one reason why the coordinatization of a (synthetically defined) elliptic plane involves difficulties.

In order to characterize pairs of points and lines being uniquely joined to each other we introduce a relation $\longmapsto$. $Ra \longmapsto Rb$ means that Rb is an affine point when Ra^* is the line at infinity. In our system of axioms we use only one basic relation $|$, standing for incidence. The relation $\longmapsto$ is derived from $|$ (compare **[10]**, where $\longmapsto$ is a basic term). Therefore, however, we must restrict our study to commutative rings where every non-unit is a zero-divisor. Then two points A, B satisfy $A \longmapsto B$ if and only if there is a pair of orthogonal lines g, h such that g is the unique perpendicular of h through A, and h is the unique perpendicular of g through B. This property will supply the definition of $\longmapsto$ in our axiomatic approach. If $A \longmapsto B$ and h is an arbitrary line through B then there is exactly one orthogonal of h through A. This property will be our first axiom. The second one is a richness condition, and the third one uses a three-reflection-theorem. A geometric property called (M) will not be used until we prove that 2 is a unit of the coordinate ring in the last section.

We need a weakened version of the property denoted by $\longmapsto$ which holds for arbitrary distinct points. Being unable to offer a suitable axiom that applies to any elliptic plane over a commutative ring, we use

(U) Given points A, B. Then there exists a line h through B such that exactly one line is incident with A and orthogonal to h.

In an elliptic plane over a commutative ring of stable rank ≤ 3 property (U) is valid. In particular (U) holds if any two points are incident with at least one common line. This plain geometric property would considerably facilitate our efforts. However we feel that this restriction is not adequate since it excludes too many rings.

In an elliptic plane each point and its polar line define a reflection. The basic concept of our system of axioms is a group G together with a subset S of involutions such that S generates G. The elements of S represent reflections, hence simultaneously also points and lines. Within this framework we formulate the axioms mentioned above.

The construction of a coordinate ring requires the existence of enough automorphisms of the geometric structure. Within our axiomatic approach the existence of sufficiently many suitable automorphisms need not be explicitly presupposed but can be proved.

Reflections of an elliptic plane are elements of its orthogonal group. Hence the study of elliptic geometry from our point of view is also a study of orthogonal groups over rings.

PART I. **Metric Planes over Rings**

Let R be a commutative ring with 1. Then the rank of a free R-module is unique. Let R^3 denote the free 3-dimensional R-module $R \times R \times R$. We write $R^{3*} := \operatorname{Hom}(R^3, R)$. Rx is called a *point* (a *line*) if x, y, z is a basis of R^3 (of R^{3*}) for some y, z. The point Rx is *incident* with the line Ry if $xy = 0$; notation: $Rx\mathrm{I}Ry$ or $Ry\mathrm{I}Rx$. Let $\mathscr{P}(R)$ denote the set of points and $\mathscr{L}(R)$ the set of lines. $\Pi(R) := (\mathscr{P}(R), \mathscr{L}(R), \mathrm{I})$ is called the projective plane over R. We shall write $\mathscr{P} = \mathscr{P}(R)$ and $\mathscr{L} = \mathscr{L}(R)$.

Every linear bijection $R^3 \to R^3$ induces a contragredient mapping $R^{3*} \to R^{3*}$. This pair of mappings induces an automorphism (collineation) of $\Pi(R)$.

Let $Rx, Ry \in \mathscr{P}$. We write Rx *distant* Ry if x, y, z is a basis of R^3 for some z. The analogous definition applies to a pair of lines. Take $Rx \in \mathscr{P}$ and $Ry \in \mathscr{L}$. We say Rx *distant* Ry or Ry *distant* Rx if $xy \in R^*$ (group of units).

Next we collect some elementary lemmas. From each of them a dual counterpart can be obtained by interchanging the words "point" and "line". I.1 and I.3 can be found in [7] and in [**10**]. Proofs of I.2 and I.5 are given in [**10**]. I.4 is due to [**2**].

I.1. *Let $A, B \in \mathscr{P}$. If A* distant *B then A, B lie on exactly one line g. We write $g = (A, B)$.*

I.2. *The following statements are equivalent.*

(i) *A* distant *B if and only if A, B have a unique common line.*

(ii) *Every non-unit of R is a zero-divisor.*

I.3. *Given $A \in \mathscr{P}$ and $g \in \mathscr{L}$. The following statements are equivalent.*
(i) A distant g.
(ii) A distant B *and* (A, B) distant g *for any point* $B \mathrm{I} g$.
(iii) A distant B *and* (A, B) distant g *for some point* $B \mathrm{I} g$.

An n-tuple $(\alpha_1, \ldots, \alpha_n) \in R^n$ is called *unimodular* if $R\alpha_1 + \cdots + R\alpha_n = R$. In other words, $\beta_1\alpha_1 + \cdots + \beta_n\alpha_n = 1$ for some $\beta_i \in R$. Another equivalent statement is that $ab = 1$ for some $b \in R^{n*}$, where $a = (\alpha_1, \ldots, \alpha_n)$.

I.4. *Any two points have at least one common line if and only if the following condition holds. Let $(\alpha, \beta) \in R \times R$. Then $(\alpha, \beta) = (\lambda\gamma, \lambda\delta)$ for some $\lambda \in R$ and some unimodular $(\gamma, \delta) \in R \times R$.*

I.5. *The following statements are equivalent*
(i) *Let $A \in \mathscr{P}$ and $g, h \in \mathscr{L}$ with $A \mathrm{I} g, h$. Then j distant g, h for some $j \in \mathscr{L}$ with $j \mathrm{I} A$.*
(ii) *R has stable rank 2; i.e. if $(\alpha, \beta) \in R \times R$ is unimodular then $\alpha + \beta\gamma$ is unimodular for some $\gamma \in R$ (i.e. $\alpha + \beta\gamma \in R^*$).*

I.6. *Suppose that R has stable rank ≤ 3; i.e. if $(\alpha, \beta, \gamma) \in R \times R \times R$ is unimodular then $(\alpha + \mu\gamma, \beta + \nu\gamma)$ is unimodular for some $\mu, \nu \in R$. Then for every $A \in \mathscr{P}$ and $g \in \mathscr{L}$ there is a line h such that $h \mathrm{I} A$ and h* distant g.

Proof. Every bijective linear mapping of R^3 induces an automorphism of $\Pi(R)$. Thus we may assume $g = R[0,0,1]$, $A = R(\alpha, \beta, \gamma)$. A set $h = R[\chi, \omega, \eta] \subseteq R^{3*}$ is a line with the asserted properties if and only if

$$(+) \qquad (\chi, \omega) \text{ is unimodular and } \alpha\chi + \beta\omega + \gamma\eta = 0.$$

The triple (α, β, γ) is unimodular since A is a point. By our assumption, $(\alpha + \mu\gamma, \beta + \nu\gamma)$ is unimodular for some $\mu, \nu \in R$. Let $\chi := -\beta - \nu\gamma$, $\omega := \alpha + \mu\gamma$, $\eta := \nu\alpha - \mu\beta$. Then $(+)$ holds.

A substructure Π' of an incidence structure Π is called *locally complete* if each line of Π which is incident with a point of Π' is a line of Π'.

I.7. *Let $u \in \mathscr{L}$. Define $\mathscr{P}' := \{A \in \mathscr{P}$: A distant $u\}$ and $\mathscr{L}' := \{g \in \mathscr{L}$: g distant $u\}$. Then the substructure $\Pi' := (\mathscr{P}', \mathscr{L}', \mathrm{I})$ of $\Pi = \Pi(R)$ is locally complete. For $g, h \in \mathscr{L}'$ define $g \| h$ (parallel) if $(g, u) = (h, u)$.*

Then for every $A \in \mathscr{P}'$ *and* $g \in \mathscr{L}'$ *there is a unique* $h \in \mathscr{L}'$ *with* $h \| g$ *and* $h \mathrm{I} A$.

The assertion follows immediately from I.1 and I.3. Π' is called the affine plane (related to the line u).

EXAMPLE. Let $u = R[0,0,1]$. Then $\mathscr{P}' = \{R(\alpha, \beta, 1)\colon \alpha, \beta \in R\}$ and $\mathscr{L}' = \{R[\alpha, \beta, \gamma]\colon \alpha, \beta, \gamma \in R$ and (α, β) unimodular$\}$.

Let $f\colon R^3 \times R^3 \to R$ be a symmetric bilinear form. *In the sequel we assume* $2 \in R^*$. Call lines Rg, Rh *orthogonal* if $(g,h)f = 0$. For $A \subseteq R^3$ let $A^\perp := \{x \in R^3\colon (a,x)f = 0$ for every $a \in A\}$.

If $a \in R^3$ satisfies $(a,a)f \in R^*$ then $R^3 = Ra \oplus Ra^\perp$, and

$$\sigma_{Ra}\colon R^3 \to R^3, \qquad x \mapsto -x + 2\frac{(x,a)f}{(a,a)f}a$$

is the linear injective mapping with $a \mapsto a$ and $x \mapsto -x$ for $x \in Ra^\perp$. σ_{Ra} is called the reflection in Ra. σ_{Ra} is an involution and satisfies $(x\sigma_{Ra}, y\sigma_{Ra})f = (x,y)f$ for $x, y \in R^3$, i.e. σ_{Ra} is orthogonal. $\sigma_{Ra} = \sigma_{Rb}$ implies $Ra = Rb$.

I.8. *Given* $a, b \in R^3$ *with* $(a,a)f$, $(b,b)f \in R^*$. *The following statements are equivalent.*

(i) σ_{Ra} *and* σ_{Rb} *commute.*

(ii) $a\sigma_{Rb} = \lambda a$ *for some* $\lambda \in R$ *with* $\lambda^2 = 1$.

(iii) $\lambda_1 a \in Rb$ *and* $\lambda_2 a \in Rb^\perp$ *for some* $\lambda_1, \lambda_2 \in R$ *with* $\lambda_1^2 = \lambda_1$, $\lambda_2^2 = \lambda_2$, $\lambda_1 + \lambda_2 = 1$ *and* $\lambda_1\lambda_2 = 0$.

(iv) $a\sigma_{Rb} \in Ra$.

Proof. Let $c := a\sigma_{Rb}$. Then $(*)$ $\sigma_{Rb}\sigma_{Ra}\sigma_{Rb} = \sigma_{Rc}$. (i) $\Rightarrow$ (ii). $(a\sigma_{Rb})\sigma_{Ra} = a\sigma_{Rb}$ implies $a\sigma_{Rb} = \lambda a$ for some $\lambda \in R$. Furthermore, $\lambda^2 = 1$ as $(a,a)f = \lambda^2(a,a)f$. (ii) $\Rightarrow$ (iii). We have $\lambda^2 = 1$ and $a\sigma_{Rb} = \lambda a$. Let $\lambda_1 := \frac{1}{2}(1+\lambda)$ and $\lambda_2 := \frac{1}{2}(1-\lambda)$. Then $\lambda_1 a \in Rb$ and $\lambda_2 a \in Rb^\perp$. (iii) $\Rightarrow$ (iv). (iii) implies $a\sigma_{Rb} = (\lambda_1 a + \lambda_2 a)\sigma_{Rb} = \lambda_1 a - \lambda_2 a = (\lambda_1 - \lambda_2)a \in Ra$, hence $Rc = Ra$ since σ_{Rb} is orthogonal. (iv) $\Rightarrow$ (i) follows immediately from $(*)$.

I.9. *Let* $(a,a)f, (b,b)f, (c,c)f \in R^*$. *The identity* $\sigma_{Ra}\sigma_{Rb} = \sigma_{Rc}$ *holds if and only if any two of the vectors* a, b, c *are orthogonal. Then* a, b, c *is a basis of* R^3.

Proof. Let us assume that a, b, c are pairwise orthogonal. Define $a^* \in R^{3*}$, $x \mapsto (x,a)f$ for $x \in R^3$; likewise b^*, c^*. We have $aa^* \in R^*$ and $ba^* = 0 = ca^*$. Let A denote the matrix whose rows are the coordinate vectors a, b, c, and let B denote the matrix whose columns are the coordinate vectors of a^*, b^*, c^* in the dual basis. Obviously,

$$AB = \begin{pmatrix} aa^* & & 0 \\ & bb^* & \\ 0 & & cc^* \end{pmatrix},$$

hence A is invertible. This means that a, b, c is a basis of R^3. We have $a\sigma_{Ra}\sigma_{Rb} = -a = a\sigma_{Rc}$, $b\sigma_{Ra}\sigma_{Rb} = b\sigma_{Rb}\sigma_{Ra} = b\sigma_{Ra} = -b = b\sigma_{Rc}$ (cf. I.8 (iii) $\Rightarrow$ (i)), $c\sigma_{Ra}\sigma_{Rb} = c = c\sigma_{Rc}$. Therefore $\sigma_{Ra}\sigma_{Rb} = \sigma_{Rc}$.

Conversely, assume $\sigma_{Ra}\sigma_{Rb} = \sigma_{Rc}$. The reflections σ_{Ra}, σ_{Rb} and also σ_{Rb}, σ_{Rc} commute. From I.8(iii) we obtain idempotent elements $\lambda, \mu \in R$ such that $\lambda a \in Rc^\perp$ and $(1-\lambda)a \in Rc$, $\mu b \in Rc^\perp$ and $(1-\mu)b \in Rc$. Pick $x \in Rc^\perp$. The assumption yields $-x\sigma_{Rb} = x\sigma_{Ra}$, hence

$$x = \frac{(b,x)f}{(b,b)f}b + \frac{(a,x)f}{(a,a)f}a.$$

Furthermore, $((1-\mu)b,x)f = 0$, thus $(b,x)f = \mu(b,x)f$. Also $(a,x)f = \lambda(a,x)f$. We conclude $x \in R\mu b + R\lambda a$. Thus we proved $Rc^\perp \subseteq R\mu b + R\lambda a$, hence $R^3 = Rc \oplus Rc^\perp \subseteq Rc + R\mu b + R\lambda a$. Therefore c, μb, λa is a basis of R^3. This implies $\mu, \lambda \in R^*$. Finally, since λ and μ are idempotent, $\mu = 1 = \lambda$ and $a, b \in Rc^\perp$.

I.10. *Let* $\sigma_{Ra'} = \sigma_{Ra}\sigma_{Re}$, $\sigma_{Rb'} = \sigma_{Rb}\sigma_{Re}$, $\sigma_{Rc'} = \sigma_{Rc}\sigma_{Re}$. *Then* $\sigma_{Ra}\sigma_{Rb}\sigma_{Rc} = \sigma_{Rd}$, *where* $d := (b,c)f \cdot a - (a,c)f \cdot b + (a,b)f \cdot c$. $\sigma_{Rd}\sigma_{Re}$ *is a reflection. Furthermore, if Ra* distant *Rb then Rc* distant *Rd.*

Proof. Let $d' := (b',c')f \cdot a' - (a',c')f \cdot b' + (a',b')f \cdot c'$. Let $s := -(a,c)f \cdot b + (a,b)f \cdot c$. Then $s \in Ra^\perp \cap Re^\perp$; cf. I.9. Since a, a', e is an orthogonal basis (cf. I.9), $Ra^\perp \cap Re^\perp = Ra'$. Also $s' \in Ra$, where s' is defined correspondingly to s. Thus $(s,s')f = 0$. This implies

$$(a,c)f \cdot (a',b')f \cdot (b,c')f + (a,b)f \cdot (a',c')f \cdot (c,b')f = 0.$$

Two similar equations arise from cyclic permutations of a, b, c. These equations immediately imply $(d,d')f = 0$. For $x_1 := a$, $x_2 := b$ and $x_3 := c$ Gram's determinant $G := \det((x_i,x_k)f)$ is zero, because $a,b,c \in Re^\perp = Ra + Ra'$. Hence $(d,d)f = (a,a)f \cdot (b,b)f \cdot (c,c)f - G \in R^*$.

Also $(d', d')f \in R^*$. Consequently e, d, d' is an orthogonal basis; cf. I.9. We assert that σ_{Rd} and $\sigma_{Ra}\sigma_{Rb}\sigma_{Rc}$ coincide on $\{e, d, d'\}$. Both of the mappings fix Re. We have $d\sigma_{Ra} = (b,c)f \cdot a + (a,c)f \cdot b - (a,b)f \cdot c$ since $-(a,c)f \cdot b + (a,b)f \cdot c \in Ra^{\perp}$. Two similar arguments yield $d\sigma_{Ra}\sigma_{Rb}\sigma_{Rc} = d = d\sigma_{Rd}$. Similarly

$$d'\sigma_{Ra}\sigma_{Rb}\sigma_{Rc} = -(b',c')f \cdot a' + (a',c')f \cdot b' - (a',b')f \cdot c'$$

since

$$-(a',c')f \cdot b' + (a',b')f \cdot c' \in Re^{\perp} \cap Ra'^{\perp} = Ra.$$

Finally we obtain $d'\sigma_{Ra}\sigma_{Rb}\sigma_{Rc} = -d' = d'\sigma_{Rd}$.

Now let us assume in addition Ra distant Rb. I.9 yields $e^{\perp} = Ra + Rb$, and a, b, e is a basis of R^3. Hence $c = \lambda a + \mu b$ for some $\lambda, \mu \in R$. The coordinates of the vectors c, d, e in the basis a, b, e are

$$\begin{array}{l} \lambda, \mu, 0 \\ (b,c)f + \lambda(a,b)f,\ -(a,c)f + \mu(a,b)f, 0 \\ 0, 0, 1, \end{array}$$

respectively. The matrix M consisting of these three rows satisfies

$$\det M = -\lambda(a,c)f - \mu(b,c)f = -(c,c)f \in R^*.$$

Therefore, c, d, e is a basis. In particular, Rc distant Rd.

I.11. Definitions. For any $a \in R^3$ let $a^* \in R^{3*}$ denote the mapping $x \mapsto (a,x)f$. We define a sub-structure $\Pi(R,f) = (\mathscr{P}(R,f), \mathscr{L}(R,f), \mathrm{I})$ of $\Pi(R)$:

$$Ra \in \mathscr{P}(R,f) \Leftrightarrow a,\ b,\ c \text{ is a regular orthogonal basis of } R^3 \text{ for some } b,\ c.$$

Regular means that $(a,a)f$, $(b,b)f$, $(c,c)f \in R^*$.

$$Rg \in \mathscr{L}(R,f) \Leftrightarrow g = a^* \quad \text{for some } Ra \in \mathscr{P}(R,f).$$

Let $Ra \in \mathscr{P}(R,f)$. The line Ra^* is called the *polar* of Ra, and Ra is called the *pole* of the line Ra^*. The pair of mappings: $Ra \mapsto Ra^*$ for $Ra \in \mathscr{P}(R,f)$ and $Ra^* \mapsto Ra$ for $Ra^* \in \mathscr{L}(R,f)$ is a polarity of $\Pi(R,f)$. A pair of points $Ra, Rb \in \mathscr{P}(R,f)$ (a pair of lines $Ra^*, Rb^* \in \mathscr{L}(R,f)$) is called *orthogonal* if $Ra \mathrm{I} Rb^*$.

I.12. *Let $Ra \in \mathscr{P}(R,f)$ and $Rb^* \in \mathscr{L}(R,f)$. Then $Ra \mathrm{I} Rb^*$ if and only if a, b, c is a regular orthogonal basis of R^3 for some c.*

Proof. If a, b, c is an orthogonal basis then $(a,b)f = 0$, hence $Ra \mathrm{I} Rb^*$. Conversely let us assume $Ra \mathrm{I} Rb^*$, i.e. $(a,b)f = 0$. Ra distant Ra^* (since $aa^* = (a,a)f \in R^*$) and $Ra \mathrm{I} Rb^*$ imply Rb^* distant Ra^*; cf. I.3(i) $\Rightarrow$ (ii). Hence $Rc \mathrm{I} Rb^*$, Ra^* for some $Rc \in \mathscr{P}(R)$; cf. I.1. Any two of the vectors a, b, c are orthogonal. From Ra distant Ra^* and Rb, $Rc \mathrm{I} Ra^*$ and Rb distant Rc follows that a, b, c is a basis of R^3.

I.13. *Let* $\mathscr{S}(R,f) := \{\sigma_{Ra}\colon Ra \in \mathscr{P}(R,f)\}$. *For* $Ra, Rb \in \mathscr{P}(R,f)$ *the following statements are equivalent.*

(i) $\sigma_{Ra}\sigma_{Rb} \in \mathscr{S}(R,f)$.

