PACIFIC JOURNAL OF MATHEMATICS
Vol. 124, No. 2, 1986

FULLY BOUNDED G-RINGS

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This paper contains a nullstellensatz for fully bounded G-rings which yields at the same time a new proof of the Amitsur-Small nullstellensatz for division rings. The result is applied to the study of fully bounded rings R for which the polynomial ring $R[x]$ is primitive.

The main result of this paper is a nullstellensatz for fully bounded G-rings (defined below) which yields at the same time a new proof of the Amitsur-Small nullstellensatz [**1**] for division rings. While our main interest is in Noetherian rings, the results of this paper apply somewhat more generally. We recall that a ring is *right bounded* if every essential right ideal contains a nonzero two-sided ideal. We call a ring *fully bounded* (or, an FB ring) if for every prime ideal P, R/P is left and right Goldie and left and right bounded. Examples of such rings are fully bounded Noetherian (FBN) rings, commutative rings, and rings satisfying a polynomial identity (PI rings). A ring R is a *G-ring* if R is prime and the intersection of the nonzero prime ideals of R is nonzero. It is worth pointing out that with the usual convention on the empty intersection, any simple Artinian ring is an FB G-ring.

The first result of the paper is that if R is an FB ring and n is a positive integer, then there is a primitive ideal P in $R[x_1, \ldots, x_n]$ such that $P \cap R = 0$ if and only if R is a G-ring. This theorem is proved in §1.

The Amitsur-Small nullstellensatz says that if D is a division ring, then for every positive integer n, every simple module over the polynomial ring $D[x_1, \ldots, x_n]$ is finite dimensional over D. This result for polynomials in n variables is used to prove the above result on G-rings, but one can also use the result on G-rings to prove the Amitsur-Small result for polynomials in $n + 1$ variables. Thus one can give a simultaneous proof of the two results, which yields a proof of the Amitsur-Small theorem which has some advantages over the original proof. This is carried out in §2.

The original purpose of this research was to find necessary and sufficient conditions on a fully bounded ring R for the polynomial ring $R[x]$ to be primitive. The results here are not complete, but there are several partial results in §3.

In this paper, all modules will be right modules unless specified otherwise. The annihilator of a right module M will be denoted by $r - \operatorname{ann}(M)$. If R is a G-ring, then the intersection of the nonzero prime ideals of R will be denoted by $G(R)$.

1. G-rings and a nullstellensatz. In this section we show that for fully bounded G-rings, one can prove results analogous to the well-known results in the commutative case (as developed, for example in **[11]**). In one direction this turns out to be easy (Lemma 1 below) but in the other it requires some new techniques.

LEMMA 1. *If R is a G-ring which is right or left Goldie, then in the polynomial ring $S = R[x_1, \ldots, x_n]$ there is a primitive ideal P such that $P \cap R = 0$.*

Proof. Let b be a regular element contained in $G(R)$ (which exists by Goldie's theorem). Let $J = (1 - bx_1)S$ and let M be a maximal right ideal containing J. Let P be the annihilator of the simple right module S/M. We claim that $P \cap R = 0$. Otherwise, if $P \cap R \neq 0$ then $P \cap R$ is prime, and so $b \in P \cap R$. It follows that $b \in M$, and hence that $bx_1 \in M$. Since $1 - bx_1 \in M$, we have $1 \in M$, a contradiction.

REMARK. It is well known that for commutative rings the converse of Lemma 1 is true, but in the non-commutative case it is only true under substantial restrictions. For example, if A_2 is the second Weyl algebra and D_2 its quotient skew field, then it follows from **[1]** that $D_2[x, y]$ is primitive although clearly $D_2[x]$ is not a G-ring.

