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SPACE CURVES THAT INTERSECT OFTEN

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In intersection theory one tries to understand $X \cap Y$ in terms of information about how X and Y lie in an ambient variety Z. When the sum of the codimensions of X and Y in Z exceeds the dimension of Z not much is known in this direction. The purpose of this note is to provide some results in perhaps the simplest nontrivial case of this—that of curves in $\mathbf{P}^3$ (projective three space). A weaker result for $\mathbf{P}^n$ is also obtained. We work over any fixed algebraically closed field of arbitrary characteristic.

(1) THEOREM. *Let X of degree d and Y of degree e be two distinct reduced, irreducible curves in $\mathbf{P}^3$ neither of which is contained in a hyperplane. Assume $d \leq e$. Let m be the number of points in $X \cap Y$ (not counting multiplicity). Then*:

(i) $m \leq (d-1)(e-1)+1$

(ii) *If $m = (d-1)(e-1)+1$ then there exists a quadric hypersurface Q containing $X \cup Y$. If furthermore $d \geq 4$ then Q is smooth and on Q X has type $(d-1,1)$ and Y has type $(1,e-1)$.*

(iii) *If $d \geq 4$ and $m \geq (d-2)e+1$ then there exists a smooth quadric Q containing $X \cup Y$.*

The key to the proof of this theorem will be a study of the ideal of the curve X. Results of [**GLP**] will be crucial.

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Before starting the proof of (1) we quote results from other sources that will be needed.

(2) DEFINITION ([**GLP**], p. 491). Let $X \subset \mathbf{P}^r$ be a reduced curve. For a given integer $n \geq 0$ we say X satisfies property (C_n) if X is cut out in $\mathbf{P}^r$ by hypersurfaces of degree n, and the homogeneous ideal of X is generated in degrees greater than or equal to n by its component of degree n.

(3) THEOREM ([**GLP**], p. 492, 504). *Let* $X \subset \mathbf{P}^r$ $(r \geq 3)$ *be a reduced, irreducible curve of degree* d, *not contained in any hyperplane. Then*:

(i) *Property* (C_{d+1-r}) *fails if and only if* X *is a smooth rational curve having a* $(d+2-r)$*- secant line.*

(ii) *Let* $\mathcal{I}_X$ *be the ideal sheaf of* X. *If* $d \geq r+2$ *and* (C_{d+1-r}) *fails then* $H^1(\mathbf{P}^r, \mathcal{I}_X(d-r))$ *is one dimensional unless both* $r = 3$ *and* X *lies on a smooth quadric surface.*

(4) DEFINITION ([**M**], lecture 14, and [**GLP**], p. 494). Let $X \subset \mathbf{P}^r$ be a reduced curve then for $n \geq 0$ we say X is n-regular if $H^i(\mathbf{P}^r, \mathcal{I}_X(n-i)) = 0$ for $i > 0$.

(5) THEOREM (*Castelnuovo, see* [**M**] *lecture* 14). *If* $X \subset \mathbf{P}^r$ *is an* n*-regular curve then* X *satisfies property* (C_n).

Proof of (1). The cases $d = 3$ and $d = 4$ follow from elementary considerations. If $d = 3$ then X lies on three independent quadrics. If Y meets X in at least $(3-1)(e-1)+1 = 2e-1$ points then at least one quadric containing X meets Y in at least $2e+1$ points. By Bézout's theorem this quadric contains $X \cup Y$. If $d = 4$ X lies on at least one quadric. If Y meets X in at least $(4-2)e+1 = 2e+1$ points, then Bézout's theorem says this quadric contains $X \cup Y$. The rest of the theorem for $d = 3$ or $d = 4$ now follows from standard knowledge about curves on quadrics (smooth or singular).

Now assume $d \geq 5$. First note that once (iii) is proven the rest easily follows from standard knowledge about curves on a smooth quadric. Assume we have X and Y as in the hypothesis of (iii). By Bézout's theorem once X lies on a smooth quadric that quadric will contain Y also. In fact any hypersurface of degree less than or equal to $d-2$ containing X must also contain Y. That means X does not satisfy property (C_{d-2}). By (3) we conclude that X is a smooth rational curve with a $(d-1)$-secant line, call it L. Bézout again tells us that the intersection of all hypersurfaces of degree $d-2$ containing X must contain at least $X \cup Y \cup L$. The following lemma now completes the proof of the theorem.

(6) LEMMA. *Let* $X \subset \mathbf{P}^r$ $(r \geq 3)$ *be a reduced, irreducible curve of degree* $d \geq r+2$ *not contained in any hyperplane that does not satisfy property* (C_{d+1-r}). *In particular by* (3) *we know that* X *is a smooth rational curve with a* $(d+2-r)$*- secant line, call it* L. *Let* W *equal the intersection of all hypersurfaces of degree* $d+1-r$ *which contain* X. *Then*:

(i) *If* $r = 3$ *and* X *lies on a smooth quadric* Q, *then* $W = Q$.

(ii) *Otherwise* $W = X \cup L$.

Proof. (i) is elementary. Among the surfaces of degree $d-2$ containing X there will certainly be every surface of the form $Q \cup$ an arbitrary surface of degree $d-4$, so $Q \supseteq W$. On the other hand one of the rulings of Q consists entirely of $(d-1)$- secant lines to X, so $W \supseteq Q$.

To prove (ii) we will use (5). That is, we want to show that if either $r \neq 3$ or X does not lie on a smooth quadric, then $X \cup L$ is $(d+1-r)$-regular. Let $\pi\colon Z \to X \cup L$ be the normalization map. Z is just the disjoint union of X and L. As usual let $\mathcal{O}$ stand for structure sheaf. We have three exact sheaf sequences.