(ii) $Ra \mathrm{I} Rb^*$.

(iii) $Ra^* \perp Rb^*$.

(iv) $Ra \perp Rb$.

(v) $(a,b)f = 0$.

(vi) a, b, c *is a regular orthogonal basis of* R^3 *for some* c.

This is obvious from I.9 and I.12.

I.14. $\Pi(R,f)$ is called an *elliptic plane* if $\Pi(R,f) = \Pi(R)$.

REMARK. $\Pi(R,f)$ is an elliptic plane if and only if any homomorphism of R onto a field $\bar{R}$ induces a homomorphism of $\Pi(R,f)$ such that the image $\Pi(\bar{R},\bar{f})$ is an elliptic plane in the usual sense.

LEMMA. $\Pi(R,f) = \Pi(R)$ *if and only if* $(a,a)f \in R^*$ *for every point* Ra *of* $\Pi(R)$.

Proof. Let $(a,a)f \in R^*$ for every basis a, b, c of R^3. Given $Ra \in \mathscr{P}(R)$. The usual vector space method can be applied in order to construct an orthogonal basis a, b, c. Hence $Ra \in \mathscr{P}(R,f)$. The mapping $Ra \mapsto Ra^*$ is a bijection of $\mathscr{P}(R)$ into $\mathscr{L}(R)$ since a regular orthogonal basis of R^3 exists. Therefore $\mathscr{L}(R) = \mathscr{L}(R,f)$.

I.15. Let $Ra, Rb \in \mathscr{P}(R,f)$. We write $\sigma_{Ra} | \sigma_{Rb}$ if $\sigma_{Ra}\sigma_{Rb} \in \mathscr{S}(R,f)$; cf. I.13. If $\sigma_{Ra}, \sigma_{Rb} | \sigma_{Rc}$ for a unique $\sigma_{Rc} \in \mathscr{S}(R,f)$ then we write $\sigma_{Ra} \mathbf{u} \sigma_{Rb}$. The abbreviation $\sigma_{Ra} \longmapsto \sigma_{Rb}$ is to denote that $\sigma_{Ra}|\sigma_{Rc}$; $\sigma_{Rc}|\sigma_{Rd}$; $\sigma_{Rd}|\sigma_{Rb}$ and $\sigma_{Ra}\mathbf{u}\sigma_{Rd}$ and $\sigma_{Rc}\mathbf{u}\sigma_{Rb}$ for some $\sigma_{Rc}, \sigma_{Rd} \in \mathscr{S}(R,f)$.

PROPOSITION. *Suppose that every non-unit of* R *is a zero-divisor and* $(a,a)f \in R^*$ *for every* $Ra \in \mathscr{P}(R)$ (*hence* $\Pi(R,f) = \Pi(R)$; cf. I.14).

(i) $Ra \mathrm{I} Rb^* \Leftrightarrow \sigma_{Ra}|\sigma_{Rb} \Leftrightarrow Ra \perp Rb \Leftrightarrow Ra^* \perp Rb^* \Leftrightarrow f(a,b) = 0$.

(ii) Ra distant $Rb \Leftrightarrow$ *there is a unique line of* $\Pi(R) = \Pi(R, f)$ *joining* Ra *to* $Rb \Leftrightarrow Ra^*$ *and* Rb^* *intersect in just one point* $\Leftrightarrow \sigma_{Ra}\mathbf{u}\sigma_{Rb}$.

(iii) Ra distant $Rb^* \Leftrightarrow Ra^*$ distant $Rb \Leftrightarrow f(a, b) \in R^* \Leftrightarrow \sigma_{Ra} \longmapsto \sigma_{Rb}$.

Proof. (i) and (ii) follow immediately from I.2 and I.13. The equivalence of the first three statements of (iii) is obvious from the definitions. The last one is a translation of I.3 (iii).

I.16. *Suppose that every non-unit of* R *is a zero-divisor, and that* $(a, a)f \in R^*$ *for every point* $Ra \in \mathscr{P}(R)$. *Let* $\mathscr{G} := \mathscr{G}(R, f)$ *denote the group generated by* $\mathscr{S} := \mathscr{S}(R, f)$. *We denote the elements of* $\mathscr{S}$ *by lowercase letters and use the notations introduced in* I.15: $a|b$ *means* $ab \in \mathscr{S}$, *and* $a\mathbf{u}b$ *indicates that* $c|a$, b *for just one* c. $a \longmapsto b$ *is to denote that* $a|c$; $c|d$; $d|b$ *and* $a\mathbf{u}d$ *and* $c\mathbf{u}b$ *for some* c, d.

$\mathscr{S}$ *is invariant under inner automorphisms of* $\mathscr{G}$, *and* $(\mathscr{G}, \mathscr{S})$ *satisfies the following properties.*

(E1) *If* $a \longmapsto b$ *and* $b|c$ *then* $a\mathbf{u}c$.

(E2) *If* $a|b$ *then* $c|a$ *and* $c\mathbf{u}b$ *and* $c \longmapsto b$ *for some* c.

(E3) a, b, $c|e$ *implies* $abc \in \mathscr{S}$.

(M) *If* $a \longmapsto b$ *then* ab *is not an involution.*

Proof. (E1) follows from I.15(ii) and I.3(i) $\Rightarrow$ (ii). (E2). Let $\sigma_{Ra}|\sigma_{Rb}$, i.e. $\sigma_{Ra}\sigma_{Rb} = \sigma_{Rd}$ for some $Rd \in \mathscr{P}(R)$. Then a, b, d is a regular orthogonal basis of R^3; cf. I.9. Thus, $a, b, c := b + d$ is a basis of R^3. In particular, $Rc \in \mathscr{P}(R)$ and Rc distant Rb. Hence $\sigma_{Rc}\mathbf{u}\sigma_{Rb}$; cf. I.15(ii). $(b, c)f = (b, b)f \in R^*$ implies $\sigma_{Rc} \longmapsto \sigma_{Rb}$; cf. I.15(iii). (E3) follows from I.15(i) and I.10. (M). Let $\sigma_{Ra} \longmapsto \sigma_{Rb}$, hence $(a, b)f \in R^*$; cf. I.15(iii). Suppose that σ_{Ra} and σ_{Rb} commute. From I.8(iv) we have $a\sigma_{Rb} \in Ra$. Now the formula for σ_{Rb} immediately shows $Ra = Rb$, hence $\sigma_{Ra} = \sigma_{Rb}$.

REMARK. *If* R *has stable rank* 3 (cf. I.6) *then* (G, S) *satisfies*

(U) *Let* $a, b \in \mathscr{S}$. *Then* $c|a$ *and* $c\mathbf{u}b$ *for some* $c \in \mathscr{S}$.

$(\mathscr{G}, \mathscr{S})$ is called the *group of motions* of the elliptic plane.

Let $\mathscr{F}(R, f) := \{\alpha \in \operatorname{Aut}(R^3, f)\colon \det\alpha = 1$ and $\det(1 + \alpha) \in R^*\}$. The following result is a "representation theorem" for $\mathscr{F}(R, f)$.

I.17. *Let* $(a, a)f \in R^*$ *for every point* $Ra \in \mathscr{P}(R)$. *Let* $Rp \in \mathscr{P}(R, f)$.

(a) *Let* $\alpha \in \mathscr{F}(R, f)$ *and* $Rg^* \in \mathscr{L}(R, f)$ *with* $Rg^*\mathrm{I}Rp$. *Then* $\alpha = \sigma_{Rq}\sigma_{Rg}\sigma_{Rh}$ *for some* $Rq \in \mathscr{P}(R, f)$ *and* $Rh^* \in \mathscr{L}(R, f)$, *where* $Rh^*\mathrm{I}Rp$ *and* Rg^* distant Rh^* *and* Rp distant Rq^*.

(b) *Given* $Rq \in \mathscr{P}(R,f)$ *and* $Rg^*, Rh^* \in \mathscr{L}(R,f)$ *such that* Rg^*, $Rh^* \mathrm{I} Rp$ *and* Rg^* distant Rh^* *and* Rp distant Rq^*. *Then* $\alpha := \sigma_{Rq}\sigma_{Rg}\sigma_{Rh} \in \mathscr{F}(R,f)$.

Proof of (a). Let $q := p(\alpha^{-1}+1)$. $\det(\alpha^{-1}+1) = \det\alpha^{-1} \cdot \det(1+\alpha) \in R^*$ implies $Rq \in \mathscr{P}(R,f)$. Select $Rj^* \in \mathscr{L}(R,f)$ with $Rj^* \mathrm{I} Rq$. Let $e = j(1+\alpha)$. Then $Re^* \in \mathscr{L}(R,f)$, and the equation

$$0 = (j,q)f = (j, p\alpha^{-1})f + (j,p)f = (j(\alpha+1),p)f = (e,p)f$$

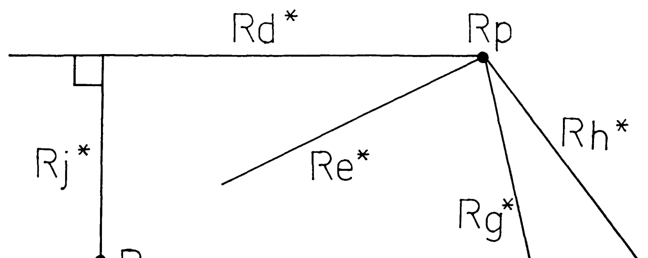
implies $Re^* \mathrm{I} Rp$.

$$R^* \ni (q,q)f = (p\alpha^{-1}+p, p\alpha^{-1}+p)f$$
$$= 2(p, p(1+\alpha^{-1}))f = 2(p,q)f.$$

Also, $R^* \ni (e,e)f = 2(j,e)f$. Hence, Rp^* distant Rq and Rj distant Re^*. In particular, Rp^* distant Rj^*, since $Rj^* \mathrm{I} Rq$. Thus, $Rd^* \mathrm{I} Rp$, Rj for a unique line $Rd^* \in \mathscr{L}(R,f)$.

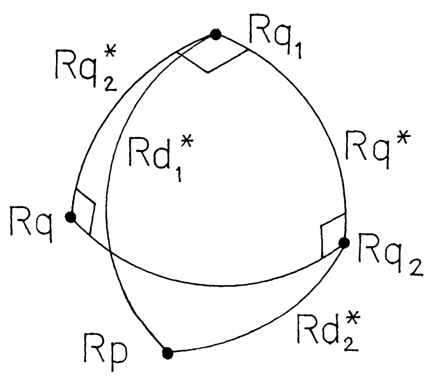
$$p\alpha^{-1}\sigma_{Rq} = \left(1/2 \cdot p(1+\alpha^{-1}) - 1/2 \cdot p(1-\alpha^{-1})\right)\sigma_{Rq}$$
$$= 1/2 \cdot p(1+\alpha^{-1}) + 1/2 \cdot p(1-\alpha^{-1}) = p,$$

since $p(1+\alpha^{-1}) = q$ and $(p(1+\alpha^{-1}), p(1-\alpha^{-1}))f = 0$. Also $j\alpha\sigma_{Re} = j$. Together with $p\sigma_{Re} = -p$ and $j\sigma_{Rq} = -j$ we conclude $p\beta = -p$ and $j\beta = -j$, where $\beta = \sigma_{Rq}\alpha\sigma_{Re}$. Therefore, $x\beta = -x$ for any $x \in Rp + Rj = d^\perp$ (apply Rp distant Rj). This yields $(Rd)\beta = Rd$, hence $d\beta = d$ as $\det\beta = 1$. Thus we proved $\beta = \sigma_{Rd}$, i.e. $\alpha = \sigma_{Rq}\sigma_{Rd}\sigma_{Re}$. From I.10 we obtain $\sigma_{Rd}\sigma_{Re} = \sigma_{Rg}\sigma_{Rh}$, where Rh^* is a line with $Rh^* \mathrm{I} Rp$ and Rg^* distant Rh^*.



Proof of (b). $\det\alpha = 1$ as α is a product of reflections. We want to prove $\det(\alpha+1) \in R^*$. Select q_1, q_2 such that q, q_1, q_2 is an orthogonal basis. The assumption Rp distant Rq^* implies Rp distant Rq_i. Let

$Rd_i^* \in \mathscr{L}(R,f)$ such that $Rd_i^* \mathrm{I} Rp, Rq_i$. Then Rd_1^* distant Rd_2^* (apply I.3, using Rp distant Rq^* and Rq_1 distant Rq_2). By I.10 there are lines



Re_i^* through Rp such that $\sigma_{Rg}\sigma_{Rh} = \sigma_{Rd_i}\sigma_{Re_i}$. The equation $\sigma_{Rd_1}\sigma_{Rd_2} = \sigma_{Re_1}\sigma_{Re_2}$ and Rd_1 distant Rd_2 imply Re_1 distant Re_2; cf. I.10. Hence $Re_1 + Re_2 = p^\perp$. We have

$$q_i(\alpha + 1) = q_i\sigma_{Rq}\sigma_{Rg}\sigma_{Rh} + q_i = -q_i\sigma_{Rd_i}\sigma_{Re_i} + q_i = q_i\sigma_{Re_i} + q_i = \lambda_i e_i,$$

where $\lambda_i := 2(q_i, e_i)f((e_i, e_i)f)^{-1} \in R^*$. Hence

$$q^\perp(\alpha + 1) = (Rq_1 + Rq_2)(\alpha + 1) = Re_1 + Re_2 = p^\perp.$$

Finally,

$$q(\alpha + 1) = (q + q\sigma_{Rp})\sigma_{Rg}\sigma_{Rh} + (-q\sigma_{Rp}\sigma_{Rg}\sigma_{Rh} + q) = \lambda p + w,$$

where w denotes the last bracket and $\lambda = 2(q,p)f((p,p)f)^{-1} \in R^*$. From the reflection formula one infers that

$$w \in 2q - 2\frac{(q,p)f}{(p,p)f}p + Rg + Rh \subseteq p^\perp.$$

We proved $R^3 = Rp + p^\perp \subseteq R^3(\alpha + 1)$. Therefore $\det(\alpha + 1) \in R^*$.

PART II. **Plane Elliptic Geometry in Terms of Reflections**

1. The system of axioms. The main theorem. Our system of axioms aims at elliptic planes over commutative rings. Under the assumptions of I.16 of Part I the group of motions of such an elliptic plane will satisfy our system of axioms, possibly apart from (U). If the ring has stable rank 3 then (U) will be fulfilled, but we do not know a nice property of the ring which is equivalent to (U).

The basic assumption. *The pair* (G, S) *consists of a group* $G = \{\alpha, \beta, \ldots\}$ *and a set* $S = \{a, b, \ldots\} \neq \varnothing$ *of involutions such that* S *is invariant under inner automorphisms of* G *and* S *generates* G.

NOTATIONS. $a|b$ if $ab \in S$. Let $a\mathbf{u}b$ denote that there is a unique c with $c|a$, b. Given a, b. If $a|c$; $c|d$; $d|b$ and $a\mathbf{u}d$ and $c\mathbf{u}b$ for some c, d then we write $a \rightarrowtail b$. We say c joins a to b if $c|a, b$. $\mathbf{u}$, and $|$ are invariant under inner automorphisms.

AXIOMS.

(E1) *If* $a \rightarrowtail b$ *and* $b|c$ *then* $a\mathbf{u}c$.

(E2) *If* $a|b$ *then* $c|a$ *and* $c\mathbf{u}b$ *and* $c \rightarrowtail b$ *for some* c.

(E3) $a, b, c|d$ *implies* $abc \in S$

(M) *If* $a \rightarrowtail b$ *then* ab *is not an involution.*

(U) *Given* a, b. *Then* $c|a$ *and* $c\mathbf{u}b$ *for some* c.

Our last axiom implies the following two statements.

(U′) *Let* $a \in S$. *Then* $a|b$ *for some* b.

(U″) *Given* a, b. *Then* $a|c$; $c|d$; $d|b$ *for some* c, d.

Axiom (M) will not be used until we study the group of motions in §11. Moreover, in §2 and §3 only (E1), (E2), (E3) and (U′) will be used.

The main purpose of this article is the proof of the following theorem. Simultaneously, the proof is a study of the group of motions of an elliptic plane over a commutative ring.

THEOREM. *Let* (G, S) *satisfy the basic assumption and* (E1), (E2), (E3), (M) *and* (U). *Then there exist a commutative ring* R, *a bilinear form* f: $R^3 \times R^3 \to R$, *and a mapping* σ *with the following properties.*

1. *Every non-unit of* R *is a zero-divisor, and* 2 *is a unit.*
2. $\Pi(R, f)$ *is an elliptic plane in the sense of Part I.*
3. σ *is a monomorphism of the group* G *into the group* $\mathcal{G}$ *such that* $S\sigma \subseteq \mathcal{S}$, *where* $(\mathcal{G}, \mathcal{S})$ *denotes the group of motions of* $\Pi(R, f)$. *Furthermore*, $\mathcal{F}(R, f) \subseteq G\sigma$.

2. Basic concepts. In this section we assume that (G, S) satisfies our basic assumption and (E1), (E2), (E3) and (U′).

2.1. (a) $|$, $\rightarrowtail$ *and* $\mathbf{u}$ *are symmetric relations on the set* S. *Furthermore, they are invariant under inner automorphisms of* G.

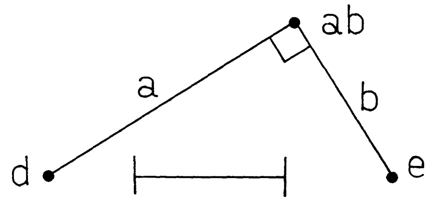
(b) $a|b$ *and* $a, b|c$ *implies* $ab = c$.

(c) *If* $a|b$ *then* $a\mathbf{u}b$.

(d) $a \rightarrowtail a$ *for every* a.

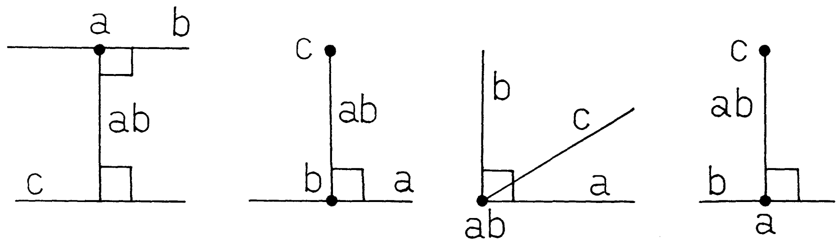
(e) *$a|b$ and $abc \in S$ implies $ab|c$.*

(f) *Let $d \longmapsto e$ and $a|d$ and $b|a, e$. The $ab \longmapsto d, e$.*



Proof. (a) and (e) are obvious. (b). Select d such that $d|b$, $d\mathbf{u}a$ and $d \longmapsto a$; cf. (E2). (E1) implies $b\mathbf{u}a$. Thus, $a, b|c$ and $a, b|ab$ yields $c = ab$. (c) is a consequence of (b). (d). Select $b \in S$ such that $b|a$; cf. (U′). Then $a|b$; $b|ab$; $ab|a$ and $a|ab$ and $b|a$. Thus, $a \longmapsto a$ by (c). (f). Our assumptions yield $d|a$; $a|b$; $b|ab$. (E1) implies $d\mathbf{u}b$. Finally, $a\mathbf{u}ab$ by (c). Hence we obtained $d \longmapsto ab$.

2.2. *Let $a|b$ and $c|ab$. Then $a\mathbf{u}c$ if and only if $b \longmapsto c$.* The figures indicate four possible interpretations of "$a|b$ and $c|ab$".



Proof. We have $b|a$; $a|ab$; $ab|c$ and $b\mathbf{u}ab$; cf. (c). Hence, $a\mathbf{u}c$ implies $b \longmapsto c$. Conversely, the assumption $b \longmapsto c$ together with $a|b$ yields $a\mathbf{u}c$; cf. (E1).

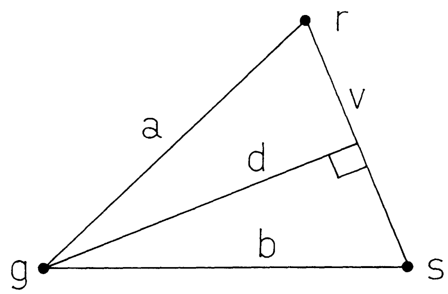
2.3. *Let $a, b, c|d$. Then $abc \in S$ and $abc|d$.*

Proof. $abc \in S$ by (E3). Also $a, b, cd \in S$ and $a, b, cd|d$. Hence $a \cdot b \cdot cd \in S$ by (E3).