LEMMA 2. *If R is an FB ring and S is a simple $R[x_1, \ldots, x_n]$-module which is faithful as an R-module, then S_R (the module restricted to R) is torsion-free injective and of finite rank.*

Proof. If such a simple module exists and P is its annihilator, then $P \cap R = 0$. Thus 0 is a prime ideal of R and R is a prime left and right Goldie ring. Suppose that S_R has a nonzero torsion submodule, and let C be a nonzero torsion cyclic submodule of S_R. Then there is an induced homomorphism $C \otimes R[x_1, \ldots, x_n] \to S$, which must be an epimorphism, since S is simple. As an R-module, $C \otimes R[x_1, \ldots, x_n]$ is a direct sum of copies of C. The hypothesis on R implies that C has a nonzero annihilator ideal, and this ideal will therefore annihilate $C \otimes R[x_1, \ldots, x_n]_R$ and S_R. Since this impossible, we conclude that S_R is torsion-free.

Since S is a torsion-free module over the prime Goldie ring R, to prove that S_R is injective it is sufficient to prove that it is divisible. Let b be a regular element of R. As R is *left* bounded, there is a nonzero ideal J of R contained in Rb. Now SJ is a fully invariant R-submodule of S and hence an $R[x_1, \ldots, x_n]$-submodule. Since S is simple over the polynomial ring, we conclude that $S = SJ \subseteq Sb$. Thus S is divisible, and hence injective, as an R-module.

Finally, we show that M has finite rank as an R-module. If Q is the Goldie quotient ring of R, then by the results of the previous two paragraphs, $S \cong S \otimes_R Q$, and the latter is a simple $Q[x_1, \ldots, x_n]$-module. According to the Amitsur-Small nullstellensatz, this module is finite dimensional, so S is torsion-free of finite rank. (Of course, in the one variable case we use a simple division algorithm, and the Amitsur-Small theorem is not involved.)

Note that only for the last step did we use the full strength of the hypotheses. Thus if R is an FB ring and S an extension of R generated by elements which centralize R, then a simple S-module which is faithful as an R-module is torsion-free and injective as an R-module, though not necessarily finite dimensional. For an illustrative infinite dimensional example see [**9**, Theorem 3].

LEMMA 3. *Let R be a prime, left and right Goldie ring, M a torsion-free injective R-module of finite rank, and L a finitely generated submodule of M. Then every nonzero prime ideal P of R is the annihilator of a finitely generated subfactor of M/L.*

Proof. In this proof, we shall use $M^{(n)}$ to denote the direct sum of a n copies of M. Since P is a prime ideal, it is an easy application of the Schreier refinement theorem to show that we may replace M and L by $M^{(n)}$ and $L^{(n)}$ for any positive integer n. Since M is a torsion-free injective R-module, it follows that M is a module over the right quotient ring Q of R, and so for suitable n and k, $M^{(n)} \cong Q(R)^{(k)}$. Thus we may assume that $M = Q(R)^{(k)}$ and $M \supseteq R^{(k)}$. Since R is left Goldie, there is a regular element b of R such that $bL \subseteq R^{(k)}$. If a is a regular element of R contained in P, then $abL \subseteq aR^{(k)} \subseteq P^{(k)} \subseteq R^{(k)}$. Thus

$$L \subseteq (ab)^{-1}P^{(k)} \subseteq (ab)^{-1}R^{(k)} \subseteq Q(R)^{(k)} = M.$$

If $A = (ab)^{-1}R^{(k)}$ and $B = (ab)^{-1}P^{(k)}$, then A/B is the required subfactor.

THEOREM 4. *If R is an FB ring, then there is a primitive ideal P of $R[x_1, \ldots, x_n]$ such that $P \cap R = 0$ if and only if R is a G-ring.*

Proof. It is immediate from Lemma 1 that if R is a G-ring then such an ideal P exists. Conversely, suppose that S is simple $R[x_1, \dots, x_n]$-module which is faithful as an R-module. By Lemma 2, S_R is torsion free injective of finite rank. Let L be a finitely generated essential R-submodule, and note that L generates S as an $R[x_1, \dots, x_n]$-module. For each i, $1 \leq i \leq n$, the action of x_i on S is an R-endomorphism α_i. Since for each index i, the factor $(L + \alpha_i(L))/L$ is finitely generated and torsion as an R-module, the right boundedness of R implies that there is a nonzero ideal I of R such that $\alpha_i(L)I \subseteq L$ for all i, $1 \leq i \leq n$. Since each α_i is a right R-homomorphism, it follows immediately that if β is a monomial in the α_i's of degree m, then $\beta(L)I^m \subseteq L$. Since S is generated as an R-module by the submodules $\beta(L)$, we conclude that for any finitely generated R-submodule N of M, there is an integer j with $NI^j \subseteq L$. From Lemma 3 we infer that every nonzero prime ideal of R contains I. Thus R is a G-ring.