(7) $$0 \to \mathcal{O}_{X \cup L}(k) \to \pi_* \mathcal{O}_Z(k) \to \mathcal{O}_{X \cap L} \to 0$$

(8) $$0 \to \mathcal{I}_X(k) \to \mathcal{O}_{\mathbf{P}^r}(k) \to \mathcal{O}_X(k) \to 0$$

(9) $$0 \to \mathcal{I}_{X \cup L}(k) \to \mathcal{O}_{\mathbf{P}^r}(k) \to \mathcal{O}_{X \cup L}(k) \to 0$$

Since $H^0(\pi_*\mathcal{O}_Z(k)) \cong H^0(\mathcal{O}_X(k)) \oplus H^0(\mathcal{O}_L(k))$ it has dimension $k(d+1)+2$. Because L is $(d+2-r)$- secant to X, $H^0(\mathcal{O}_{X \cup L})$ has dimension $d+2-r$.

What is presented from here to (11) is a simplification of our original proof. This was suggested by the referee and independently by P. Rao to whom we are grateful.

$$H^1(\pi_*\mathcal{O}_Z(k)) \cong H^1(\mathcal{O}_X(k)) \oplus H^1(\mathcal{O}_L(k)) = 0, \quad \text{for } k \geq 1.$$

Putting this into the cohomology sequence for (7) we see that if we wish to show that $H^1(\mathcal{O}_{X \cup L}(k)) = 0$ it is sufficient to show that the map

(10) $$H^0(\pi_*\mathcal{O}_Z(k)) \to H^0(\mathcal{O}_{X \cap L})$$

is surjective. But this map is just the difference of the homomorphisms $H^0(\mathcal{O}_X(k)) \to H^0(\mathcal{O}_{X \cap L})$ and $H^0(\mathcal{O}_L(k)) \to H^0(\mathcal{O}_{X \cap L})$ arising from the inclusions of $X \cap L$ in X and L. So it is enough to show that either of these, in particular the first, is surjective. But this is clear, since

$$\mathcal{O}_X(-(X \cap L)) \cong \mathcal{O}_{\mathbf{P}^1}(r-2-d)$$

and hence $H^1(\mathcal{O}_X(-(X \cap L)) \otimes \mathcal{O}_{\mathbf{P}^r}(k)) = 0$ for $k > 0$.

(11) $$H^1(\mathcal{O}_{X \cup L}(k)) = 0 \quad \text{for } k \geq 1.$$

Putting (11) and the other cohomology groups calculated just after (9) into the exact cohomology sequence for (7) we obtain:

(12) $$\dim H^0(\mathcal{O}_{X \cup L}(k)) = (k-1)d + k + r, \quad \text{for } k \geq 1.$$

Compare the cohomology sequences for (8) and (9) with $k = d - r$. Using (12), (3)(ii), and the facts that $H^0(\mathcal{I}_X(d-r)) = H^0(\mathcal{I}_{X \cup L}(d-r))$ (Bézout) and $\dim H^0(\mathcal{O}_X(k)) = kd + 1$ one proves that

$$H^1(\mathcal{I}_{X \cup L}(d-r)) = 0.$$

Using the cohomology sequence for (9) together with (11) one shows that $H^2(\mathcal{I}_{X\cup L}(d-r-1)) = 0$. Finally that $H^i(\mathcal{I}_{X\cup L}(d+1-r-i)) = 0$ for $i \geq 3$ comes directly from the cohomology sequence for (9).

(13) COROLLARY. *Let X of degree d and Y of degree e be two reduced curves in $\mathbf{P}^r$ ($r \geq 4$), with X irreducible and not contained in any hyperplane, and no component of Y equal to a line or X. Assume $r + 2 \leq d$. Let m be the number of points in $X \cap Y$ (not counting multiplicity). Then $m < (d - r + 1)e + 1$.*

Proof. Otherwise some component of Y would be contained in every hypersurface of degree $d - r + 1$ that contained X contradicting (6).

REMARKS (14). In intersection theory one usually likes to count intersections with appropriate multiplicities. In the case of a hypersurface H intersecting a reduced, irreducible curve X in finitely many points $p_1, \ldots, p_n$ there is more or less universal agreement on the definition for $i(X, H, p_j)$ the intersection multiplicity of X and H at p_j. Let $\mathcal{O}_{p_j,X}$ be the local ring of p_j in X and r_j the image of an equation for H in $\mathcal{O}_{p_j,X}$, then $i(X, H, p_j) =$ the length over $\mathcal{O}_{p_j,X}$ of $\mathcal{O}_{p_j,X}/(r_j)$. For a reducible curve, add up the multiplicities for each component. For the case of two curves X and Y intersecting in finitely many points $p_1, \ldots, p_n$ a possible definition for an intersection multiplicity is: $i(X, Y, p_j) =$ the minimum over all hypersurfaces H which contain X but do not contain any component of Y of $i(Y, H, p_j)$.

If in either (1) (with $d \geq 5$) or (13) we replace m with $\sum_{p \in X \cap Y} i(X, Y, p)$, then the proofs as stated go through without change. This multiplicity has the disadvantage that it is not symmetric. Also it is not known whether this is the largest multiplicity one can use and have these results remain true.

(15) In (1) the assumptions that X and Y are irreducible are necessary. For instance if X consisted of d lines from one ruling of a smooth quadric and Y was e lines from the other ruling, then $X \cap Y$ would consist of de points.

(16) When X and Y in (1) achieve the maximum possible number of intersections both X and Y are smooth rational curves. This follows immediately from standard knowledge about curves on a quadric.

(17) For most values of d and e in (1) there is a gap between the largest possible value of m as counted in (14) and the second largest. This gap gets wider as d and e get larger. This follows immediately from standard knowledge about curves on a smooth quadric.

References

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