2.4. (i) *Let $a, b|g$ and $a\mathbf{u}b$. Then $c|g$ and $c\mathbf{u}a, b$ for some c.*

(ii) *Let $a, b|g$ and $a \longmapsto b$. Then $c|g$ and $c\mathbf{u}a$ and $c — b$ for some c.*

Proof. (i). Select r, s with $r|a$ and $s|b$ and $r, s\mathbf{u}g$ and $r, s \longmapsto g$; cf. (E2). Then $r \longmapsto b$ and $s \longmapsto a$, hence $r\mathbf{u}s$. Let $v|r, s$. We have $v\mathbf{u}a, b$ and conclude $v \mapsto g$ and $v\mathbf{u}g$. Let $d|g, v$. $v|r$; $r|a$; $a|ag$ and $v\mathbf{u}a$ and $r\mathbf{u}ag$ (as $r \mapsto g$ by 2.2) imply $v \mapsto ag$. Hence $d\mathbf{u}ag$. Applying 2.2 twice we get $c := dg\mathbf{u}a$ and, likewise, $c\mathbf{u}b$.



Proof of (ii). From 2.2 we know $a\mathbf{u}bg$. From (i) we obtain $c|g$ with $c\mathbf{u}a, bg$. Thus $c\mathbf{u}a$ and $c \mapsto b$; cf. 2.2.

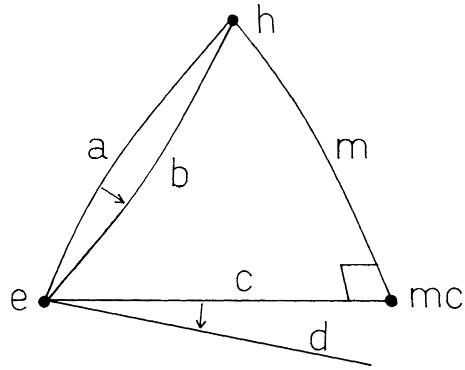
2.5. *If* $ac = bd \in S$ *and* $a\mathbf{u}b$ *then* $c\mathbf{u}d$.

Proof. 2.2 yields $a \mapsto bac = d$, hence $c\mathbf{u}d$; cf. (E1).

2.6. *Let* $a, b|c$. *If* $d|a, b$ *and* $d \mapsto c$ *implies* $d = c$ *then* $a\mathbf{u}b$.

Proof. We proceed in a number of steps.

(i) *Let* $ab = cd$ *and* $a, b, c, d|e$. *Suppose that* $g|c, d$ *and* $g \mapsto e$ *implies* $g = e$. *If* $h|a, b$ *and* $h \mapsto e$ *then* $h|ec$, ed.



Proof of (i). (E1) implies $h\mathbf{u}c, d$. Let $m|c, h$. Since $m, a, b|h$, (E3) implies $mcd = mab \in S$, i.e. $mc|c, d$. $h \mapsto e$ yields $e\mathbf{u}m$; cf. (E1). Therefore, $e \vdash\!\!\dashv mc$ by 2.2. Our assumptions imply $e = mc$, i.e. $ec = m|h$. Similarly $ed|h$.

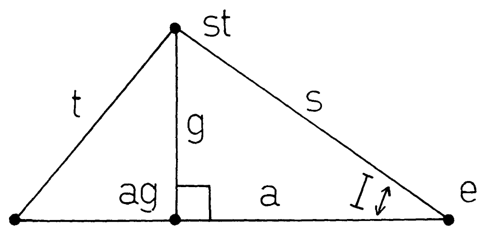
(ii) *Let* $ab = cd$ *and* $a, b, c, d|e$. *Suppose that* $g|c, d$ *and* $g \vdash\!\!\dashv e$ *implies* $g = e$. *Then* $h|a, b$ *and* $h \vdash\!\!\dashv e$ *implies* $h = e$.

Proof. Let $h|a, b$ and $h \vdash\!\!\dashv e$. Then $h|ec, ed$ by (i). Select v such that $v|e$ and $v\mathbf{u}c$, ec; cf. 2.4 (i). (E3) implies $w := vab \in S$; furthermore, $w|e$ by 2.3. We claim

$$(*) \qquad \text{If } f \mid v, w \text{ and } f \vdash\!\!\dashv e \text{ then } f = e.$$

Namely, $f|v, w$ and $f \vdash\!\!\dashv e$ implies $f|ec, ed$ by (i) (take v, w, f instead of a, b, h). Hence $f, e|v, ec$, and the assumption $v\mathbf{u}ec$ yields $f = e$. Thus, $(*)$ is true. Applying (i) once more, we obtain $h|ev, ew$. 2.5 implies $ev\mathbf{u}ec$, since $ev \cdot v = ec \cdot c \in S$ and $v\mathbf{u}c$. Therefore, $h|ec, ev$ implies $h = e$.

Now we are ready to prove the proposition. Let $a, b|c$ and suppose that $a\mathbf{u}b$ is not true. Then $e|a, b$ for some $e \neq c$. Select s with $s|e$ and $s\mathbf{u}a$ and $s \vdash\!\!\dashv a$; cf. (E2).

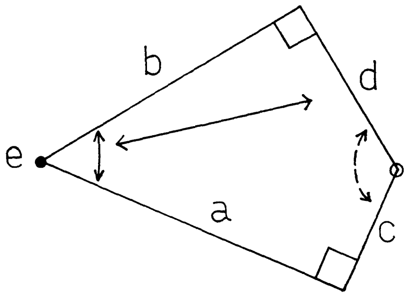


(E1) yields $c\mathbf{u}s$. Let $t|c, s$. Then $p := sab \in S$ and $p|e$ by 2.3. The equation $tsp = tab \in S$ shows $p|st$, and $s \vdash\!\!\dashv a$ implies $st\mathbf{u}a$; cf. (E1). Let $g|st, a$. Since $p, s, g|st$, (E3) yields $r := bag = psg \in S$, hence $b, r|ag$. Now $a \vdash\!\!\dashv s$ implies $c\mathbf{u}s$, hence $c \vdash\!\!\dashv st$ by 2.2. As $g|st$, (E1) and 2.2 yield $c\mathbf{u}g$ and $c \vdash\!\!\dashv ag$. Furthermore $ag|a, b$. The assumption in the proposition implies $ag = c$. In particular, $b, a, g|c$, hence $r|c$ by 2.3. We have $a, b, g, r|c$ and $ab = gr$. Furthermore, $st|g, r$ and $st \vdash\!\!\dashv c$ and $st \neq c$ (the last statement is true, since $st = c$ would imply $a, s|c, e$, contradicting $a\mathbf{u}s$ and $c \neq e$). We apply (ii) and obtain an element $d \neq c$ with $d|a, b$ and $d \vdash\!\!\dashv c$. Thus we reach a contradiction.

Statement (ii) in the proof of 2.6 and the assertion of 2.6 yield

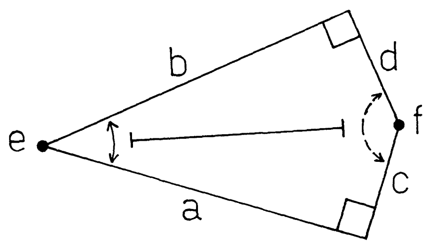
2.7. *Let* $a, b, c, d|e$ *and* $ab = cd$. *If* $a\mathbf{u}b$ *then* $c\mathbf{u}d$.

2.8. *Let* $a|c, e$ *and* $b|d, e$. *If* $d\mathbf{u}e$ *and* $a\mathbf{u}b$ *then* $c\mathbf{u}d$.



Proof. $a\mathbf{u}b$ and $e\mathbf{u}d$ mean $a \leftrightarrow d$ by definition. Hence $c\mathbf{u}d$.

2.9. *Let* $a, b|e$ *and* $c|a, f$ *and* $d|b, f$. *If* $e \leftrightarrow f$ *and* $a\mathbf{u}b$ *then* $c\mathbf{u}d$. This follows immediately from 2.8.



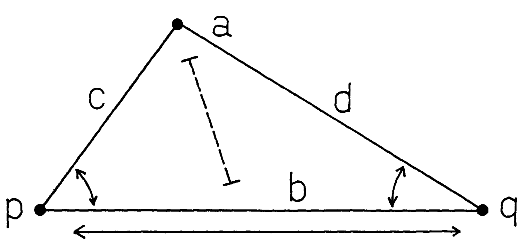
2.10. *Let* $e|a, b$; $abc \in S$; $a\mathbf{u}b$ *and* $c\mathbf{u}e$. *Then* $c|e$.

Proof. Let $d := abc$; $h|e, c$ and $g := abh$. Then $g|e$ and $g\mathbf{u}h$; cf. 2.3 and 2.7. We have $g, h|e$, $gd = hc$, and $g\mathbf{u}h$. Hence $e = gd = hc|g, h, c, d$.

A similar argument shows

2.10′. *Suppose that* (U″) *holds. Let* $ab = cd$ *and* $q|c, d$ *and* $a\mathbf{u}b$. *Then* $q|a, b$. *In particular, if* $ab = cd$ *and* $a\mathbf{u}b$ *and* $c\mathbf{u}d$ *then* $q|a, b, c, d$ *for some* q.

2.11. *Suppose* $a|c, d$; $p|b, c$; $q|b, d$; $b\mathbf{u}c, d$ *and* $p\mathbf{u}q$. *Then* $a \leftrightarrow b$.



Proof. The assumptions yield $c \mapsto q$, hence $a\mathbf{u}q$. Furthermore, $a|d$; $d|q$; $q|b$ and $d\mathbf{u}b$. Thus $a \mapsto b$.

2.12. *Let* $a, b \in S$. *Then* $a \mapsto b \Leftrightarrow c\mathbf{u}b$ *for every* $c|a$.

Proof. " $\Rightarrow$ " is (E1). " $\Leftarrow$ ". Select any $c|a$. Then $c\mathbf{u}b$ by our assumption. Let $p|b, c$. Select $g|b$ such that $g\mathbf{u}p$; cf. (E2). Then $c \mapsto g$, hence $a\mathbf{u}g$. Let $d|a, g$. Our assumption yields $d\mathbf{u}b$. Now $b \mapsto a$ by definition.

Without success we made attempts to avoid (U) in our system of axioms, since an analytic model may fail to satisfy (U). Observe that (U) is valid in the particular case that $a \mapsto b$ or $c|a, b$ for some c holds for every pair a, b; cf. (E1) and 2.2(c).

3. The elliptic plane and the pseudo-plane. Let again (G, S) satisfy the basic assumption and (E1), (E2), (E3), (U$'$).

In Part I we introduced the group of motions $(\mathscr{G}, \mathscr{S})$ of an elliptic plane over a ring. $(\mathscr{G}, \mathscr{S})$ satisfies our system of axioms (except perhaps (U)). Every pair consisting of a point Ra and its polar line Ra^* corresponds to a reflection σ_{Ra}. With this identification, incidence and orthogonality of the elliptic plane both correspond to the relation "|" on $\mathscr{S}$; cf. I.15. Hence, in order to reconstruct the elliptic plane, we must assign both a point and a line to each element of $\mathscr{S}$. Having this in mind we perform the following construction.

Select a bijection $\hat{}$ of S onto a set $\mathscr{P} = \hat{S}$ such that $\mathscr{P} \cap G = \varnothing$. This bijection will be maintained throughout the rest of this article. Also for the rest of this article we fix an element $u \in S$. Let $\mathscr{L} := S$, $\mathscr{L}' := \{g \in \mathscr{L}\colon g\mathbf{u}u\}$ and $\mathscr{P}' := \{\hat{a} \in \mathscr{P}\colon a \mapsto u\}$.

We regard $\mathscr{P}$ as a *set of points*, $\mathscr{L}$ as a *set of lines*. A point $\hat{a}$ is *incident* with a line g, abbreviated $\hat{a}|g$ or $g|\hat{a}$, if $a|g$ holds. Lines g, h are called *orthogonal*, abbreviated $g|h$, if $g|h$ holds. The incidence structure $(\mathscr{P}, \mathscr{L}, |)$, together with this orthogonality, is called the *elliptic plane* assigned to (G, S). The incidence structure $(\mathscr{P}', \mathscr{L}', |)$ is called the *affine plane* (with respect to u). $(\mathscr{P}', \mathscr{L}', |)$ is a locally complete substructure of $(\mathscr{P}, \mathscr{L}, |)$, i.e. if $\hat{a} \in \mathscr{P}'$ and $g \in \mathscr{L}$ with $\hat{a}|g$ then $g \in \mathscr{L}'$ (apply (E1)). $\hat{a}$ is called the *polar point* of the line a.

Let $\mathscr{D} := \{xy\colon x, y|u\}$ and $\mathscr{D}^* := \{xy\colon x, y|u \text{ and } x\mathbf{u}y\}$. $\mathscr{D}$ is an abelian subgroup of G; cf. (E3). For brevity, the elements of $\mathscr{D}$ will be called *angles*. u is an element of $\mathscr{D}$ and is called the *right angle*. (More precisely, the elements of $\mathscr{D}$ will serve as quantities assigned to pairs of lines by an angle measure.) We define an *angle measure*

$$\mathrm{w}\colon \mathscr{L}' \times \mathscr{L}' \to \mathscr{D}, \qquad \mathrm{w}(g,h) := cd,$$

where $c|g,u$ and $d|h,u$. $(\mathscr{P}', \mathscr{L}', |, \mathrm{w})$ is called he *pseudo-plane* (of (G,S) with respect to u). Two lines $g,h \in \mathscr{L}'$ are called *pseudo-parallel* if $\mathrm{w}(g,h) = 1$, and *pseudo-orthogonal* if $\mathrm{w}(g,h) = u$.

An angle $\alpha \in \mathscr{D}$ is called *regular* if for all $g,h \in \mathscr{L}'$ $\mathrm{w}(g,h) = \alpha$ implies that g, h intersect in just one point of $\mathscr{P}'$.

3.1. $\mathrm{w}(g,h)\mathrm{w}(h,j) = \mathrm{w}(g,j)$ *and* $\mathrm{w}(g,h) = \mathrm{w}(h,g)^{-1}$ *for all* $g,h,j \in \mathscr{L}'$.

3.2. *Given* $\hat{a} \in \mathscr{P}'$, $g \in \mathscr{L}'$ *and* $\alpha \in \mathscr{D}$ *there is exactly one* $h \in \mathscr{L}'$ *such that* $\hat{a}|h$ *with* $\mathrm{w}(g,h) = \alpha$.

Proof. Let $c \in S$ with $c|g,u$, and $d := c\alpha$. $a \longleftarrow u$ implies $a\mathbf{u}d$. Hence $h|a,d$ for exactly one $h \in S$. Furthermore, $h \in \mathscr{L}'$.

3.3. $\mathscr{D}^*$ *is the set of regular angles*.

Proof. Let $g,h \in \mathscr{L}'$ and $\alpha := \mathrm{w}(g,h) \in \mathscr{D}^*$, hence $c\mathbf{u}d$ (by 2.7), where $c|g,u$ and $d|h,u$. Then $g\mathbf{u}h$ by 2.8, i.e. there is exactly one $a \in S$ with $a|g,h$. We have $h|d$; $d|u$; $u|c$ and $h\mathbf{u}u$ and $d\mathbf{u}c$, hence $h \longleftarrow c$ and $a\mathbf{u}c$. Furthermore, $a|g$; $g|c$; $c|u$ and $g\mathbf{u}u$. Therefore, $a \longleftarrow u$, i.e. $\hat{a} \in \mathscr{P}'$. Conversely, let us assume $\alpha \in \mathscr{D} \setminus \mathscr{D}^*$. We claim that α is not a regular angle. Select $c,d \in S$ with $c,d|u$ and $\alpha = cd$. Then $c\mathbf{u}d$ is not true, and 2.6 produces an element $a \in S$ with $a \neq u$ and $a|c,d$ and $a \longleftarrow u$. Thus, $c,d|\hat{u},\hat{a}$, i.e. the lines $c,d \in \mathscr{L}'$ do not intersect uniquely in the pseudo-plane and satisfy $\mathrm{w}(c,d) = \alpha$.

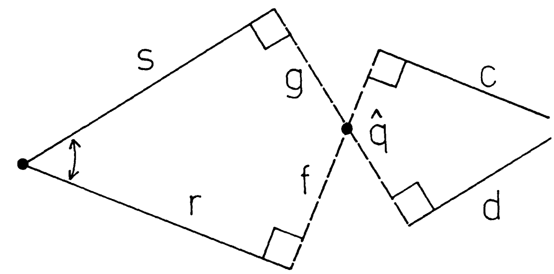
3.4. *Let* $g,h \in \mathscr{L}'$ *and* $\hat{a} \in \mathscr{P}'$ *with* $\hat{a}|g,h$. *If* $g\mathbf{u}h$ *then* $\mathrm{w}(g,h) \in \mathscr{D}^*$.
The assertion follows immediately from 2.9.

REMARK. In 3.4 we were not able to replace the assumption $g\mathbf{u}h$ by the property that $\hat{a}$ is the only common point of g and h in the pseudo-plane.

4. Products of two elements of S. From now on we assume that (G,S) satisfies (E1), (E2), (E3) and (U) of §2.

4.1. *Let* $ab = cd$ *and* $a\mathbf{u}b$. *Then* $c\mathbf{u}d$ *and* $e|a,b,c,d$ *for some* e. *In particular*, $abc \in S$ *and* $a,b|e$ *and* $a\mathbf{u}b$ *imply* $c|e$.

Proof. Let $e|a, b$. Select f such that $f|c$ and $f\mathbf{u}e$; cf. (U). Let $r|e, f$ and $s := rab$. Then $s|e$ and $r\mathbf{u}s$; cf. 2.3, 2.7. Select q such that $q|f$ and $q\mathbf{u}d$; cf. (U). Then $q, e, r|f$, hence $m := qcr = qcabs = qds \in S$ and $m|f$.



$s\mathbf{u}r$ and $e\mathbf{u}f$ yield $s \mapsto f$. In particular, $m\mathbf{u}s$. Also, $ms = qd$ and $q\mathbf{u}d$. Hence, $g|m, s, q, d$ for some g; cf. 2.10′. From $s \mapsto f$ and $g|s$ follows $g\mathbf{u}f$. Together with $q\mathbf{u}d$ follows $c\mathbf{u}d$; cf. 2.8. Thus, $ab = cd$ and $a\mathbf{u}b$ and $c\mathbf{u}d$. 2.10′ yields immediately the assertion.

REMARK. 4.1 subsumes the statements 2.5 and 2.10.
Due to 4.1 the following definition makes sense.

DEFINITION. Let $\mathscr{E} := \{ab\colon a\mathbf{u}b\}$. To every $ab \in \mathscr{E}$ corresponds a unique c with $c|a, b$, called the *support* of ab. Let $\mathscr{F} := \{ghq\colon g\mathbf{u}h$ and $\text{support}(gh) \mapsto q\}$. $\mathscr{F}_0 := \{pq\colon p \mapsto q\}$.

The following remarks are obvious.
$a\mathbf{u}b$ if and only if $ab \in \mathscr{E}$, for any pair a, b; cf. 4.1. *$\mathscr{E}$, $\mathscr{F}$ and $\mathscr{F}_0$ are invariant under inner automorphisms of G. Furthermore, $\mathscr{F}_0 \subseteq \mathscr{F}$. If $\alpha \in \mathscr{E}$, $\mathscr{F}$ or $\mathscr{F}_0$, then $\alpha^{-1} \in \mathscr{E}$, $\mathscr{F}$, $\mathscr{F}_0$, respectively.*

For arbitrary subsets $\mathscr{A}, \mathscr{B} \subseteq \mathrm{G}$ let $\mathscr{A}\mathscr{B} := \{\alpha\beta\colon \alpha \in \mathscr{A},\ \beta \in \mathscr{B}\}$.