Recall that a ring is *Jacobson* if every prime ideal is the intersection of primitive ideals.

COROLLARY 4.1. *Let R be an FB ring which is a Jacobson ring, and let S be a simple right module over the ring $R[x_1, \dots, x_n]$. Then there is a maximal ideal M of R such that $SM = 0$, and as an R-module, S is finitely generated and semi-simple.*

Proof. Let $M = R \cap (r - \operatorname{ann} S)$. Theorem 4 implies that R/M is a G-ring. Since M is also the intersection of primitive ideals, this is only possible if M itself is primitive. As R is FB, M must be maximal, and R/M must be simple Artinian. Lemma 2 now implies that S is finitely generated as an R-module.

In the case in which R is a division ring, this, of course, is the Amitsur-Small nullstellensatz.

We close this section by remarking that despite the above results, we know remarkably little about noncommutative G-rings, even if they are fully bounded and Noetherian. For example, if R is a right Noetherian G-ring which satisfies a polynomial identity, then it is known that R is one-dimensional and has finite spectrum, as in the commutative case [**2**, Theorem 5.1]. However, while there are Noetherian G-rings of arbitrary classical Krull dimension (for example, among primitive factors rings of enveloping algebras of simple Lie algebras, or see [**12**, 4.7(iii)], we know of no example of a fully bounded Noetherian G-ring which has Krull dimension greater than one.

2. On the Amitsur-Small nullstellensatz. We may regard the Amitsur-Small nullstellensatz as a sequence of statements as follows:

(α_n) If D is a division ring then every simple module over the ring $D[x_1, \ldots, x_n]$ is finite dimensional as a D-module.

We may similarly restate Theorem 4 as a sequence of results:

(γ_n) If R is an FB ring and I a primitive ideal of $R[x_1, \ldots, x_n]$, then $R/I \cap R$ is a G-ring.

We first remark that α_1 is trivial since it is just the division algorithm. In proving Theorem 4, we showed that α_n implies γ_n. In this section we will use the result γ_n on G-rings to prove α_{n+1}. Thus, in addition to showing that the two forms of the nullstellensatz are equivalent, we will have given a new proof of the Amitsur-Small nullstellensatz. (This proof, while not shorter than the original one, has the advantage of being more conceptual.)

LEMMA 5. *If R is a left and right principal ideal ring and $\mathscr{C}$ is the set of elements in R which are regular modulo all nonzero prime ideals of R, then $\mathscr{C}$ is a left and right Ore set, and the localization of R at $\mathscr{C}$ is a fully bounded ring.*

Proof. To show $\mathscr{C}$ is a right Ore set, we let $r \in R$ and $c \in \mathscr{C}$ and we must find $c' \in \mathscr{C}$ with $rc' \in cR$. So let $U = \{s \in R\colon rs \in cR\}$. For each nonzero prime ideal P of R, $\mathscr{C}(P)$, (the elements regular modulo P), is an Ore set (cf. [6]). Hence $U \cap \mathscr{C}(P) \neq \varnothing$. Since U is a right ideal, $U = aR$ for some $a \in R$. Now if $ab \in \mathscr{C}(P)$, then $a \in C(P)$, so $a \in \cap \mathscr{C}(P) = \mathscr{C}$. This shows that $\mathscr{C}$ is a right Ore set, and similarly $\mathscr{C}$ is a left Ore set. If $R\mathscr{C}^{-1}$ denotes the localization of R at $\mathscr{C}$, then it is clear that $R\mathscr{C}^{-1}$ is a principal ideal ring, and hence of Krull dimension one. Therefore, to show that $R\mathscr{C}^{-1}$ is fully bounded, we need only show that a maximal right ideal is bounded.