4.2. *$\mathscr{E} = S\mathscr{F} = \mathscr{F}S = S\mathscr{F}_0 = \mathscr{F}_0 S$. Furthermore,* $\text{support}(x\mu) \mapsto x$ *for every $\mu \in \mathscr{F}$ and every $x \in S$.*

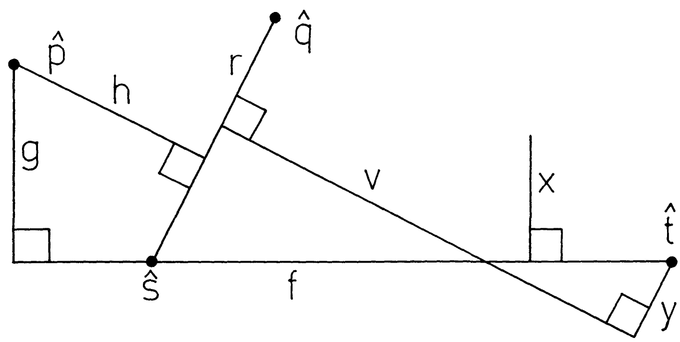
Proof. If $\mu \in \mathscr{F}$ then $S\mathscr{F} \ni a\mu = \mu^a \cdot a \in \mathscr{F}S$ for any a. Thus the second and the last equality are clear. Also $\mathscr{F}S \supseteq \mathscr{F}_0 S$. So it suffices to prove

(1) $\mathscr{F}_0 S \supseteq \mathscr{E}$ and

(2) $\mathscr{E} \supseteq S\mathscr{F}$.

Proof of (1). Let $ab \in \mathscr{E}$. Then $ac \mapsto b$, where $c|a, b$; cf. 2.2. Hence $ab = c(ac)b \in S\mathscr{F}_0$.

Proof of (2). Let $g, h | p$ and $g\mathbf{u}h$ and $p \mapsto q$. Let $x \in S$. We claim $xghq \in \mathscr{E}$, and $y \mapsto x$, where $y :=$ support$(xghq)$. Select $f | x$ with $f\mathbf{u}p$; cf. (U). Applying (E3) one can assume $g | f$. $p \mapsto q$ yields $h\mathbf{u}q$. Let $r | h, q$. From 2.8 we obtain $f\mathbf{u}r$. Let $s | f, r$. Then $t := xgs \in S$ and $t | f$. Also $v := shq \in S$ and $v | r$. We have $r | v, s$; $f | t, s$; $v\mathbf{u}s$ (namely $vs = qh \in \mathscr{E}$ since $q \mapsto p$) and $f\mathbf{u}r$. Hence $tv \in \mathscr{E}$ by 2.8. Therefore, $xghq = xgsshq = tv \in \mathscr{E}$. From $g | p$; $p | h$; $h | r$; $g\mathbf{u}h$; $p\mathbf{u}r$ (since $p \mapsto q$) follows $g \mapsto r$. Thus $xt = gs \in \mathscr{E}$. We have observed $v\mathbf{u}s$ and $f\mathbf{u}r$. 2.8 yields $f\mathbf{u}y$. Together with $x | f$; $f | t$; $t | y$ follows $x \mapsto y$.



4.3. *Let $ab = cd$. If $a \mapsto b$ then $c \mapsto d$. In other words*: $cd \in \mathscr{F}_0$ *if and only if* $c \mapsto d$, *for any pair* c, d.

Proof. Let $g | c$. Then $gd = (gc)ab \in S\mathscr{F}_0 = \mathscr{E}$; cf. 4.2. 2.12 yields the assertion.

4.4. $\mathscr{F}_0 = SS \cap \mathscr{F}$.

Proof. Let $ab \in \mathscr{F}$. Let $c | a$. Then $cb = ca \cdot ab \in S\mathscr{F} = \mathscr{E}$, hence $c\mathbf{u}b$; cf. the remarks preceding 4.2. 2.12 yields the assertion.

4.5. $\mathscr{F} = \{\gamma \in G\colon \gamma x \in \mathscr{E} \text{ for all } x\} = \{\gamma \in G\colon x\gamma \in \mathscr{E} \text{ for all } x\}$.

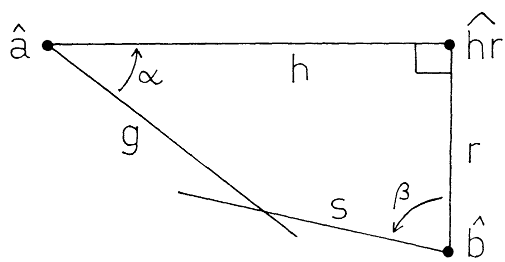
Proof. "$\subseteq$" is an assertion of 4.2. Let γ be an element of the right-hand set. Select an arbitrary q. Then $\gamma q = gh \in \mathscr{E}$. Let $p | g, h$. We want to prove $p \mapsto q$. By 2.12 it is enough to verify $xq \in \mathscr{E}$ for any $x | p$. So let $x | p$. We can assume $h = x$; cf. (E3). Then $xq = g\gamma = (\gamma g)^g \in \mathscr{E}$ by our assumption.

REMARK. We shall prove a "representation theorem" for $\mathscr{F}$; cf. 6.4. This theorem is the synthetic counterpart of I.17.

5. $G = \mathscr{E}S$. We continue to assume that (G, S) satisfies (E1), (E2), (E3) and (U).

5.1. $\mathscr{E}\mathscr{E} \subseteq S\mathscr{E} = \mathscr{E}S$.

Proof. Let $\alpha, \beta \in \mathscr{E}$ and $a :=$ support(α), $b :=$ support(β). Due to (U) we can assume $\alpha = gh$, $\beta = rs$, $h|r$ and $h\mathbf{u}b$. Then $\alpha\beta = g(hr)s$ and $b \mapsto hr$. Hence $s\mathbf{u}hr$, and $\alpha\beta \in S\mathscr{E}$.

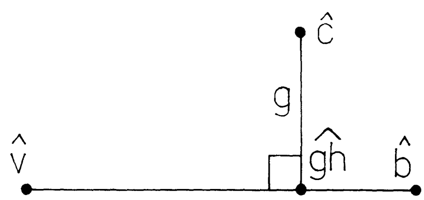


5.2. $SSS \subseteq \mathscr{E}\mathscr{E} \subseteq \mathscr{E}S$.

Proof. Given a, b, c. Select g, h such that $g|c$; $g\mathbf{u}b$; $h|b, g$; cf. (U). From 2.4(i) and 2.2 we obtain v such that $v|h$ and $v \mapsto b, g$. Then $v\mathbf{u}gh$ and $v\mathbf{u}c$ since $v \mapsto g$ and $g|c$. Thus,

$$abc = (abv)(vc) \in S\mathscr{F}\mathscr{E} \subseteq \mathscr{E}\mathscr{E};$$

cf. 4.2. 5.1 finally yields the conclusion $abc \in \mathscr{E}S$.



5.3. $G = SSSS = \mathscr{E}SS$.

Proof. 5.1 and 5.2 yield

$$SSSSS \subseteq \mathscr{E}S \cdot SS \subseteq \mathscr{E}\mathscr{E}S \subseteq \mathscr{E}SS \subseteq SSSS \subseteq SSSSS.$$

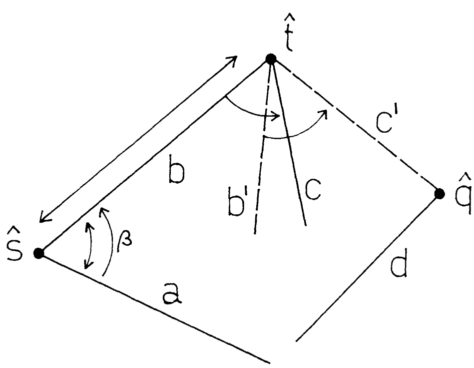
This proves the assertion.

5.4. $G = \mathscr{E}S = S\mathscr{E}$.

Proof. Let $\alpha \in G$. We can write $\alpha = \beta cd$, where $\beta \in \mathscr{E}$; cf. 5.3. Let $s := \text{support}(\beta)$. Select t such that $t|c$ and $t\mathbf{u}s$; cf. (U). Then $\beta = ab$ for some a, b with $b|s, t$ and $a \longmapsto t$. Select q such that $q|d$ and $q\mathbf{u}t$. Using (E3) we obtain $bc = b'c'$, where $b', c'|t$ and $c'|q$. $q\mathbf{u}t$ implies $q \longmapsto tc'$, and $a \longmapsto t$ implies $a\mathbf{u}b't$. Thus,

$$\alpha = abcd = ab'c'd = a(b't)(tc')d \in \mathscr{E}\mathscr{E} \subseteq S\mathscr{E};$$

cf. 5.1.



5.5. *If* (M) *holds then* $C_G = \{\alpha \in G\colon x^\alpha = x \text{ for every } x\} = 1$.

Proof. Let $\alpha \in C_G$. We have $\alpha = \beta c$, where $\beta \in \mathscr{E}$; cf. 5.4. Select t such that $t|c$ and $t\mathbf{u}s := \text{support}(\beta)$; cf. (U). Then $\beta = ab$ for some a, b with $b|s, t$. $t^\alpha = t$ yields $t^a = t$, hence $t = a$ by (M), since $t \longmapsto a$. Let $d := ab$. Then $d \in S$ and $c, d|a$ and $\alpha = dc$. Select e such that $e|d$ and $e\mathbf{u}a$ and $e \longmapsto a$; cf. (E2). Then $e\mathbf{u}c$. Let $s|e, c$. We have $e \longmapsto cs$ and $e = e^\alpha = e^c = e^{cs}$, hence $e = cs$; cf. (M). Finally, from $a, e|d, c$ and $a\mathbf{u}e$ follows $d = c$, i.e. $\alpha = 1$.

6. A class of automorphisms of $(\mathscr{P}, \mathscr{L}, |)$.

6.1. Definition. To every $\gamma \in \mathscr{F}$ (cf. §4.1) we define a mapping T_γ: $S \to S$, $x \mapsto \text{support}(x\gamma)$; cf. §4.1. We collect some properties of T_γ.

6.2. *Let* $\gamma \in \mathscr{F}$.
(a) $x \longmapsto xT_\gamma$ *for every* x.
(b) T_γ *is a bijection*.
(c) $a|b \Leftrightarrow aT_\gamma|bT_\gamma^{-1}$ *for all* a, b. *In other words*: $a|bT_\gamma \Leftrightarrow aT_{\gamma^{-1}}|b$.
(d) $a\mathbf{u}b \Leftrightarrow aT_\gamma\mathbf{u}bT_\gamma$ *for all* a, b.

Proof. (a) follows from 4.2.

(b) Given $y \in S$ and $\gamma = ghq$ with $p|g, h$ and $g\mathbf{u}h$ and $p \longmapsto q$. We want to show $y = \mathrm{support}(x\gamma)$ for just one x. Consider the figure of 4.2. From (U) we obtain r, v such that $v|y$ and $v\mathbf{u}q$ and $r|q, v$. Then $p\mathbf{u}r$ since $p \longmapsto q$. Using (E3) we may therefore assume $h|p, r$. Let $s := hqv$. Then $s|r$ and $sh = vq \in \mathscr{E}$. From $s\mathbf{u}h$ and $r\mathbf{u}p$ we obtain $s \longmapsto p$, hence $s\mathbf{u}g$. Let $f|s, g$. $g\mathbf{u}h$ and $p\mathbf{u}r$ yield $f\mathbf{u}r$; cf. 2.8. Together with $s\mathbf{u}v$ follows $f \longmapsto v$, hence $f\mathbf{u}y$. Let $t|f, y$. Now $x := tsg$ satisfies $x\gamma = tv$ and $\mathrm{support}(x\gamma) = \mathrm{support}(tv) = y$. Let us assume support $(x'\gamma) = y$ for some x'. Perform the construction in the proof of (2) in 4.2 with $g', h', r', \ldots$ instead of $g, h, r, \ldots$ Then $x'\gamma = t'v'$ and $x\gamma = tv$, where $t', v', t, v|y = y'$. Let $t'' := t'v'v$. Then $t''|y$ and $t''x' = tx \in \mathscr{E}$ as $y \longmapsto x$ and $t|y$. Therefore $t, t''|y, f$. Furthermore, $y\mathbf{u}f$ as $y \longmapsto x$. Hence $t = t''$ and $x = x'$.

(c) Let $x := aT_\gamma$ and $y := bT_{\gamma^{-1}}^{-1}$; i.e. $x = \mathrm{support}(a\gamma)$ and $b = \mathrm{support}(y\gamma^{-1})$. The following statements are equivalent. $a|b$; $y\gamma^{-1}a \in S$; $a\gamma y \in S$; $y|x$.

(d) follows immediately from (b) and (c).

6.3. *For every $\gamma \in \mathscr{F}$ the mapping $\mathscr{L} \to \mathscr{L}$, $g \mapsto gT_\gamma$ induces an automorphism of $(\mathscr{P}, \mathscr{L}, |)$.*

Proof. Define a mapping $\mathscr{P} \to \mathscr{P}$, $\hat{x} \mapsto \hat{y}$, where $y = xT_{\gamma^{-1}}^{-1}$. The pair, consisting of the two mappings, is an automorphism of $(\mathscr{P}, \mathscr{L}, |)$; cf. 6.2(b) and (c).

6.4. *Let $\gamma \in \mathscr{F}$ and $p \in S$.*

(a) *Let $ghq = \gamma = g'h'q'$, where $g, h, g', h'|p$. Then $gh = g'h' \in \mathscr{E}$ and $q = q' \longmapsto p$.*

(b) *$\gamma = \beta q$ for some $\beta \in \mathscr{E}$ and q such that* $\mathrm{support}(\beta) = p$ *and $p \longmapsto q$.*

Proof. (a) We have $gh = \gamma q \in \mathscr{F}S = \mathscr{E}$; cf. 4.5.

$$qT_{\gamma^{-1}} = \mathrm{support}\left(q\gamma^{-1}\right) = \mathrm{support}(hg) = p = \mathrm{support}\left(h'g'\right)$$

$$= \mathrm{support}\left(q'\gamma^{-1}\right) = q'T_{\gamma^{-1}}.$$

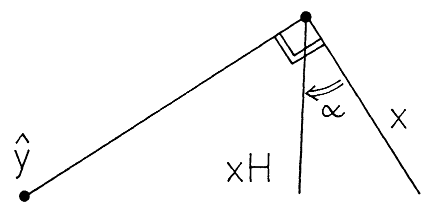
Hence $q = q' \longmapsto p$ by 6.2(a) and (b).

(b) Let $\delta := \gamma^{-1}$. The mapping T_δ is surjective; cf. 6.2(b). Hence $qT_\delta = p$ for some q; i.e. $\mathrm{support}(\beta) = p$, where $\beta = q\delta$. Furthermore, $p \longmapsto q$ by 6.2(a).

7. Pseudo-semirotations. We refer to the notions and notations introduced in §3. In particular, $u \in S$ is a fixed element, and the point set of the pseudo-plane is $\mathscr{P}' = \{\hat{a}\colon a \rightarrowtail u\}$; the set of lines is $\mathscr{L}' = \{g\colon g\mathbf{u}u\}$.

Our next aim is to show that the pseudo-plane is a generalized semi-rotation plane in the sense of [**8**].

Let $\alpha \in \mathscr{D}$ and $\hat{y} \in \mathscr{P}'$. The pseudo-semi-rotation $H = H_{\hat{y},\alpha}$ assigned to the center $\hat{y}$ and the angle α is a mapping of $\mathscr{L}'$ into $\mathscr{L}'$, where xH is the line passing through the foot of the pseudo-orthogonal of x through $\hat{y}$

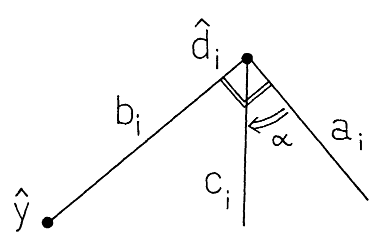


such that $\mathrm{w}(x, xH) = \alpha$. This definition yields immediately:

7.1. $\mathrm{w}(x, xH) = \alpha$ *for every* $x \in \mathscr{L}'$ *and* $H = H_{\hat{y},\alpha}$. *In particular,* $\mathrm{w}(x, z) = \mathrm{w}(xH, zH)$ *for all* $x, z \in \mathscr{L}'$.

7.2. *Let* $a_1, a_2, a_3 \in \mathscr{L}'$ *and* $\hat{x} \in \mathscr{P}'$ *with* $a_1, a_2, a_3 | \hat{x}$. *Let us assume that* $\mathrm{w}(a_1, a_2) \in \mathscr{D}^*$. *Then every pseudo-semi-rotation* H *satisfies* $a_3H | \hat{z}$, *where* $\hat{z} \in \mathscr{P}'$ *is the unique point with* $\hat{z} | a_1H, a_2H$.

Proof. Let $H = H_{\hat{y},\alpha}$. For $h \in \mathscr{L}'$ let $h' \in S$ be the element with the property $h'|u, h$. For $i = 1, 2, 3$ let $c_i = a_iH$ and let $b_i \in \mathscr{L}'$ denote the pseudo-orthogonal of a_i through $\hat{y}$. Let $\hat{d}_i$ denote the foot. With these



notations the definition of H implies for $i = 1, 2, 3$:

$$(1) \qquad a_i'b_i' = u \quad \text{and} \quad a_i'c_i' = \alpha$$
$$x_i := xa_i'd_i \in S \quad \text{as } x, a_i', d_i \mid a_i \qquad (\text{cf. (E3)}),$$
$$y_i := yb_i'd_i \in S \quad \text{as } y, b_i', d_i \mid b_i,$$

and for $i = 1, 2$

$$(2) \qquad z_i := zc_i'd_i \in S \quad \text{as } z, c_i', d_i \mid c_i.$$

Furthermore, $a_1'a_2' \in \mathscr{D}^*$.

An elementary calculation yields

$$(3) \quad x_1x_2 = y_1y_2 = z_1z_2 \quad \text{and} \quad x_2x_3 = y_2y_3 = z_2z_3,$$

where $z_3 := zc_3'd_3$ but $z_3 \in S$ is still unknown.

We have $a_2|d_2, x_2$; $b_2|d_2, y_2$; $a_2b_2 \in \mathscr{E}$ (since $\mathrm{w}(a_2, b_2) = u \in \mathscr{D}^*$) and $x_2d_2 = xa_2' \in \mathscr{E}$ as $a_2'|u$ and $u \mapsto x$. 2.8 implies

$$(4) \qquad x_2y_2 \in \mathscr{E}.$$

We apply 2.8 once more, using the relations $a_1|x, x_1$; $a_2|x, x_2$; $xx_1 = a_1'd_1 \in \mathscr{E}$ (since $a_1'|u$ and $u \mapsto d_1$); $a_1a_2 \in \mathscr{E}$ (since $\mathrm{w}(a_1, a_2) \in \mathscr{D}^*$). Thus

$$(5) \qquad x_1x_2 \in \mathscr{E}.$$

Let $e :=$ support(x_2y_2). (3) implies $x_2y_2x_1 \in S$ and $x_2y_2y_3 \in S$. (4) and 5.1 yield $x_1, y_3|e$. Therefore, support$(x_1x_2) = e$. Since $x_1x_2z_2 \in S$ we obtain $z_2|e$; cf. 4.1. A simple calculation involving only (1) and (2) yields $z_3 = z_2y_2y_3$. We have proved $z_2, y_2, y_3|e$. Hence $z_3 \in S$. This stands for $zc_3'd_3 \in S$. Moreover, $c_3'd_3 \in \mathscr{E}$ and $c_3 =$ support$(c_3'd_3)$ since $c_3'|u$ and $u \mapsto d_3$. 4.1 implies $z|c_3$.

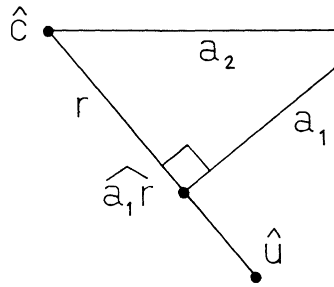
7.3. *The following property holds in the pseudo-plane.* (Δ) *There are regular angles $\alpha, \beta, \gamma \in \mathscr{D}^*$ such that $\alpha\beta = \gamma$.*

Proof. Select g, h, j with $g, h, j|u$ and $g|h$ and $j\mathbf{u}g, h$. This choice is possible by 2.4(i). Let $\alpha := gj$, $\beta := jh$, $\gamma := u$. α, β, γ fulfill the desired property; cf. 3.3.