Thus, let M be a maximal right ideal, and let z be a generator of M, where we may assume that $z \in M \cap R$. Since z is not a unit in $R\mathscr{C}^{-1}$, there is a nonzero prime P such that $z \notin \mathscr{C}(P)$. Now if it were true that $zR\mathscr{C}^{-1} + P\mathscr{C}^{-1} = R\mathscr{C}^{-1}$, then by clearing denominators we would have $zr + p = c$ for some $r \in R$, $p \in P$ and $c \in \mathscr{C}$. This would imply (by reduction modulo P) that $z \in \mathscr{C}(P)$, which is false. Since M is maximal and $M = zR\mathscr{C}^{-1}$, it follows that the ideal $P\mathscr{C}^{-1}$ is contained in M.

We now show that γ_n implies α_{n+1}. It obviously suffices to show that if M is a maximal right ideal of $D[x_1, \ldots, x_n]$, then $M \cap D[x_k] \neq 0$ for every k, $1 \leq k \leq n + 1$. In $D[x_k]$, let $\mathscr{C}$ be the set of elements which are regular modulo every nonzero prime ideal of $D[x_k]$, and let $R_k = D[x_k]\mathscr{C}^{-1}$. Since $D[x_k]$ has infinitely many centrally generated prime ideals, R_k is a fully bounded principal ideal domain with an infinite number of prime ideals. Hence the only G-ideals of R_k are the maximal ideals. If $S_k = D[x_1, \ldots, x_{n+1}]\mathscr{C}^{-1}$ and $M_k = M\mathscr{C}^{-1}$, then S_k is a polynomial ring in n variables over R_k, and M_k is either a maximal right ideal of S_k or all of S_k. In either case, since R_k is not a G-ring, $M_k \cap R_k \neq 0$. Clearing denominators, we have $M \cap D[x_k] \neq 0$, as required.

3. Primitivity of $R[x]$. As was noted in the introduction, the original purpose of this research was to determine those fully bounded Noetherian rings for which the polynomial ring $R[x]$ is primitive. The existence of such R, other than division rings, was first observed by Hodges [7], who proved that if R is a noncommutative discrete valuation ring whose quotient division ring Q is transcendental over its center, then Q itself is a simple faithful $R[x]$-module. This example, together with Theorem 4, suggests that if R is a prime FB ring with quotient ring Q, then $R[x]$ should be primitive iff R is a G-ring and $Q[x]$ is primitive. While the necessity of these conditions follows immediately from the developments of §1, we can prove their sufficiency (Theorem 8) only under the additional, assumption that the center of Q is algebraic over the quotient field of the center of R.

The first proposition of this section, which will play a key role in the proof of Theorem 8, can be viewed as a generalization of the well-known characterization of commutative G-rings [**11**, Theorem 19].

PROPOSITION 6. *Let R be an FB G-ring, let b be a regular element contained in $G(R)$ and let $\mathscr{B} = \{b^n\colon n \geq 0\}$. Then $\mathscr{B}$ is a left and right Ore set in R, and the localization of R at $\mathscr{B}$ equals the full quotient ring $Q(R)$ of R. Conversely, if R is a right bounded prime right Goldie ring containing a regular element b such that every element of $Q(R)$ is of the form ab^{-n}, then R is a G-ring.*

Proof. To show that $\mathscr{B}$ is right Ore, we must show that if $r \in R$ and if $U = \{u \in R\colon ru \in bR\}$, then $U \cap \mathscr{B}$ is nonempty. Since bR is an essential right ideal in R, U is an essential right ideal of R and, since R is right bounded, U must contain a nonzero two-sided ideal V of R. If $V \cap \mathscr{B}$ is empty, then V is contained in an ideal P maximal with respect to the property that $P \cap \mathscr{B} = \varnothing$, and such an ideal is clearly prime. This contradicts the choice of b, and establishes our claim. A symmetric argument shows that $\mathscr{B}$ is also left Ore.

Let $R\mathscr{B}^{-1}$ denote the localization of R at $\mathscr{B}$. Since $R\mathscr{B}^{-1}$ is both bounded and simple, it is Artinian, and hence equals $Q(R)$.

To prove the converse (which, by the way, is not of importance in the applications) note that bR is essential R and thus contains a nonzero ideal I. We show that every prime ideal contains I. If $P \not\supset I$ then $I + P > P$ so $I \cap \mathscr{C}(P) \neq \varnothing$. Thus $bR \cap \mathscr{C}(P) \neq \varnothing$, and so $b \in \mathscr{C}(P)$. This, as usual, is contradictory. (If z is a regular element in P then z is invertible in $Q(R)$ so for some $a \in R$, $zab^{-n} = 1$. Thus $za = b^n$, where $za \in P$ and $b^n \notin P$.)

If R is a prime Goldie ring with quotient ring Q, then it is well-known [**10**, p. 241] that $Q[x]$ is primitive iff the matrix ring $M_m(Q)$ is transcendental over its center, for some positive integer m. If $I = G(R)$, then in every known example of a G-ring for which $M_m(Q)$ is transcendental over its center, there is already a regular transcendental element in the subset $M_m(I)$ of $M_m(Q)$. We conjecture that this is always the case. The next lemma shows that this is sufficient to solve the general problem raised in this section.

LEMMA 7. *Let R be an FB G-ring, with full quotient ring Q, and let $I = G(R)$. Suppose there exists a positive integer m such that $M_m(I)$ contains a regular element which is transcendental over the center of Q. Then $R[x]$ is primitive.*

Proof. Since both being an FB G-ring and being a primitive ring are Morita invariants, we may assume, upon replacing R by $M_m(R)$ if necessary, that I itself contains a regular element which is transcendental over the center of Q. Let $b \in I$ be such an element, and consider the polynomial $f = 1 - bx$. Since b is transcendental over the center of Q, $fQ[x] = (x - b^{-1})Q[x]$ is an unbounded right ideal of $Q[x]$, hence is contained in an unbounded maximal right ideal M of $Q[x]$ (one obtains M by looking at a composition series for $Q[x]/fQ[x]$, see [**1**, p. 357]). Let N be a maximal right ideal of $R[x]$ which contains $M \cap R[x]$. If $NQ[x] = Q[x]$, then in view of Proposition 6, we have $nb^i = 1$ for some integer $i \in \mathbf{Z}$ and some $n \in N$. Multiplying this relation by a sufficiently large positive power of b, we obtain $b^j \in N$ for some $j \geq 1$. In as much as $1 - b^jx^j$ is a multiple of f, this implies that $1 \in N$, which is absurd. Thus $NQ[x]$ is a proper right ideal of $Q[x]$, and we must have $NQ[x] = M$. Let $J = r - \operatorname{ann}(R[x]/N)$. In as much as $JQ[x]$ is an ideal of $Q[x]$ contained in M, the choice of M forces $JQ[x] = 0$, whence $J = 0$.

THEOREM 8. *Let R be a prime FB ring and let Q be the quotient ring of R. If the center of Q is algebraic over the quotient field of the center of R, the $R[x]$ is primitive iff R is a G-ring and $Q[x]$ is primitive.*

Proof. The necessity of the two conditions follows from Theorem 4 and Lemma 2, respectively. To prove their sufficiency we shall argue by contradiction that R satisfies the hypotheses of Lemma 7.