7.4. *To any x there exist a_1, a_2 such that $a_1, a_2|x$; $a_1\mathbf{u}a_2$; $a_1, a_2\mathbf{u}u$; and $c|a_2$ with $c \mapsto u, a_1$ for some c.*

Proof. From (U) we get $a_1|x$ with $a_1\mathbf{u}u$. Let $r|u, a_1$. As $a_1r \mapsto u$ (cf. 2.2) there is $c|r$ with $c — u$ and $c\mathbf{u}a_1r$; cf. 2.4(ii). From $c \mapsto a_1$ and $a_1|x$

follows $c\mathbf{u}x$. Let $a_2|x, c$. We conclude $a_1\mathbf{u}a_2$ and $a_2\mathbf{u}u$; cf. (E1).

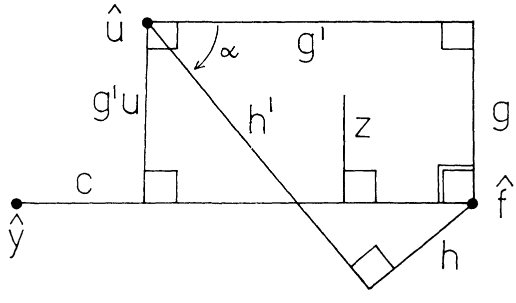


7.2 and 7.4 assert that every pseudo-semirotation $H = H_{\hat{y},\alpha}$ induces a (unique) mapping $\mathscr{P}' \to \mathscr{P}'$ such that $g|\hat{a}$ implies $gH|\hat{a}H$ for each $g \in \mathscr{L}'$ and $\hat{a} \in \mathscr{P}'$. Hence we shall regard a pseudo-semi-rotation as a mapping of $\mathscr{L}'$ and $\mathscr{P}'$.

It is easy to check that $H_{\hat{y},\alpha}$ *is injective if and only if* $u\alpha \in \mathscr{D}^*$, *i.e.* $\alpha \in \mathscr{F}$. *Then* $H = H_{\hat{y},\alpha}$ *is already a bijection* (*of* $\mathscr{L}'$ *and* $\mathscr{P}'$) *with* $g|\hat{a} \Leftrightarrow gH|\hat{a}H$.

7.5. *Let* $\alpha \in \mathscr{D} \cap \mathscr{F}$ *and* $\hat{y} \in \mathscr{P}'$. *Then* $gH_{\hat{y},\alpha} = gT_{yu}^{-1}T_{yu\alpha}$ *for any* $g \in \mathscr{L}'$. *Each bijective pseudo-semirotation can be uniquely extended to an automorphism of* $(\mathscr{P}, \mathscr{L}, |)$ (the elliptic plane where the orthogonality is not taken into account).

Proof. $yu \in \mathscr{F}$ since $y \mapsto u$. Write $\alpha = bc$ with $b, c|u$ and $u = ab$. Then $u\alpha \in \mathscr{E}$ by 4.2 and $u\alpha = ac$, so $a\mathbf{u}c$. Further, $yu\alpha = (acy)^y \in \mathscr{F}$, since support$(ac) = u \mapsto y$. Let $H := H_{\hat{y},\alpha}$, $g \in \mathscr{L}'$ and $h := gH$. Take $g'|u, g$ and $h'|u, h$. Then $\alpha = g'h'$. Let c denote the pseudo-orthogonal of g through $\hat{y}$ and let $\hat{f}$ be the foot. Then $y, g'u, f|c$, hence $z := f(g'u)y \in S$ and $z|c$. We have $zyu = fg'$. Therefore $zT_{yu} = g$. Now $zyu\alpha = fg'\alpha = fh'$ yields $zT_{yu\alpha} = $ support$(fh') = h$. The second assertion follows from the first one together with 6.3 and 7.4.



8. Transvections. Let $g, h \in \mathscr{L}'$. We write $g \textcircled{u} h$ if $x|g$ and $x \multimap h$, or $x|h$ and $x \multimap g$ for some $\hat{x} \in \mathscr{P}'$. Clearly, $g \textcircled{u} h$ implies $g\mathbf{u}h$.

We need this technical term in the proof of 9.1.

8.1. *Let $g, h \in \mathscr{L}'$ and $\hat{a} \in \mathscr{P}'$ such that $a|g, h$ and $g\mathbf{u}h$. Then $g \textcircled{u} h$.*

Proof. Let $b|g, u$. Then $b\mathbf{u}a$. Select $c|g$ such that $c\mathbf{u}a, b$; cf. 2.4(i). Then $\hat{c} \in \mathscr{P}'$ and $c \multimap h$.

8.2. *Let $a \in S$. Then $g, h|a$ for some $g, h \in \mathscr{L}'$ with $g \textcircled{u} h$.*

This is a reformulation of 7.4.

8.3. *Let $g, h|u$; $a|h$; $a\mathbf{u}u$; $g\mathbf{u}h$. Define $\mathscr{N}^{(i)}$ recursively:*

$$\mathscr{N}^{(0)} := \{\hat{a}\} \cup \{\widehat{ug}\} \cup \{\hat{y} \in \mathscr{P}' \colon y|g\} \cup \{x \in \mathscr{L}' \colon x|u\},$$

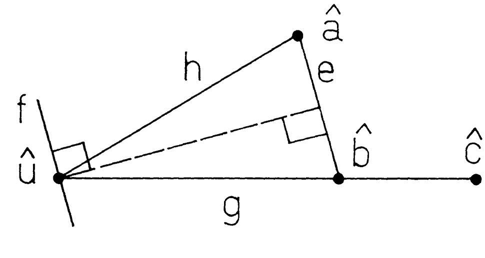
$$\mathscr{N}^{(i+1)} := \mathscr{N}^{(i)} \cup \left\{\hat{y} \in \mathscr{P}\ y|x_1, x_2 \text{ and } x_1 \textcircled{u} x_2 \text{ for some } x_1, x_2 \in \mathscr{N}^{(i)}\right\}$$

$$\cup \left\{x \in \mathscr{L}' \colon x|y_1, y_2 \text{ and } y_1\mathbf{u}y_2 \text{ for some } \hat{y}_1, \hat{y}_2 \in \mathscr{N}^{(i)}\right\}.$$

Then $\mathscr{N}^{(8)} = \mathscr{P} \cup \mathscr{L}'$.

Proof. Let $c := ug$. Select $b|g$ such that $b \multimap u$ and $b\mathbf{u}u$; cf. (E2). Then $c\mathbf{u}b, u$ and $\hat{b}, \hat{c} \in \mathscr{N}^{(0)}$. Then $a\mathbf{u}b$ since $a \multimap g$. Let $e|a, b$ and let f denote the pseudo-parallel of e through $\hat{u}$. Then $g\mathbf{u}e, f$; $e \multimap u$; $c \multimap e, f$.

(1) $$\{\hat{y} \in \mathscr{P} \colon y|e\} \subseteq \mathscr{N}^{(2)}.$$



Proof. Let $y|e$. As $u \multimap e$ there is a unique $x \in \mathscr{L}'$ with $x|\hat{u}, \hat{y}$. Then $x \textcircled{u} e$. From $e \in \mathscr{N}^{(1)}$ and $x \in \mathscr{N}^{(0)}$ it follows that $\hat{y} \in \mathscr{N}^{(2)}$.

(2) $$\{\hat{y} \in \mathscr{P} \colon y|f \text{ and } y \multimap u\} \subseteq \mathscr{N}^{(4)}.$$

Proof. Let $y|f$ and $y \longmapsto u$. Then $x|y,c$ for a unique x. We have $c \longmapsto e$, hence $z|e,x$ and $z\mathbf{u}c$ for some $\hat{z} \in \mathscr{P}$. Then $\hat{z} \in \mathscr{N}^{(2)}$ by (1). Also, $\hat{c} \in \mathscr{N}^{(0)}$ and $z\mathbf{u}c$. Hence $x \in \mathscr{N}^{(3)}$. $c \longmapsto f$ implies $x\mathbf{u}f$. Select $w|f$ such that $w \longmapsto u$ and $w\mathbf{u}y$; cf. 2.4(ii). Then $\hat{w} \in \mathscr{P}'$ and $w \longmapsto x$. Hence $f \textcircled{$u$} x$. Together with $f \in \mathscr{N}^{(0)}$ follows $\hat{y} \in \mathscr{N}^{(4)}$.

(3) $\{x \in \mathscr{L}' \colon y \mid f, x \text{ and } z \mid e, x \text{ for some } y,\, z \text{ with } y \longmapsto u\} \subseteq \mathscr{N}^{(5)}$.

Proof. From $y\mathbf{u}uf$ and $f\mathbf{u}e$ follows $y \longmapsto e$, hence $y\mathbf{u}z$. Together with (1) and (2) this yields $x \in \mathscr{N}^{(5)}$.

(4) $$\mathscr{P}' \subseteq \mathscr{N}^{(6)}.$$

Proof. Let $y \longmapsto u$. Select $j|u$ with $j\mathbf{u}f$ and $j \longmapsto f$; cf. (E2). Let $m := ju$. Then $m\mathbf{u}f$. Let j' and m' denote the pseudo-parallels of j, m, respectively, through the point $\hat{y}$. Then any two of the lines e, j', m', and also of the lines f, j', m', intersect uniquely in a point of the pseudo-plane. Therefore, $j', m' \in \mathscr{N}^{(5)}$; cf. (3). Also, $j'\mathbf{u}m'$. Therefore $j' \textcircled{$u$} m'$ by 8.1. Thus, $\hat{y} \in \mathscr{N}^{(6)}$.

Clearly, (4) implies $\mathscr{L}' \subseteq \mathscr{N}^{(7)}$. Therefore $\mathscr{P} \subseteq \mathscr{N}^{(8)}$ by 8.2.

8.4. Definition. Let $\hat{z} \in \mathscr{P}$ and $g \in \mathscr{L}$ such that $g|z$. An automorphism τ of $(\mathscr{P}, \mathscr{L}, |)$ is called a *transvection* whose center is $\hat{z}$ and whose axis is g if $\hat{x}\tau = \hat{x}$ and $y\tau = y$ for any point $\hat{x}$ of g and any line y through $\hat{z}$.

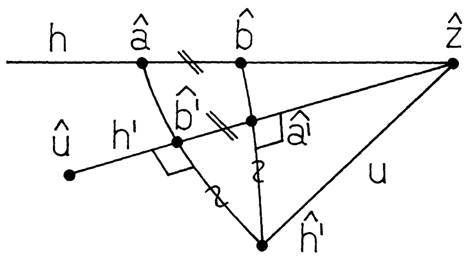
8.5. *Let $z|g$ and $a, b, z|h$ such that $a, b\mathbf{u}z$ and $h\mathbf{u}g$. There is at most one transvection τ having axis g and center $\hat{z}$ such that $\hat{a}\tau = \hat{b}$.*

Proof. Suppose, τ_1 and τ_2 satisfy the above properties. We apply 8.3 with z instead of u. The automorphism $\bar{\tau} := \tau_1\tau_2^{-1}$ of $(\mathscr{P}, \mathscr{L}, |)$ fixes each point and each line of $\mathscr{N}^{(0)}$. 8.3 implies that $\bar{\tau}$ is the identity on the set $\mathscr{N}^{(8)} \supseteq \mathscr{P}$.

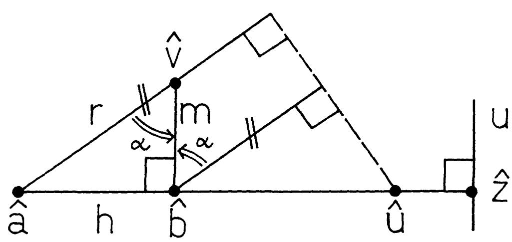
8.6. *Let $z|g$ and $a, b, z|h$ such that $a, b\mathbf{u}z$ and $h\mathbf{u}g$. There is a transvection τ of $(\mathscr{P}, \mathscr{L}, |)$ (cf. 8.4) whose center is $\hat{z}$ and whose axis is g such that $\hat{a}\tau = \hat{b}$. τ is unique. Moreover, if $y\mathbf{u}g$ then $y\tau\mathbf{u}g$ for any line $y \in \mathscr{L}$; and if $x \longmapsto g$ then $x' \longmapsto g$, where $\hat{x}\tau = \widehat{x'}$, for any point $\hat{x} \in \mathscr{P}$.*

Proof. 8.5 states the uniqueness.

In order to construct τ we may assume $g = u$, since our notion of the pseudo-plane may be introduced with respect to an arbitrary element $g \in S$. Then $\hat{a}, \hat{b} \in \mathscr{P}'$ and $h \in \mathscr{L}'$.



Secondly, we may assume $h|u$. Namely, let h' denote the pseudo-parallel of h through $\hat{u}$; let $\widehat{a'}, \widehat{b'}$ be the feet of the pseudo-orthogonals of h' through $\hat{a}, \hat{b}$, respectively. If τ is a transvection with center $\hat{z}$ and axis u such that $\widehat{a'}\tau = \widehat{b'}$ then clearly τ will also fulfill $\hat{a}\tau = \hat{b}$. Let $m := bh$. Select v such that $v|m$; $v \mapsto h$; $v\mathbf{u}h$; cf. (E2). Then $v\mathbf{u}a, u$ and $v \mapsto u$. Thus, $\hat{v} \in \mathscr{P}'$ and $r|a, v$ for some r with $r\mathbf{u}h$. $\alpha := \mathrm{w}(r, m) \in \mathscr{F}_0 \cap \mathscr{D}$. Let $\tau := T_{vu}^{-1} T_{vu\alpha} T_{bu\alpha}^{-1} T_{bu}$; cf. §6 and 7.5. We regard τ as a mapping $\mathscr{L} \to \mathscr{L}$. τ



induces an automorphism of $(\mathscr{P}, \mathscr{L}, |)$; cf. 6.3. Let τ also denote this automorphism. We contend that τ satisfies our requirements. 7.5 implies

(1) $H_{\hat{v},\alpha} H_{\hat{b},\alpha}^{-1}$ is the restriction of τ to the pseudo-plane $(\mathscr{P}', \mathscr{L}', |)$.
From this we conclude

(2) $y\tau = y$ for any $y \in \mathscr{L}'$ with $y|z$,

(3) y pseudo-parallel $y\tau$ for any $y \in \mathscr{L}'$,

(4) $\hat{a}\tau = \hat{b}$.

Since $uT_{vu}^{-1} = v$, $vT_{vu\alpha} = u$, $uT_{bu\alpha}^{-1} = b$ and $bT_{bu} = u$ one obtains

(5) $u\tau = u$.

The statements (3) and (5) imply

(6) $\hat{x}\tau = \hat{x}$ for each point $\hat{x}$ of the line u.

Finally, we want to prove

(7) $y\tau = y$ for each line y through the point $\hat{z}$.

(If y is an affine line or if $y = u$ then this is clear; cf. (2) and (5). For an arbitrary y however the arguments are not at all obvious.) Let $y|z$. We have $h = uz = bm$. Hence $mz = bu \in \mathscr{F}$ and $y\mathbf{u}m$. Let $e|y, m$ and $y' := yT_{bu}^{-1}$. The identity

$$\operatorname{support}(em\,bu) = \operatorname{support}(ez)$$

together with $e, z|y$ yields $y' = em$. Furthermore, $y', b, v|m$. Hence, $y'' := y'bv \in S$ and $y''|m$. Thus, by the above observation,

$$\operatorname{support}(y''vu) = \operatorname{support}(y'bvvu) = \operatorname{support}(y'bu) = y.$$

In other words, $y'' = yT_{vu}^{-1}$. We have proved

$$yT_{vu}^{-1}T_{vu\alpha} = y''T_{vu\alpha} = \operatorname{support}(y''vu\alpha) = \operatorname{support}(y'bvvu\alpha)$$
$$= \operatorname{support}(y'bu\alpha) = y'T_{bu\alpha} = yT_{bu}^{-1}T_{bu\alpha}.$$

This means that $y\tau = y$.

The additional assertion of 8.6 is clear from (1).

9. Coordinates. If R is a commutative ring with 1 then $\Pi(R) = (\mathscr{P}(R), \mathscr{L}(R), \mathrm{I})$ will denote the projective plane over R, $\Pi'(R) = (\mathscr{P}'(R), \mathscr{L}'(R), \mathrm{I})$ will denote the affine plane over R, viewed as a substructure of $\Pi(R)$, where $R[0,0,1]$ is the line of infinity.

9.1. *There is a commutative ring R with* 1 *and an embedding ξ of the pseudo-plane $(\mathscr{P}', \mathscr{L}', |)$ (viewed as an incidence-structure) into the affine plane $\Pi'(R)$, and an element $k \in R^*$ (group of units) with the following properties.*

(i) $\mathscr{P}'\xi = \mathscr{P}'(R)$. $R[0,1,0], R[1,0,0] \in \mathscr{L}'\xi$, $\hat{u}\xi = R(0,0,1)$.

(ii) $D_0 := \{R(\alpha, \beta)\colon \alpha, \beta \in R$ *and* $\alpha^2 + k\beta^2 \in R^*\}$ *is a commutative group, where multiplication is given by*

$$R(\alpha, \beta) \cdot R(\gamma, \delta) = R(\alpha\gamma - k\beta\delta, \alpha\delta + \beta\gamma).$$

Let m: $\mathscr{L}'(R) \times \mathscr{L}'(R) \to D_0$ *denote the mapping*

$$(R[\alpha, \beta, \varepsilon], R[\gamma, \delta, \nu]) \mapsto R(k\alpha\gamma + \beta\delta, \alpha\delta - \beta\gamma).$$

Then m *is an angle measure of the affine plane $\Pi'(R)$. The mapping*

$$\iota\colon \mathscr{D} \to D_0, \quad \mathrm{w}(g, h) \mapsto \mathrm{m}(g\xi, h\xi), \qquad \textit{where } g, h \in \mathscr{L}',$$

is well defined (i.e. does not depend on the choice of g, h). ι is a monomorphism of the group $\mathscr{D}$ into the group D_0. Call $\Omega \in D_0$ regular if any two lines Σ, Γ of $\Pi'(R)$ with $\mathrm{m}(\Sigma, \Gamma) = \Omega$ *intersect in exactly one point of $\Pi'(R)$. Let D_0^* denote the set of these regular angles Ω. Then $\mathscr{D}^*\iota = D_0^*$. Furthermore, $u\iota = R(0,1)$.*

(iii) *Every non-unit of R is a zero-divisor.*

(iv) *Let* $\hat{a} \in \mathscr{P}'$ *and* $g \in \mathscr{L}'$ *be such that* $a \longmapsto g$. *Then* $\hat{a}\xi$ distant $g\xi$. ("distant" is defined in Part I.) *Let* $g, h \in \mathscr{L}'$ *and* $g(u)h$. *Then* $g\xi$ distant $h\xi$ (definition of (u) see §8). *Let* $\hat{a}, \hat{b} \in \mathscr{P}'$ *and* $a\mathbf{u}b$. *Then* $\hat{a}\xi$ distant $\hat{b}\xi$. *If* $\hat{a}, \hat{b} \in \mathscr{P}'$, $g|a, b$ *for some* g *and* $\hat{a}\xi$ distant $\hat{b}\xi$ *then* $a\mathbf{u}b$.

Proof. First, let us recollect

(1) If $y|a$ and $\hat{a} \in \mathscr{P}'$ then $y \in \mathscr{L}'$.

The main result of 9.1 will be obtained from 5.5 of **[8]**. We apply this theorem to the pseudo-plane. In **[8]**, the technical denotion "$\hat{a}$ fern g" for a point $\hat{a} \in \mathscr{P}'$ and a line $g \in \mathscr{L}'$ means: Each point $\hat{x} \in \mathscr{P}'$ of g has a unique joining line y to $\hat{a}$; and $\mathrm{w}(g, y) \in \mathscr{D}^*$; i.e. g, y define a regular angle.

(2) "$\hat{a}$ fern g" if and only if $a \longmapsto g$, for any $\hat{a} \in \mathscr{P}'$ and $g \in \mathscr{L}'$.

Proof. Suppose that "$\hat{a}$ fern g" holds. Let h denote the pseudo-orthogonal of g through $\hat{a}$. Let $\hat{c}$ be the foot. The assumption together with (1) yields $a\mathbf{u}c$ and $\mathrm{w}(g, h) \in \mathscr{D}^*$. 2.9 implies $h\mathbf{u}g$. Hence $a \longmapsto g$. Conversely, let us assume $a \longmapsto g$. Let $\hat{x} \in \mathscr{P}'$ with $x|g$. Then $a\mathbf{u}x$. Therefore, $y|a, x$ for just one $y \in \mathscr{L}'$; cf. (1). Furthermore $y\mathbf{u}g$. 3.4 yields $\mathrm{w}(g, y) \in \mathscr{D}^*$.