Let L denote the center of Q, let $I = G(R)$ and suppose that for every $m \geq 0$, every regular element of $M_m(I)$ is algebraic over L. Let C be the center of R with quotient field K. Then given any regular element $b \in M_m(I)$ there exist a positive integer j and elements $c_0, c_1, \ldots, c_j \in C$ with $c_j \neq 0$ such that

$$c_0 + c_1 b + \cdots + c_j b^j = 0.$$

Since b is not a zero-divisor, the usual minimal degree argument shows that we may assume that $c_0 \neq 0$. It follows that $c_0 \in M_m(I) \cap C$, and so $Q \cong R \otimes_c C[c_0^{-1}]$ by Proposition 6. Thus, if q is any element of $M_m(Q)$, then we can write $q = rc_0^i$, where $r \in M_m(I)$ and $i \in \mathbf{Z}$. If q is a unit in $M_m(Q)$, moreover, then r is necessarily regular. Since r is algebraic over L by our assumption on $M_m(I)$ and since q is a central multiple of r, it follows that every unit q is algebraic over L.

We have also assumed that $Q[x]$ is primitive, however, so we know that there exists some integer l such that $M_l(Q)$ is not algebraic over L. To obtain the desired contradiction, therefore, it suffices to observe that if D is a division ring for which there exists a positive integer i such that $M_i(D)$ is transcendental over its center k, then there also exists an (apparently unrelated) integer j such that $M_j(D)$ contains a *unit* which is transcendental over k. This is well known. (For example, by [**10**, p. 241] the polynomial ring $D[x]$ is primitive. If S is a simple $D[x]$-module, then S is finite dimensional over D, say $\dim(S_D) = j$, and so $\mathrm{End}_{D[x]}(S)$ embeds in $M_j(D)$. If S is also faithful, on the other hand, then $k[x]$ embeds in $\mathrm{End}_{D[x]}(S)$. As this endomorphism ring is a division ring, $k(x)$ embeds in $M_j(D)$ and the image of x is a transcendental unit in $M_j(D)$.)

To indicate not only the limitations of Theorem 8, but also the independent utility of Lemma 7, we shall now construct, for any pair of commutative fields $K \subseteq L$, an FB G-rings whose center is K and whose quotient ring has center L.

EXAMPLE. Let K be a commutative field and let L be an arbitrary extension field of K. Let $X = \{x_i \mid -\infty < i < \infty\}$ be a countable set of commuting indeterminates indexed by $\mathbf{Z}$, and form the rational function fields $F = K(X)$, $E = L(X)$. Let σ: $E \to E$ denote the L-automorphism of E defined by $\sigma(x_i) = x_{i+1}$, and let $T = E[[s; \sigma]]$ be the ring of twisted power series over E. Let $I = sT$ be the unique nonzero prime ideal of T, and define a subring $R \subseteq T$ by $R = F + I$. Then it is easy to verify (using the fact that T and R have an ideal in common) that (i) $Q(R) = Q(T)$ and $Q(R)$ has center L; (ii) R has center K; and (iii) R is an FB G-domain.

In closing we remark that the ring R constructed above is Noetherian iff the extension L/K is finite dimensional. Thus, Theorem 8 does apply to the FBN case of this construction. Indeed, we know of no example of an FBN G-ring R for which the center of the quotient ring of R is not algebraic over the quotient field of the center of R. (Examples for FB rings which are not G-rings are in [**4**].)

4. Polynomials over hereditary rings. In this section we take a slightly different approach to that of the last section and consider the case when R is a prime hereditary FBN ring, with quotient ring Q. In this case we completely solve the problem of the primitivity of $R[x]$ by proving that $R[x]$ is primitive if and only if R is a G-ring and $Q[x]$ is primitive. Some partial results in this direction were obtained in [**8**]. This is an additional section, added in proof. We only became aware of [**8**] after the first version of this paper (which consisted of the first three sections of the present paper) was accepted for publication. The results of this section were proved by combining the ideas of [**8**] with those of the earlier sections.

The following facts will be used frequently in this section, without specific reference:

(9.1) If R is a Dedekind prime ring and P a maximal ideal of R, then P is localizable [**6**].

(9.2) If R is a semilocal Dedekind prime ring then R is a principal ideal ring. (If I is an essential right ideal and J is the Jacobson radical of R, then the Dedekind property implies that $I/IJ \cong R/J$, from which the result follows.)

(9.3) If R is a principal ideal ring and I an ideal of R, say $I = aR$, then a is normalizing element and $I = Ra$. (Otherwise an easy computation shows that $a = gaf$ where g and f are elements of R and g is not a unit. We thus have $R \supseteq gR \supseteq gaR \supseteq gafR = aR$, and we note that $gR/gaR \cong R/aR$, while $R/gR \neq 0$. This is impossible since all of these modules have finite length.)

(9.4) Let R be a semilocal prime principal ideal ring. Let $I = aR$, and $J = bR$ be two semiprime ideals such that $I \cap J = J(R)$ and $I + J = R$. Then, as a is a normal element, $\mathscr{C}_a = \{a^n: n \geq 0\}$ is an Ore set in R. Further, $R_J = R\mathscr{C}_a^{-1}$.

The first part of the next lemma comes from the proof of [**8**, Theorem 3.5].

LEMMA 10. *Let S be a semilocal Dedekind prime ring, with quotient Artinian ring Q, and let F be the center of Q. Then*

(i) *For every maximal ideal P of S, either $F \subseteq S_P$ or $F \cap J(S)_P \neq 0$.*

(ii) *If for every maximal ideal P of S, $F \subseteq S_P$, then $F \subseteq S$.*

(iii) *If for every maximal ideal P of S, $F \cap J(S_P) \neq 0$, then $F \cap J(S) \neq 0$.*

Proof. (i) Suppose that $F \not\subseteq S_P$. If $f \in F - S_P$, then fS_P is a fractional S_P-ideal (i.e. it is closed under right and left multiplication by elements of S_P.) Since S_P is a local principal ideal ring, this implies that $fS_P = J(S_P)$ for some integer m. As $f \notin S_P$, we conclude that $m < 0$, and so $f^{-1}S_P \subseteq J(S_P)$, as required.

(ii) Part (ii) is trivial, since $S = \bigcap S_P$.

(iii) Let $P_1, \ldots, P_n$ be the maximal ideals of S. The proof is by induction on n. Thus set $T = P_1 \cap \cdots \cap P_{n-1}$ and $P = P_n$. By induction, there exists a nonzero element f, with $f \in F \cap J(S_T)$. Now as S is a principal ideal ring, $P = pS$ for some normalizing element p, and $S_T = S\mathscr{C}_p^{-1}$ by (9.4). Thus $f = p^{-n}t$ for some $n \geq 0$ and some $t \in S$. Of course, here $t \in J(S_T) \cap S = T$. Choose such a representation for f with n as small as possible.

Suppose first that $n = 0$. By hypothesis, there is an element $g \in F \cap J(S_P)$. Let $T = aS$ for some $a \in S$. Then, again, $S_P = S\mathscr{C}_a^{-1}$. Thus for some integer r, $a^r g \in S$, and hence $f^r g \in S$. Now clearly $f^{r+1}g \in F \cap J(S_T) \cap J(S_P) = F \cap J(S)$, as required. Thus we may assume that $n > 1$. In this case, if $t \in P$, say $t = pt_1$ for some $t_1 \in S$, then $f = p^{-n}t = p^{-n+1}t_1$; a contradiction. Thus $t \notin P$. We now show that, in fact, $t \in \mathscr{C}(P)$. If not, then $tr \in P$ for some $r \in S$. Then, as f and p are normal, $tSr = p^n fSr = p^n Sfr = Sp^n fr = Str \subseteq P$. Thus $r \in P$, and this shows that $t \in \mathscr{C}(P)$.

We now calculate in the Artinian ring Q:

$$(1+f)^{-1} = (1+p^{-n}t)^{-1} = (p^n+t)^{-1}p^n.$$

Now, $t \in \mathscr{C}(P) \cap T$ and $p^n \in P \cap \mathscr{C}(T)$, and so $t + p^n \in \mathscr{C}(P) \cap \mathscr{C}(T)$, whence $t + p^n$ is a unit of S. Thus, $(p^n + t)^{-1} \in S$. But this implies that $(1 + f)^{-1} \in P$. Further,

$$(1+f)^{-1}f = (p^n+t)^{-1}t \in T.$$

Finally, this says that $(1 + f)^{-2}f \in PT \subseteq J(S)$; as required.