(E) Given $g \in \mathscr{L}'$. Then "$\hat{a}$ fern g" for some $\hat{a} \in \mathscr{P}'$.

Proof. Let $h|u, g$. Then $gh \longmapsto u$. Select a such that $a|h$ and $a\mathbf{u}gh$ and $a \longmapsto u$; cf. 2.4(ii). Then $\hat{a} \in \mathscr{P}'$ and $a \longmapsto g$, hence "$\hat{a}$ fern g"; cf. (2).

Now we have proved that the pseudo-plane fulfills all of the requirements of 5.5 in **[8]**. This theorem yields the main assertion of 9.1, together with (i) and (ii), after an easy conversion into projective terms. This conversion is carried out in the following small type section.

5.5 of **[8]** provides an embedding ζ of $(\mathscr{P}', \mathscr{L}', |)$ into a structure $\mathscr{H}(A, R)$, where A is an algebra over the commutative ring R such that $A = R + R\omega$ for some $\omega \in A$ with $k := -\omega^2 \in R^*$. The point set of $\mathscr{H}(A, R)$ is A. A line is a set $a + Rb$, where $a \in A$ and $b \in A^*$ (group of units of A). Incidence is given by inclusion. The angle-measure of a pair of lines is $\mathrm{M}(a + Rb, c + Rd) := R^* b^{-1} d \in A^*/R^*$. The mapping λ_M: $A^*/R^* \to D_0$, $R^*(\alpha + \beta\omega) \mapsto R(\alpha, \beta)$ is an isomorphism (D_0 is defined in the theorem). $R^*(\alpha + \beta\omega) \in A^*/R^*$ is regular if and only if $\beta \in R^*$; cf. **[8]**, 2.1. Let λ denote the following embedding of $\mathscr{H}(A, R)$ into $\Pi'(R)$.

$$\varepsilon + \eta\omega \mapsto R(\varepsilon, \eta, 1)$$

$$(\gamma + \delta\omega) + R(\alpha + \beta\omega) \mapsto R[-\beta, \alpha, -\alpha\delta + \beta\gamma].$$

Then $(\mathrm{M}(g, h))\lambda_{\mathrm{M}} = \mathrm{m}(g\lambda, h\lambda)$ for any two lines g, h of $\mathscr{H}(A, R)$ (m is defined in 9.1(ii)). Let $\xi := \zeta\lambda$. Now we obtain 9.1 together with (i) and (ii) from Theorem 5.5 of [**8**].

Proof of (iii). Suppose $\mu \in R \setminus R^*$. We have $g\xi = R[0, 1, 0]$ for some $g \in \mathscr{L}'$ and $\hat{a}\xi = R(\mu, 0, 1)$ for some $\hat{a} \in \mathscr{P}'$; cf. (i). A line $h \in \mathscr{L}'$ through $\hat{a}$ will never satisfy $h \leftarrow\!\!\rightarrow u$; namely, suppose the contrary. Then (2) implies "$\hat{u}$ fern h". From 5.5 and 5.2 of [**8**] we obtain $R(\lambda\mu, 0, 1) = R(1, 0, 1)$ for some $\lambda \in R$; a contradiction. Consequently, $u\mathbf{u}a$ fails to hold. Hence $j|a, u$ for some $j \neq g$. Let $j\xi = R[\nu, \chi, 0]$. Then $\nu \neq 0$ and $\mu\nu = 0$.

Proof of (iv). Pick $\hat{a} \in \mathscr{P}'$ and $g \in \mathscr{L}'$ such that $a \leftarrow\!\!\rightarrow g$. We want to prove $\hat{a}\xi$ distant $g\xi$. Let h be the pseudo-orthogonal of g through $\hat{a}$. Let $\hat{b}$ denote the foot. Then "$\hat{a}$ fern g" by (2). 5.5 of [**8**] implies that $\hat{a}\xi$ and $\widehat{b\xi}$ have a unique joining line (in the affine plane $\Pi'(R)$), hence also in $\Pi(R)$ and that $g\xi$ and $h\xi$ intersect uniquely in $\Pi'(R)$, hence also in $\Pi(R)$. Due to (iii) this yields $\hat{a}\xi$ distant $\hat{b}\xi$ and $g\xi$ distant $h\xi$; cf. I.2. Thus $\hat{a}\xi$ distant $g\xi$; cf. I.3.

Let $g, h \in \mathscr{L}'$ and $g \textcircled{u} h$. We want to prove $g\xi$ distant $h\xi$. We can assume $a \leftarrow\!\!\rightarrow h$ and $a|g$ for some $\hat{a} \in \mathscr{P}'$. Then $\hat{a}\xi$ distant $h\xi$ by the first statement of (iv). In particular, $g\xi$ distant $h\xi$.

Let $g, h \in \mathscr{L}'$ and $a|g, h$ for some $\hat{a} \in \mathscr{P}'$. If $g\mathbf{u}h$ then $g \textcircled{u} h$ (cf. 8.1), hence $g\xi$ distant $h\xi$ according to the above proof.

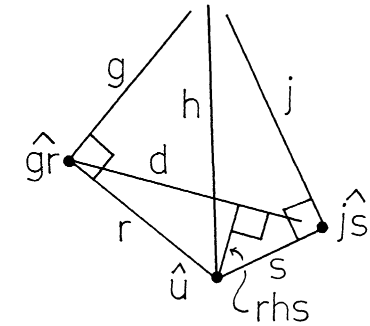
Let $\hat{a}, \hat{b} \in \mathscr{P}'$ and $a\mathbf{u}b$. We assert $\hat{a}\xi$ distant $\hat{b}\xi$. Let $g|a, b$. Select $h|b$ such that $h\mathbf{u}g$. Then $a \leftarrow\!\!\rightarrow h$ and $h \in \mathscr{L}'$, hence $\hat{a}\xi$ distant $h\xi$; cf. first statement of (iv). In particular, $\hat{a}\xi$ distant $\hat{b}\xi$.

Conversely, let us assume $\hat{a}, \hat{b} \in \mathscr{P}'$, $g|a, b$ and $\hat{a}\xi$ distant $\hat{b}\xi$. Then $a\mathbf{u}b$, since $h|a, b$ implies $h \in \mathscr{L}'$ and $g\xi$, $h\xi \mathrm{I} \hat{a}\xi$, $\hat{b}\xi$, hence $g\xi = h\xi$ and $g = h$.

REMARK. Let $\Omega = R[\alpha_1, \alpha_2, \alpha_3] \in \mathscr{L}'(R)$ and $\omega = R(\omega_1, \omega_2) \in D_0$. Then the line $\Gamma := R[\alpha_1\omega_1 - \alpha_2\omega_2, \alpha_2\omega_1 + k\alpha_1\omega_2, 0]$ satisfies $m(\Omega, \Gamma) = R(\omega_1, \omega_2)$.

9.2. *Let $g|r$ and $j|s$ and $r, h, s|u$ (hence $rhs \in S$). Suppose $g\mathbf{u}u$. There is a unique $d \in S$ with $d|gr, rhs$. Furthermore,*

$$ghj \in S \Leftrightarrow d \mid js.$$



Proof. We have $\alpha := ghj = (gr)(rhs)(sj)$. If $d | gr$, rhs, sj for some d, (E3) yields $\alpha \in S$. Conversely let us assume $\alpha \in S$. $g\mathbf{u}u$ implies $u \hookrightarrow gr$ (cf. 2.2), hence $rhs\mathbf{u}gr$. Let $d | rhs, gr$. Then $d | sj$ by 4.1.

9.3. (Algebraical description of pseudo-semi-rotations). *Let $\hat{y} \in \mathscr{P}'$ and $\omega \in \mathscr{D}$. Let $\hat{y}\xi = R(y_1, y_2, 1)$ and $\omega\iota = R^*(\omega_1, \omega_2)$. Let $H = H_{\hat{y},\omega}$. Then*

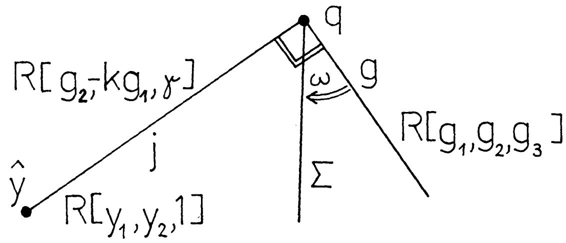
$$(gH)\xi = R[g_1, g_2, g_3]\begin{pmatrix} \omega_1 & k\omega_2 & -k\omega_2 y_2 \\ -\omega_2 & \omega_1 & \omega_2 y_1 \\ 0 & 0 & \omega_1 \end{pmatrix}$$

for every $g \in \mathscr{L}'$ with $g\xi = R[g_1, g_2, g_3]$. If $\omega \in \mathscr{F}$ then $\omega_1 \in R^*$; hence the linear transformation given by the matrix is bijective.

Proof. Let j denote the pseudo-orthogonal of g through $\hat{y}$. Then $j\xi = R[g_2, -kg_1, \gamma]$, where $\gamma = ky_2 g_1 - y_1 g_2$. The foot $\hat{q}$ is

$$\hat{q}\xi = R\big(-\gamma g_2 - kg_1 g_3, \gamma g_1 - g_2 g_3, kg_1^2 + g_2^2\big).$$

Obviously the line Σ obtained by multiplication of $g\xi$ with the above matrix is incident with $\hat{q}\xi$ and satisfies $\mathrm{m}(g\xi, \Sigma) = R(\omega_1, \omega_2)$.



If $\omega \in \mathscr{F}$ then $\omega u \in \mathscr{E}$. Thus ωu is a regular angle of $(\mathscr{P}', \mathscr{L}', \|)$. Hence, $(\omega u)\iota = R(-k\omega_2, \omega_1)$ is a regular angle of $\Pi'(R)$; cf. 9.1 (ii). This means that $\omega_1 \in R^*$.

9.4. *Let $g, h, j \in \mathscr{L}'$ and $h|u$. Let $g\xi = R\Gamma$, $h\xi = R\Sigma$, $j\xi = R\Omega$, where $\Gamma, \Sigma, \Omega \in R^{3*}$. Then*

$$ghj \in S \Leftrightarrow \det(\Gamma, \Sigma, \Omega) = 0.$$

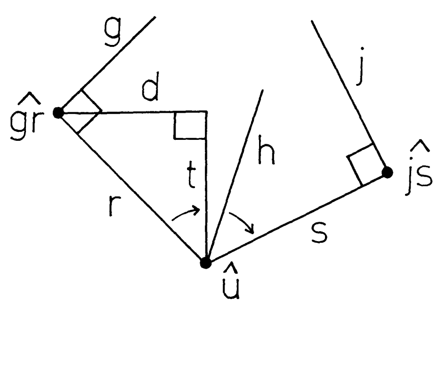
Proof. Let $r|u, g$; $s|u, j$ and $t := rhs$. Let $d|gr, t$; cf. 9.2. Let $\Gamma = [g_1, g_2, g_3]$, $\Sigma = [h_1, h_2, 0]$, $\Omega = [j_1, j_2, j_3]$. We have

$$(hs)\iota = R\big(h_2 j_1 - h_1 j_2, h_1 j_1 + k^{-1} h_2 j_2\big)$$

and $d = gH_{\hat{u},hs}$. 9.3 yields $d\xi = R\Lambda$, where

$$\begin{aligned}\Lambda = \big[&-g_1h_1j_2 + g_1h_2j_1 - g_2h_1j_1 - k^{-1}g_2h_2j_2,\\ &g_2h_2j_1 - g_2h_1j_2 + g_1h_2j_2 + kg_1h_1j_1, g_3(h_2j_1 - h_1j_2)\big].\end{aligned}$$

Moreover, $(\widehat{js})\xi = R\theta$, where $\theta = (kj_1j_3, j_2j_3, kj_1^2 + j_2^2)$. Hence, $\Lambda\theta = (kj_1^2 + j_2^2) \cdot \det(\Gamma, \Sigma, \Omega)$. Since $kj_1^2 + j_2^2$ is a unit, the following statements are equivalent: $d|js$; $\Lambda\theta = 0$; $\det(\Gamma, \Sigma, \Omega) = 0$. Now, 9.2 yields the assertion.



9.5. *Let $g, h, j \in \mathscr{L}'$ and $g, h|c$ and $g\mathbf{u}h$. Let $g\xi = R\Sigma$, $h\xi = R\Omega$, $j\xi = R\Gamma$, where $\Sigma, \Omega, \Gamma \in R^{3*}$. Then $j|c \Leftrightarrow \det(\Sigma, \Omega, \Gamma) = 0$.*

Proof. Let $r|g, u$ and $s|h, u$. Select $y|h$ such that $y\mathbf{u}s$ and $y \mapsto s$; cf. (E2). Then $\hat{y} \in \mathscr{P}'$ as $y \mapsto u$. From $y \mapsto s$ and $s|u$ follows $y\mathbf{u}u$. Let $d|y, u$ and $e := du$. $s \mapsto y$ and $y|d$ implies $sd \in \mathscr{E}$, hence $\alpha := se \in \mathscr{F}$. Let $H := H_{\hat{y},\alpha}$. H is a bijective pseudo-semi-rotation; cf. 7.3. Moreover, $hH = d|u$. From 9.3 we obtain a linear bijective mapping ψ: $R^{3*} \to R^{3*}$

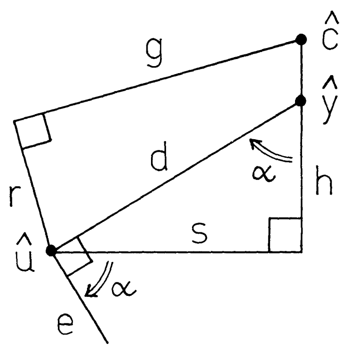
such that $(gH)\xi = R(\Sigma\psi), (hH)\xi = R(\Omega\psi), (jH)\xi = R(\Gamma\psi)$. 9.3 yields

(1) $\quad gH \cdot hH \cdot jH \in S \Leftrightarrow \det(\Sigma\psi, \Omega\psi, \Gamma\psi) = 0 \Leftrightarrow \det(\Sigma, \Omega, \Gamma) = 0.$

We can view H as an automorphism of the incidence structure $(\mathscr{P}, \mathscr{L}, |)$; cf. 7.5. Since $\hat{c}$ is the only intersection of g and h, $\hat{c}H$ will be the only intersection of gH and hH. Therefore, 4.1 implies

(2) $$j \,|\, \hat{c} \Leftrightarrow jH \,|\, \hat{c}H \Leftrightarrow gH \cdot hH \cdot jH \in S.$$

(1) and (2) together yield that $j|\hat{c}$ holds if and only if $\det(\Sigma, \Omega, \Gamma) = 0$.



9.6. *There is a unique extension of ξ to an embedding of $(\mathscr{P}, \mathscr{L}', |)$ into $(\mathscr{P}(R), \mathscr{L}'(R), \mathrm{I})$.*

Let ξ also denote this extension. If $c \in \mathscr{L}'$, $a, b|c$ and $a\mathbf{u}b$ then $\hat{a}\xi$ distant $\hat{b}\xi$.

Proof. Let $c \in \mathscr{P}$. According to 7.4 we may select lines $g, h \in \mathscr{L}'$ such that $g, h|c$ and $g \textcircled{u} h$. Let $g\xi = R\Sigma$ and $h\xi = R\Omega$. 9.1 (iv) implies $g\xi$ distant $h\xi$. Thus, $g\xi$ and $h\xi$ intersect in just one point $R(\Sigma \times \Omega)$ of $\Pi(R)$. Let $j \in \mathscr{L}'$ and $j\xi = R\Gamma$. By 9.5 the following statements are equivalent.

$$j \,|\, c; \quad \det(\Sigma, \Omega, \Gamma) = 0; \quad j\xi \mathrm{I} R(\Sigma \times \Omega).$$

Hence we can define $\hat{c}\xi := R(\Sigma \times \Omega)$, and there is no other choice.

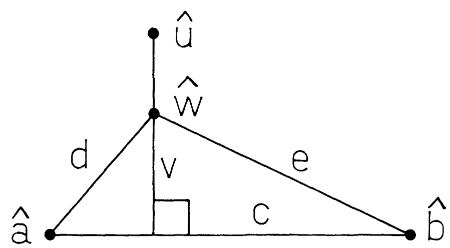
Proof of the last statement. Let $v|c, u$. Select w such that $w|v$ and $w \vdash\!\dashv c, u$; cf. 2.4. Then $\hat{w} \in \mathscr{P}'$ and $w\mathbf{u}a, b$. Let $d|a, w$ and $e|b, w$. Then $d, e \in \mathscr{L}'$ and $\hat{w}\xi$ distant $c\xi$; cf. 9.1(iv). In particular,

(1) $$\hat{w}\xi \text{ distant } \hat{a}\xi;$$

cf. 1.3. We have $d\mathbf{u}e$ (since $d \longmapsto b$) and $d, e \in \mathscr{L}'$ and $w \in \mathscr{P}'$; hence $d \textcircled{u} e$ by 8.1 and

(2) $$d\xi \text{ distant } e\xi;$$

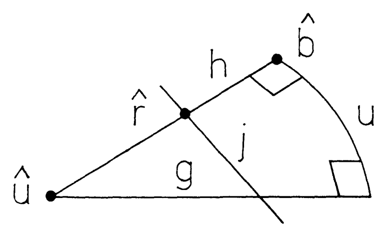
cf. 9.1(iv). (1) and (2) implies $\hat{a}\xi$ distant $e\xi$; cf. I.3. In particular, $\hat{a}\xi$ distant $\hat{b}\xi$.



9.7. Finally, we want to attach coordinates to every line of the elliptic plane.

Let $j \in \mathscr{L}$.

Select r such that $r|j$ and $r\mathbf{u}u$ (cf. (U)). Let $h|u, r$ and $b := hu$. Select $g|u$ such that $g\mathbf{u}h$ and $g \longmapsto h$.



(i) There is a transvection τ and a point $\hat{a}$ with the following properties. $a|h$ and $a\mathbf{u}u$; $\hat{u}$ is the center and g is the axis of τ; $\hat{a}\tau = \hat{b}$; $j\tau \in \mathscr{L}'$.

Proof. Select t such that $t|h$ and $t\mathbf{u}u$ and $t \longmapsto u$; cf. (E2). Then $t\mathbf{u}b$. There is a transvection τ of $(\mathscr{P}, \mathscr{L}, |)$ with center $\hat{u}$ and axis g such that $\hat{r}\tau = \hat{t}$; cf. 8.6. Then $j\tau \in \mathscr{L}'$ as $j\tau|\hat{t}$ and $\hat{t} \in \mathscr{P}'$. Let $\hat{a} := \hat{b}\tau^{-1}$. Then $a|h$ and $a \longmapsto g$ by 8.7, hence $a\mathbf{u}u$.

For the rest of this section we select τ and a according to (i).

(ii) There is an invertible matrix A such that

$$(*)\quad (\hat{y}\tau)\xi = (\hat{y}\xi)A, \qquad \text{and} \qquad (**)\quad (y\tau)\xi = A^{-1}(y\xi)$$

for every point $\hat{y} \in \mathscr{P}$ and every line $y \in \mathscr{L}'$.

Proof. Observe that a point $\hat{y} \in \mathscr{P}$, where $\hat{y}\xi = R(y_1, y_2, y_3)$, lies on the line u if and only if $y_3 = 0$.

Let $g\xi = R[g_1, g_2, 0]$, $\hat{a}\xi = R(a_1, a_2, a_3)$, $\hat{b}\xi = R(a_1, a_2, 0)$, $h\xi = R[a_2, -a_1, 0]$. Then $h \textcircled{u} g$ by 8.1, hence $h\xi$ distant $g\xi$; cf. 9.1 (iv). Thus we can assume $a_1 g_1 + a_2 g_2 = 1$. Let

$$A := \begin{pmatrix} 1 & 0 & -a_3 g_1 \\ 0 & 1 & -a_3 g_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We apply 8.3 in order to prove $(*)$ and $(**)$. According to 8.3 it is enough to prove $(*)$ and $(**)$ for $\hat{y} \in \mathscr{N}^{(i)} \cap \mathscr{P}$ and $y \in \mathscr{N}^{(i)} \cap \mathscr{L}'$, respectively, (we use the notations introduced in 8.3). We proceed by induction.

For $i = 0$ the assertion is easily verified.

Now let us step from i to $i + 1$.

Let $\hat{y} \in \mathscr{N}^{(i+1)} \cap \mathscr{P}$. If $\hat{y} \in \mathscr{N}^{(i)}$ then there is nothing to prove. Otherwise, $\hat{y} | x_1, x_2$ and $x_1 \textcircled{u} x_2$ for some $x_1, x_2 \in \mathscr{N}^{(i)} \cap \mathscr{L}'$. Then $x_1\xi$ distant $x_2\xi$ by 9.1(iv), and $(x_j\tau)\xi = A^{-1}(x_j\xi)$ by our assumption. Also $0 = (\hat{y}\tau)\xi \cdot (x_j\tau)\xi = (\hat{y}\tau)\xi \cdot A^{-1}(x_j\xi)$ and $0 = \hat{y}\xi \cdot x_j\xi$. Therefore $(\hat{y}\tau)\xi \cdot A^{-1} = \hat{y}\xi$, i.e. $(\hat{y}\tau)\xi = (\hat{y}\xi)A$.

Let $y \in \mathscr{N}^{(i+1)} \cap \mathscr{L}'$. We want to prove $(**)$ and may assume $y | \hat{x}_1, \hat{x}_2$ for some points $\hat{x}_1, \hat{x}_2 \in \mathscr{N}^{(i)}$ with $x_1 \mathbf{u} x_2$. From 9.6 follows $\hat{x}_1\xi$ distant $\hat{x}_2\xi$. The assumption yields $(\hat{x}_j\tau)\xi = (\hat{x}_j\xi)A$. Finally, $0 = (\hat{x}_j\tau)\xi \cdot (y\tau)\xi = (\hat{x}_j\xi)A \cdot (y\tau)\xi$ and $\hat{x}_j\xi \cdot y\xi = 0$ for $j = 1,2$ implies $A \cdot (y\tau)\xi = y\xi$, hence $(y\tau)\xi = A^{-1}(y\xi)$.

Now let A be a matrix such that (ii) holds. Since $j\tau \in \mathscr{L}'$, $(j\tau)\xi \in \mathscr{L}'(R)$ is well-defined. $(j\tau)\xi \cdot A^{-1}$ is a line of $\Pi(R)$. We define $j\xi := (j\tau)\xi \cdot A^{-1}$. Then for any $\hat{y} \in \mathscr{P}$ the following statements are equivalent: $\hat{y}\xi \mathrm{I} j\xi$; $\hat{y}\xi \mathrm{I} (j\tau)\xi \cdot A^{-1}$; $(\hat{y}\xi)A \mathrm{I} (j\tau)\xi$; (by $(*)$) $(\hat{y}\tau)\xi \mathrm{I} (j\tau)\xi$; $\hat{y}\tau | j\tau$; $\hat{y} | j$. Hence the above definition extends ξ to an embedding of $(\mathscr{P}, \mathscr{L}, |)$ into $\Pi(R)$. Clearly, the extension is unique. We summarize our results.

PROPOSITION. *There is an embedding ξ of $(\mathscr{P}, \mathscr{L}, |)$ into $\Pi(R)$. ξ is an extension of the mapping ξ of* 9.1. *If* $a\mathbf{u}b$ *then* $\hat{a}\xi$ distant $\hat{b}\xi$ *for any* $\hat{a}, \hat{b} \in \mathscr{P}$. *If* $a \text{—} j$ *then* $\hat{a}\xi$ distant $j\xi$ *for all* $\hat{a} \in \mathscr{P}$ *and* $j \in \mathscr{L}$. *If* $g\mathbf{u}h$ *then* $g\xi$ distant $h\xi$, *for all* $g, h \in \mathscr{L}$.

Proof of the last three assertions. Let $a\mathbf{u}b$. Let $j|a, b$. If $j \in \mathscr{L}'$ then $\hat{a}\xi$ distant $\hat{b}\xi$ by 9.6. Otherwise $j\tau \in \mathscr{L}'$ for some transvection τ according to the previous construction. Then $\hat{a}\tau\xi$ distant $\hat{b}\tau\xi$, i.e. $\hat{a}\xi$A distant $\hat{b}\xi A$, where A is a matrix as in (ii). Thus $\hat{a}\xi$ distant $\hat{b}\xi$.

Let $a \rightarrowtail j$. We want to prove $\hat{a}\xi$ distant $j\xi$. By the previous consideration we may assume $j \in \mathscr{L}'$. Select a point $\hat{b}$ such that $\hat{b}|j$ and $b \rightarrowtail u$. Then $a\mathbf{u}b$, hence

(1) $$\hat{a}\xi \text{ distant } \hat{b}\xi$$

by our first statement. Let $h|a, b$. Since $b \in \mathscr{P}'$ and $j\mathbf{u}h$ 8.1 yields $j\,ⓤ\,h$, hence

(2) $$j\xi \text{ distant } h\xi$$

by 9.1(iv). (1) and (2) implies $\hat{a}\xi$ distant $j\xi$; cf. I.3.

Finally, let us assume $g\mathbf{u}h$. Let $b|g, h$ and $a|h$ such that $a\mathbf{u}b$. Then $a \rightarrowtail g$ and $\hat{a}\xi$ distant $g\xi$ by what we have proved. In particular, $h\xi$ distant $g\xi$; cf. I.3.

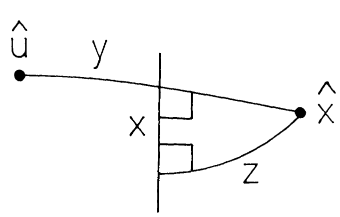
10. The bilinear form. We want to describe the polarity $\mathscr{L} \to \mathscr{P}$, $x \mapsto \hat{x}$ in terms of a bilinear form. Assumptions and notations of previous sections are preserved; in particular, $k \in R^*$ is the element introduced in 9.1.

10.1. *Let* $x \in \mathscr{L}'$ *with* $x \rightarrowtail u$ *and* $x\xi = R[x_1, x_2, x_3]$. *Then* $\hat{x}\xi = R(kx_1, x_2, lx_3)$ *for some* $l \in R^*$.

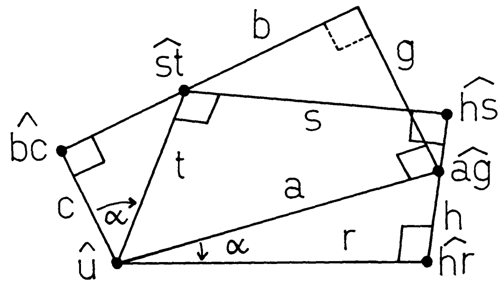
Proof. Let $y|u, x$. Then $y\xi = R[x_2, -kx_1, 0]$ and $y\xi \mathrm{I} \hat{x}\xi$. Choose $z|x$ with $z \rightarrowtail y$ and $z\mathbf{u}y$. This implies $z\mathbf{u}u$ and $z \rightarrowtail u$, hence $z\xi = R[z_1, z_2, 1]$; cf. 9.1 (iii). Obviously, $y\xi$ distant $z\xi$. Thus the unique point of intersection is

$$\hat{x}\xi = R(kx_1, x_2, -x_2z_2 - kx_1z_1) = R(kx_1, x_2, lx_3),$$

where $l := -x_3^{-1}(x_2z_2 + kx_1z_1)$. Moreover, $\hat{x} \in \mathscr{P}'$, hence $\hat{x}\xi \in \mathscr{P}'(R)$. This is equivalent to $lx_3 \in R^*$.



10.2. *Let* $g \in \mathscr{L}'$ *with* $g \leftarrow\!\!\rightarrow u$; $g\xi = R[g_1, g_2, g_3]$, $\hat{g}\xi = R(kg_1, g_2, lg_3)$. *Let* $H = H_{\hat{u},\alpha}$ *be a pseudo-semi-rotation with center* $\hat{u}$, $h := gH$, $h\xi = R[h_1, h_2, h_3]$. *Then* $\hat{h}\xi = R(kh_1, h_2, lh_3)$.



Proof. Let $a|g, u$. Then $h|\widehat{ag}$ and $ag \leftarrow\!\!\rightarrow u$. Let $r|h, u$, hence $\alpha = ar$. Choose $s \in S$ with $s|h$ and $s \leftarrow\!\!\rightarrow r$ and $\mathbf{s}u\mathbf{r}$. This choice implies $\mathbf{suu}$ and $s \leftarrow\!\!\rightarrow u$. Let $t|u, s$ and $c := \alpha t$. $st \leftarrow\!\!\rightarrow u$ and $c|u$ implies $st\mathbf{uc}$. Let $b|c, st$. From $s, r, ag|h$ follows $(st)cg = sr(ag) \in S$; cf. (E3). Furthermore, $st\mathbf{uc}$ and $b|st, c$. Hence $g|b$ by 4.1. Therefore, $a, b|g$, hence $a\xi, b\xi \mathrm{I} \hat{g}\xi$. The rest is a simple calculation involving coordinates. We have

$$g\xi = R[g_1, g_2, g_3], \qquad a\xi = R[-g_2, kg_1, 0];$$
$$h\xi = R[h_1, h_2, h_3], \qquad r\xi = R[-h_2, kh_1, 0].$$

Let

$$s\xi = R[s_1, s_2, s_3], \quad \text{hence} \quad t\xi = R[-s_2, ks_1, 0];$$
$$b\xi = R[b_1, b_2, b_3], \quad \text{hence} \quad c\xi = R[-b_2, kb_1, 0].$$

Then $\widehat{st}\xi = R(-ks_1s_3, -s_2s_3, ks_1^2 + s_2^2)$, and $\widehat{st}\xi \mathrm{I} b\xi$ means that

(1) $$(ks_1b_1 + s_2b_2)s_3 - (ks_1^2 + s_2^2)b_3 = 0.$$

Likewise, $\widehat{ag}|h$ implies

(2) $$(kh_1g_1 + h_2g_2)g_3 - (kg_1^2 + g_2^2)h_3 = 0.$$

From 9.1(ii) and $ct = \alpha = ar$ we have

(3) $$R^*(ks_1b_1 + s_2b_2, -s_1b_2 + s_2b_1)$$
$$= R^*(kh_1g_1 + h_2g_2, -g_2h_1 + g_1h_2).$$

In other words

(3′) $$ks_1b_1 + s_2b_2 = \lambda(kg_1h_1 + g_2h_2) \quad \text{and}$$
$$-s_1b_2 + s_2b_1 = \lambda(-g_2h_1 + g_1h_2) \quad \text{for some } \lambda \in R^*.$$

(3′) implies

(4) $\quad (kg_1b_1 + g_2b_2)(ks_1^2 + s_2^2) = \lambda(kh_1s_1 + h_2s_2)(kg_1^2 + g_2^2).$

In the next line, (1), (3′) and (2) yield the first, the second and the third equality:

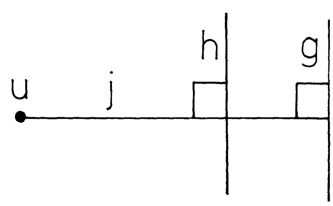
(5) $$b_3g_3(ks_1^2 + s_2^2) = (ks_1b_1 + s_2b_2)s_3g_3$$
$$= \lambda(kg_1h_1 + g_2h_2)g_3s_3 = \lambda h_3s_3(kg_1^2 + g_2^2).$$

Finally, (4) and (5) imply

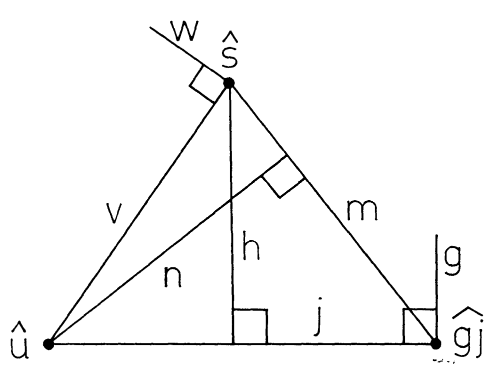
$$\lambda(kh_1s_1 + h_2s_2 + lh_3s_3)(kg_1^2 + g_2^2) = (kg_1b_1 + g_2b_2 + lb_3g_3)(ks_1^2 + s_2^2).$$

The right side is zero because $b\xi \mathrm{I} \hat{g}\xi = R(kg_1, g_2, lg_3)$. Therefore, $kh_1s_1 + h_2s_2 + lh_3s_3 = 0$. We conclude $\hat{h}\xi = r\xi \times s\xi = R(kh_1, h_2, lh_3)$.

10.3. *Let $j|u, g, h$ and $g, h\mathbf{u}u$ and $g \mapsto u$. Let $g\xi = R[g_1, g_2, g_3]$ and $h\xi = R[h_1, h_2, h_3]$. If $\hat{g}\xi = R(kg_1, g_2, lg_3)$ then $\hat{h}\xi = R(kh_1, h_2, lh_3)$.*

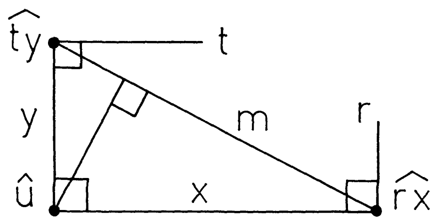


Proof. Select $s|h$ with $s\mathbf{u}j$ and $s \mapsto j$. Then $s \mapsto u$ and $s \mapsto j$, hence also $s\mathbf{u}u$. Let $v|u, s$ and $w := sv$. We have $s\mathbf{u}gj$. Let $m|s, gj$ and $n|u, m$. There are pseudo-semi-rotations mapping g to m and w to m. Thus, by the previous lemma, $\hat{m}\xi = R[km_1, m_2, lm_3]$ and $\hat{w}\xi = R[kw_1, w_2, lw_3]$ ($g, m, w \mapsto u$ implies $g_3, m_3, w_3 \in R^*$). Since $wH = h$ for some pseudo-semi-rotation H whose center is $\hat{u}$, 10.2 finally yields $\hat{h}\xi = R(kh_1, h_2, lh_3)$.



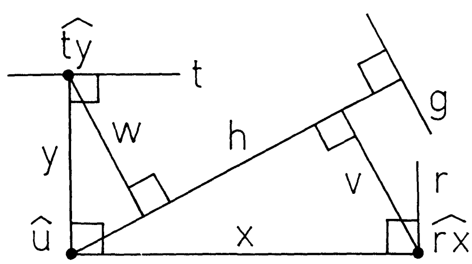
10.4. *There is an* $l \in R^*$ *such that* $\hat{g}\xi = R(kg_1, g_2, lg_3)$ *for every* $g \in \mathscr{L}$ *with* $g\xi = R[g_1, g_2, g_3]$.

Proof. Let $x, y \in S$ with $x\xi = R[0,1,0]$ and $y\xi = R[1,0,0]$. Select $r|x$ with $r \mapsto u$ and $r\mathbf{u}u$, and $t|y$ with $t \mapsto u$ and $t\mathbf{u}u$. From 9.1 follows $r\xi = R[1,0,\alpha]$ and $t\xi = R[0,1,\beta]$, where $\alpha, \beta \in R^*$. We have $m|rx, ty$ for some m. Moreover, $m \mapsto u$ and $m\mathbf{u}u$. Let $m\xi = R(m_1, m_2, m_3)$ and $l \in R^*$ with $\hat{m}\xi = R(km_1, m_2, lm_3)$; cf. 10.1. From 10.2 we conclude $\hat{r}\xi = R(k, 0, l\alpha)$ and $\hat{t}\xi = R(0, 1, l\beta)$.



We claim $\hat{g}\xi = R(kg_1, g_2, lg_3)$ for every $g \in \mathscr{L}$, where $g\xi = R[g_1, g_2, g_3]$.

First, let us assume $g \in \mathscr{L}'$, i.e. $g\mathbf{u}u$. Let $h|u, g$. Then $h\xi = R[-g_2, kg_1, 0]$. Let $v|rx, h$ and $w|ty, h$. Then $v\xi = R[g_1, g_2, \alpha g_1]$ and $w\xi = R[g_1, g_2, \beta g_2]$. By 10.1, $\hat{g}\xi = R(kg_1, g_2, l'g_3)$ for some $l' \in R$. Thus $\hat{v}\xi = R(kg_1, g_2, l'\alpha g_1)$ and $\hat{w}\xi = R(kg_1, g_2, l'\beta g_2)$; cf. 10.3.



On the other hand, v is the image of r under a pseudo-semi-rotation with center $\hat{u}$. Hence $\hat{v}\xi = R(kg_1, g_2, l\alpha g_1)$; cf. 10.2. The same conclusion, with t instead of r, yields $\hat{w}\xi = R(kg_1, g_2, l\beta g_2)$. Using $kg_1^2 + g_2^2 \in R^*$ and $\alpha \in R^*$ we obtain $lg_1 = l'g_1$. Likewise, $lg_2 = l'g_2$. Hence $l = l'$.

Finally, let g be an arbitrary line $\in \mathscr{L}$. From 7.4 we get $v, w \in \mathscr{L}'$ and $c \in \mathscr{P}'$ with $v, w|g$; $v\mathbf{u}w$; $c|w$; $c \mapsto v$. From 9.1(iv) we obtain $\hat{c}\xi$ distant $v\xi$. Hence $v\xi$ distant $w\xi$. Thus, in $\Pi(R)$ the lines $v\xi$, $w\xi$ have just one point of intersection, namely $\hat{g}\xi$. Let $v\xi = R[v_1, v_2, v_3]$ and $w\xi = R[w_1, w_2, w_3]$. Then $\hat{v}\xi = R(kv_1, v_2, lv_3)$ and $\hat{w}\xi = R(kw_1, w_2, lw_3)$ by the previous special case. From $v\xi$ distant $w\xi$ and $k, l \in R^*$ follows $\hat{v}\xi$ distant $\hat{w}\xi$. Hence

$$\begin{aligned} g\xi &= \hat{v}\xi \times \hat{w}\xi \\ &= R\big[l(v_2w_3 - v_3w_2),\ kl(v_3w_1 - v_1w_3),\ k(v_1w_2 - v_2w_1)\big]. \end{aligned}$$

Finally,

$$\begin{aligned} \hat{g}\xi &= v\xi \times w\xi \\ &= R\big[kl(v_2w_3 - v_3w_2),\ kl(v_3w_1 - v_1w_3),\ kl(v_1w_2 - v_2w_1)\big]. \end{aligned}$$

10.5. We define a symmetric bilinear form

$$f\colon R^3 \times R^3 \to R, \quad ((a_1, a_2, a_3), (b_1, b_2, b_3)) \mapsto la_1b_1 + kla_2b_2 + ka_3b_3,$$

where k and l stem from 10.4.

PROPOSITION. *Let $a, b \in S$ and $\hat{a}\xi = RA$, $\hat{b}\xi = RB$.*
(a) *$a|b$ if and only if $f(A, B) = 0$.*
(b) *$f(A, A) \in R^*$.*

Proof. (a) Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. Then $b\xi = R[lb_1, klb_2, kb_3]$; cf. 10.4. The following statements are equivalent. $a|b$; $\hat{a}|b$; $\hat{a}\xi \mathrm{I} b\xi$; $f(A, B) = 0$. (b) We have $a \multimap a$ and thus $\hat{a}\xi$ distant $a\xi$; cf. the proposition in §9.7. This means that $f(A, A) \in R^*$; cf. part I.

10.6. *$f(A, A) \in R^*$ for any point RA of $\Pi(R)$; i.e. $\Pi(R, f)$ is an elliptic plane in the sense of* §1.

Proof. Let RA be a point of $\Pi(A)$ and let $\mathfrak{m}$ be an arbitrary maximal ideal of R. We want to prove

$$(*) \qquad f(A, A) \notin \mathfrak{m}.$$

Let $^-$ denote the canonical homomorphisms

$$\begin{aligned} ^-\colon R &\to \overline{R} := R/\mathfrak{m} \\ ^-\colon R^3 &\to \overline{R}^3, \quad R^{3*} \to \overline{R}^{3*}. \end{aligned}$$

Let α denote the canonical homomorphism of $\Pi(R)$ onto the projective plane $\Pi(\overline{R})$ over the field $\overline{R}$:

$$\begin{aligned} \alpha\colon RX &\mapsto \overline{R}\overline{X} \\ \alpha\colon RY &\mapsto \overline{R}\overline{Y}, \end{aligned}$$

where RX is an arbitrary point and RY a line of $\Pi(R)$. First we will show

$$(**) \qquad \mathscr{P}\xi\alpha = \mathscr{P}(R)\alpha = \mathscr{P}(\overline{R}).$$

Proof of $(**)$. Let $\overline{R}\overline{B} \in \mathscr{P}(\overline{R})$, where $B \in R^3$. We claim $\overline{R}\overline{B} \in \mathscr{P}\xi\alpha$. One can assume $B = (b_1, b_2, 1)$ or $B = (1,0,0)$ or $B = (\lambda, 1, 0)$ for some $b_1, b_2 \in R$ or $\lambda \in R$, respectively. In the first case $RB \in \mathscr{P}'(R) = \mathscr{P}'\xi \subseteq \mathscr{P}\xi$, hence $\overline{R}\overline{B} = (RB)\alpha \in \mathscr{P}\xi\alpha$; cf. 9.1(i). Let $g, h \in \mathscr{L}$ be lines such that $g\xi = R[0,1,0]$ and $h\xi = R[1,0,0]$; cf. 9.1(i). In the second case follows $RB = R(1,0,0) = \widehat{(gu)}\xi \in \mathscr{P}\xi$, hence $(RB)\alpha \in \mathscr{P}\xi\alpha$. Now we consider the last case $B = (\lambda, 1, 0)$. The line $J := R[1, -\lambda, 0] \in \mathscr{L}(R)$ joins the point $\hat{u}\xi$ to the point RB. Furthermore, $m(g\xi, J) = R(\lambda, 1)$ is a regular angle, since $(1,0)$, $(\lambda, 1)$ constitute a R-basis of R^2. Thus $\mathrm{m}(h\xi, J) = \mathrm{w}(h, j)\iota = \mathrm{m}(h\xi, j\xi)$ for some $j \in \mathscr{L}'$ with $j|u$; cf. 9.1(ii), $\mathscr{D}^*\iota = D_0^*$. This implies $j\xi = J$. Finally, $RB = \widehat{(ju)}\xi \in \mathscr{P}\xi$ since $\widehat{(ju)}\xi IJ, u\xi$. Thus we have proved $\overline{R}\overline{B} = (RB)\alpha \in \mathscr{P}\xi\alpha$ in each of the three cases.

Now we deduce $(*)$. From $(**)$ follows that $(RA)\alpha = \overline{R}\overline{A} = (\hat{a}\xi)\alpha$ for some $\hat{a} \in \mathscr{P}$. Let $\hat{a}\xi = RB$, where $B \in R^3$. Then $\overline{R}\overline{A} = \overline{R}\overline{B}$ and $\overline{f(B,B)} \in R^*$; cf. 10.5. Thus, $\overline{A} = \bar{\mu}\overline{B}$ for some $\mu \in R \setminus \mathfrak{m}$, and $\overline{f(A,A)} = \bar{\mu}^2 \cdot \overline{f(B,B)} \neq \bar{0}$. This concludes the proof.

11. The group of motions. In this section we assume that (G, S) satisfies all of our five axioms. For the first time we shall also use (M).

11.1. *If* $g\mathbf{u}h$ *and* $g \mapstochar\!\!\longrightarrow\!\!\!\mapstochar h$ *then* $g\mathbf{u}g^h$.

Proof. Let $a|g, h$. Suppose that $g\mathbf{u}g^h$ does not hold. Then $b|g, g^h$ and $b \mapstochar\!\!\longrightarrow\!\!\!\mapstochar a$ for some $b \neq a$; cf. 2.6. In particular, $b\mathbf{u}h$. Let $j|b, h$. Then $b \mapstochar\!\!\longrightarrow\!\!\!\mapstochar hj$, and $g\mathbf{u}j$ since $g \mapstochar\!\!\longrightarrow\!\!\!\mapstochar h$. Together with $b, b^h|g, j$ follows $b = b^h = b^{hj}$. (M) yields $b = hj$. Thus, $a, b|g, h$. Finally, $g\mathbf{u}h$ implies the contradiction $a = b$.

11.2. $2 \in R^*$.

Proof. Let $g \in \mathscr{L}$ such that $g\xi = R[0,1,0]\mathrm{I}\hat{u}\xi = R(0,0,1)$. Select j such that $j|\hat{u}$ and $j\mathbf{u}g$ and $j \mapstochar\!\!\longrightarrow\!\!\!\mapstochar g$. Then $j\mathbf{u}g, h$, where $h := ug$. Thus, $j\xi$ distant $g\xi$, $h\xi$; cf. the proposition in §9.7. Therefore $j\xi = R[1, \lambda, 0]$ for some $\lambda \in R^*$. Let $q := j^g$. Then

$$R(\lambda, 1) = \mathrm{m}(j\xi, g\xi) = (\mathrm{w}(j, g))\iota = (\mathrm{w}(g, q))\iota = \mathrm{m}(g\xi, q\xi);$$

cf. 9.1(ii). Thus, $q\xi = R[1, -\lambda, 0]$. 11.1 implies $j\mathbf{u}q$, hence $j\xi$ distant $q\xi$; cf. 9.7. This means that $\lambda + \lambda \in R^*$.

We have proved that $\Pi(R, f)$ is an elliptic plane over the ring R in the sense of Part I and that 2 is a unit (10.6 and 11.2). In particular, the reflection σ_Ω in any point Ω of $\Pi(R, f)$ is well-defined.

Every $\alpha \in G$ induces an automorphism $\tilde{\alpha}$ of the elliptic plane $(\mathscr{P}, \mathscr{L}, \mathrm{I})$:

$$\tilde{\alpha}: \quad \begin{array}{l} \hat{x} \mapsto \widehat{x^\alpha} \text{ for } \hat{x} \in \mathscr{P}, \quad \text{and} \\ y \mapsto y^\alpha \text{ for } y \in \mathscr{L}. \end{array}$$

$\tilde{\alpha}$ is called the *motion induced by* α. The mapping $\tilde{}$ is a homomorphism of the group G into the group of automorphisms of $(\mathscr{P}, \mathscr{L}, |)$, and the kernel is the center C_G of G. In 5.5 we have proved $C_G = 1$. This yields

11.3. *The mapping* $\tilde{}: G \to \tilde{G}$ *is an isomorphism of the group* G *onto the subgroup* $\tilde{G}$ *of the group of automorphisms of* $(\mathscr{P}, \mathscr{L}, |)$.

Let $(\mathscr{G}, \mathscr{S})$ denote the group of motions of the elliptic plane $\Pi(R, f)$; cf. Part I. Hence $\mathscr{S} = \{\sigma_\Omega : \Omega \in \mathscr{P}(R)\}$, and $\mathscr{G}$ is the group generated by $\mathscr{S}$.

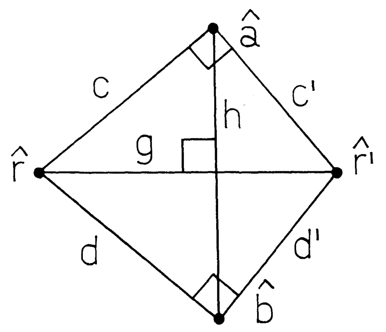
If Ω is a point of $\Pi(R, f)$ and Γ is its polar line (cf. Part I) then we define $\sigma_\Gamma := \sigma_\Omega$. In particular, $\sigma_{a\xi} = \sigma_{\hat{a}\xi}$ for any $a \in \mathscr{L}$.

11.4. *Let* $g \in S$ *and* $\hat{x} \in \mathscr{P}$. *Then* $(\widehat{x^g})\xi = (\hat{x}\xi)\sigma_{g\xi}$. *In other words,* $\tilde{g}\xi = \xi\sigma_{g\xi}$ *for every* g.

The proof consists of four steps.

11.4(a). *Let* $c\mathbf{u}g$ *and* $c \sqcap g$. *Then* $(c^g)\xi = (c\xi)\sigma_{g\xi}$.

Proof. Let $r|c, g$. Select r' such that $r'|g$; $r'\mathbf{u}r$ and $r' \sqcap r$; cf. (E2). Then $c\mathbf{u}r'$. Let $c'|c, r'$ and $a := cc'$. Then $a\mathbf{u}g$. Let $h|a, g$. $r' \sqcap r$ implies $c'\mathbf{u}r$. Together with $c\mathbf{u}g$ follows $c' \sqcap g$. Furthermore, $c'\mathbf{u}g$ since $c \sqcap g$.



Hence $d' := c'^g \mathbf{u} c'$; cf. 11.1. The proposition of §9.7 yields $d'\xi$ distant $c'\xi$. This implies

(1) $$\sigma_{d'\xi}\mathbf{u}\sigma_{c'\xi} \quad \text{in } (\mathscr{G}, \mathscr{S}).$$

Let $b := a^g$ and $d := c^g$. Then $b = dd'|h$. Hence $\hat{a}\xi$, $\hat{g}\xi$, $\hat{b}\xi \mathrm{I} h\xi$ and

(2) $$\sigma_{a\xi}\sigma_{g\xi}\sigma_{b\xi} \in \mathscr{S},$$

by I.15 and I.10. The analogue conclusion applies to the lines $c\xi$, $g\xi$, $d\xi$ since they pass through the point $\hat{r}\xi$. Thus, $\sigma_{c\xi}\sigma_{g\xi}\sigma_{d\xi} = \sigma_\Gamma$ for some line Γ of $\Pi(R, f)$ through $\hat{r}\xi$. The identity

(3) $$\sigma_{c'\xi}\sigma_\Gamma\sigma_{d'\xi} = \sigma_{c'\xi}\sigma_{c\xi}\sigma_{c\xi}\sigma_\Gamma\sigma_{d\xi}\sigma_{d\xi}\sigma_{d'\xi} = \sigma_{a\xi}\sigma_{g\xi}\sigma_{b\xi}$$

together with (2) yields

(4) $$\sigma_{c'\xi}\sigma_\Gamma\sigma_{d'\xi} \in \mathscr{S}.$$

$r \longmapsto r'$ implies $\hat{r}\xi$ distant $r'\xi$; cf. 9.7. In particular, Γ distant $r'\xi$; cf. I.3. Thus

(5) $$\sigma_\Gamma\mathbf{u}\sigma_{r'\xi} \quad \text{in } (\mathscr{G}, \mathscr{S}).$$

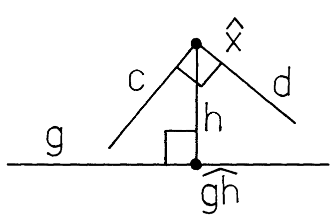
Now we use (1), (4) and (5) in order to apply 2.10 to $(\mathscr{G}, \mathscr{S})$. (Note that 2.10 does not require (U)). 2.10 yields $\sigma_\Gamma | \sigma_{r'\xi}$, hence $\Gamma \mathrm{I} \hat{r}'\xi$; cf. the proposition in I.15. Thus we obtain

$$\Gamma, g\xi \mathrm{I} \hat{r}\xi, \widehat{r'}\xi \quad \text{and} \quad \hat{r}\xi \text{ distant } \widehat{r'}\xi.$$

This implies $\Gamma = g\xi$, hence $\sigma_{d\xi} = \sigma_{c\xi}^{\sigma_{g\xi}}$. I.8(iv) yields $d\xi = (c\xi)\sigma_{g\xi}$. This is the assertion.

11.4(b). $\widehat{(x^g)}\xi = (\hat{x}\xi)\sigma_{g\xi}$ *for each pair* x, g *with* $x\mathbf{u}g$.

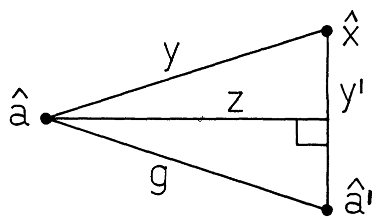
Proof. Let $h|g, x$. Select c such that $c|x$; $c\mathbf{u}h$ and $c \longmapsto h$. Then $d := cx$ satisfies $d|x$; $d\mathbf{u}h$; $d \longmapsto h$. Also $c\mathbf{u}g$ and $d\mathbf{u}g$. From $c|x$; $x|h$; $h|g$; $c\mathbf{u}h$ and $x\mathbf{u}g$ follows $c \longmapsto g$. Likewise, $d \longmapsto g$. 11.4(a) shows $(c^g)\xi = (c\xi)\sigma_{g\xi}$ and $(d^g)\xi = (d\xi)\sigma_{g\xi}$. Hence $\widehat{(x^g)}\xi$, $(\hat{x}\xi)\sigma_{g\xi}\mathrm{I}(c^g)\xi$, $(d^g)\xi$. Furthermore, $(c^g)\xi$ distant $(d^g)\xi$. Thus, $\widehat{(x^g)}\xi = (\hat{x}\xi)\sigma_{g\xi}$.



11.4(c). *$(y^g)\xi = (y\xi)\sigma_{g\xi}$ for each y, g with $y\mathbf{u}g$.*

Proof. Let $x|g, y$. Select x' such that $x'|y$ and $x'\mathbf{u}x$ and $x' \longmapsto x$. Then $x, x'\mathbf{u}g$. x, g and x', g fulfill the requirements of 11.4(b). Furthermore, $\hat{x}\xi$ distant $\widehat{x'}\xi$. This yields the assertion.

Now we are ready to prove 11.4. Given g, x. Select y such that $y|x$ and $y\mathbf{u}g$; cf. (U). Let $a|g, y$. Select z such that $z|a$ and $z \longmapsto g, y$; cf. 2.4(i). Then $x\mathbf{u}z$. Let $y'|x, z$. The two pairs y, g and y', g satisfy the assumptions of 11.4(c). Furthermore, $y\mathbf{u}y'$ since $y \longmapsto z$. Hence $y\xi$ distant $y'\xi$, and 11.4(c) implies $\widehat{(x^g)}\xi = (\hat{x}\xi)\sigma_{g\xi}$.



11.5. *There is a monomorphism $\sigma\colon G \to \mathscr{G}$ such that*

$$\sigma\colon g \mapsto \sigma_{g\xi} \text{ for every } g,$$

and $\mathscr{F}(R, f) \subseteq G\sigma$.

Proof. We want to extend the mapping $\sigma\colon S \to \mathscr{S}$, $g \mapsto \sigma_{g\xi}$, to a homomorphism of G into $\mathscr{G}$. Let $a_1, \ldots, a_n \in S$. Let $\alpha := a_1 \cdots a_n$ and $\gamma := \sigma_{a_1\xi} \cdots \sigma_{a_n\xi}$. Applying 11.4 n times we obtain

$(*)\quad \tilde{\alpha}\xi = \tilde{a}_1 \cdots \tilde{a}_n\xi = \xi\gamma;\qquad$ i.e. $\widehat{x^{a_1 \cdots a_n}}\xi = (\hat{x}\xi)\gamma\quad$ for every x.

Suppose $\alpha = 1$. We want to prove $\gamma = \mathrm{id}$. From $(*)$ follows that $\Omega = \Omega\gamma$ for every $\Omega \in \mathscr{P}\xi$. γ is a linear mapping $R^3 \to R^3$, and the points $R(0,0,1)$, $R(1,0,0)$, $R(0,1,0)$, $R(1,1,1)$ are elements of $\mathscr{P}\xi$; cf. 9.1(i) and 9.7. Hence they are fixed points of γ. It follows that there is some $\lambda \in R^*$ such that $X\gamma = \lambda X$ for every $X \in R^3$. We obtain $\lambda^2 = 1$ since γ is an orthogonal mapping, and $\lambda^3 = 1$ since $\det\gamma = 1$. Thus $\lambda = 1$ and $\gamma = \mathrm{id}_{R^3}$. Now we have proved that σ can be extended to a homomorphism σ of G into $\mathscr{G}$, namely

$$\sigma\colon \alpha = a_1 \cdots a_n \mapsto \gamma = \sigma_{a_1\xi} \cdots \sigma_{a_n\xi}.$$

If $\alpha \in \operatorname{kernel}\sigma$ then $\gamma = \mathrm{id}$, and $(*)$ implies $\hat{x}\tilde{\alpha} = \hat{x}$ for every $\hat{x} \in \mathscr{P}$. Since the mapping $\tilde{}$ is injective (cf. 11.3) we conclude $\alpha = 1$. Thus we have proved that σ is injective.

Finally, let $\mu \in \mathscr{F}(R, f)$. We have $\hat{u}\xi = R(0,0,1)$ and $g\xi = R[0,1,0]$ for some g; cf. 9.1(i). From 1.17(a) one obtains $\mu = \sigma_\Omega \sigma_{g\xi} \sigma_\Gamma$, where Γ is a line through $\hat{u}\xi$ with $g\xi$ distant Γ, and Ω is a point with Ω distant $u\xi$. The angle $\mathrm{m}(g\xi, \Gamma)$ is regular (cf. 9.1(ii)) since the lines Γ and $g\xi$ intersect in just one point of the affine plane $\Pi'(R)$ and since every non-unit of R is a zero-divisor; cf. 9.1(iii) and [3], 2.3. 9.1(ii) yields $\Gamma = j\xi$ for some $j \in \mathscr{L}$. Also $\Omega = \hat{q}\xi$ for some q, since each affine point of $\Pi(R)$ is the ξ-image of some point of $(\mathscr{P}, \mathscr{L}, \mathrm{I})$; cf. 9.1(i). We conclude $\mu = (qgj)\sigma \in G\sigma$.

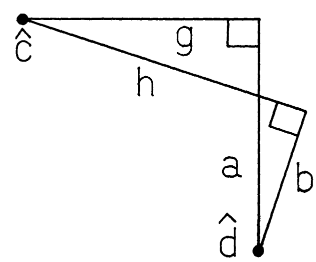
Appendix: Fixed point theorems. Let (G, S) satisfy the system of axioms introduced in §1 of Part II. We reformulate 4.1:

1. *Let* $ab = cd$ *and* $c, d | q$. *If* $a\,\mathbf{u}\,b$ *then* $c\,\mathbf{u}\,d$ *and* $q | a, b$
2. *Let* $a, b | d$ *and* $a\,\mathbf{u}\,b$ *and* $c\,\mathbf{u}\,d$. *Then* $c^{ab} = c$ *implies* $ab = d | c$.

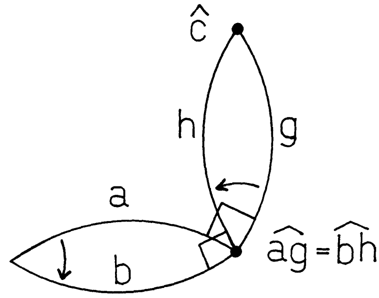
Proof. Let $g | c, d$. Then $h := gab \in S$ since $g, a, b | d$. Also $h | d$. 1. implies $h\,\mathbf{u}\,g$. Furthermore, $c\,\mathbf{u}\,d$. Hence $c \longmapsto h$, and $c = c^{ab} = c^{gh} = c^h$ yields $c = h | d$; cf. (M). Therefore, $g | h = c$ and $g, h | d$. Hence $d = gh = ab$; cf. 2.2(b).

3. *Let* $a, b | d$ *and* $a\,\mathbf{u}\,b$ *and* $c \longmapsto d$. *If* $c^{ab} = c$ *then* $c = d$.

Proof. (E1) implies $c\,\mathbf{u}\,a, b$. Let $g | a, c$ and $h | b, c$. We have $c | g, h$ and $c^a = c^b | g, h$. Furthermore, $g\,\mathbf{u}\,h$ by 2.9. Thus $c = c^a = c^{ga}$, and $c \longmapsto ga$ by 2.2. Therefore $c = ga$ by (M). Likewise, $c = hb$. Thus $c | a, b$, and $a\,\mathbf{u}\,b$ yields $c = d$.



4. *Let* $a \longmapsto b$ *and* $c\,\mathbf{u}\,a, b$. *If* $c^{ab} = c$ *then* $ab = gh$, *where* $g | a, c$ *and* $h | b, c$.



Proof. $a \mapsto b$ implies $ga\mathbf{u}b$. Let $d|ga, b$. Then $c\mathbf{u}d$, since $c\mathbf{u}a$ implies $c \mapsto ga$. Furthermore, $c = c^{ab} = c^{gab}$. We apply 2 (with ga, b, c, d instead of a, b, c, d) and obtain $gab = d|c$. Thus, $d|c, b$ and $h|c, b$ and $c\mathbf{u}b$. Hence $d = h$, and finally $ab = gh$.

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Received November 14, 1984 and in revised form June 25, 1985. The first-named author is indebted to the University of Toronto for support during a stay in 1983.

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