PROPOSITION 11. *Let S be a semilocal Dedekind prime ring, with quotient ring Q, and let F be the center of Q. Suppose that Q is transcendental over F. Then there exists an element $d \in J(S)$ such that d is transcendental over F.*

Proof. Let $\mathfrak{U}$ be the set of maximal ideals P such that $F \subseteq S_P$, and let $U = \bigcap\{P\colon P \in \mathfrak{U}\}$. Similarly, let $\mathfrak{V}$ be the set of maximal ideals P such that $F \cap J(SP) \neq 0$, and let $V = \bigcap\{P\colon P \in \mathfrak{V}\}$. Note that $\mathfrak{U} \cap \mathfrak{V} = \varnothing$, as F is a field, and Lemma 10 implies that $\mathfrak{U} \cup \mathfrak{V} = \operatorname{Maxspec}(S)$. We assume for the moment that both $\mathfrak{U}$ and $\mathfrak{V}$ are nonempty. According to Lemma 10 we may pick nonzero elements u and v with $u \in F \cap S_U$ and $v \in F \cap J(S_V)$. Write $U = aS$. Observe that F is the center of S_U and thus, as F is a field and a is not a unit in S_U, a must be transcendental over F. Once again, $S_V = S\mathscr{C}_a^{-1}$. Thus, $va^m \in S$ for some m. Finally, if $d = va^{m+1}$, then

$$d \in J(S_V) \cap S \cap U = V \cap U = J(S).$$

But as $v \in F$ and a is transcendental over F, d is transcendental over F, as required. (If either $\mathfrak{U}$ or $\mathfrak{V}$ is empty, then a condensed version of the above proof will suffice to obtain the desired conclusion.)

COROLLARY 12. *Let R be a prime Goldie ring that is equivalent to a semilocal Dedekind prime ring $S \supseteq R$. Suppose further that the quotient Artinian ring Q of R is transcendental over its center F. Then there exists $d \in J(R)$ such that d is transcendental over F.*

Proof. We have $aSb \subseteq R$ for some regular elements a, b of R. Since S is bounded and semilocal, for some integers n and m, $aS \supseteq J(S)^n$ and $Sb \supseteq J(S)^m$. Thus $R \supseteq J(S)^{m+n}$. It follows easily that $J(S)^{m+n} \subseteq J(R)$. By Proposition 11, there is an element $d \in J(S)$ which is transcendental over F. Thus $d^{m+n} \in J(S)^{m+n} \subseteq J(R)$ and certainly d^{m+n} is transcendental over F.

THEOREM 13. *Let R be a prime FB ring that is equivalent to a Dedekind prime ring S, with $S \supseteq R$, and let Q be the Goldie quotient ring to R. Then $R[x]$ is primitive if and only if R is a G-ring and $Q[x]$ is primitive.*

Proof. The necessity of the conditions follows from Theorem 4 and Lemma 2. Conversely, suppose that the conditions hold. Since $Q[x]$ is primitive, there is an integer n such that the matrix ring $M_n(Q)$ contains an element that is transcendental over its center F. Replacing R by $M_n(R)$, we may suppose that the same is true for Q.

We aim to apply Corollary 12. Note that, as in the proof of Corollary 12, R contains a non-zero ideal V of S. Also, as R is a fully bounded G-ring, $J(R) \neq 0$. Thus $VJ(R)V$ is an ideal of S that, since it is radical as an ideal of R, is radical as an ideal of S. Hence $J(S) \neq 0$. Thus we may apply Corollary 11 to conclude that $J(R)$ contains an element transcendental over F. Finally, Proposition 7 implies that $R[x]$ is primitive.

In the special case of hereditary Noetherian prime (HNP) rings, Theorem 13 can be strengthened as follows.

COROLLARY 14. *Let R be a bounded HNP ring, with quotient Artinian ring Q. Then $R[x]$ is primitive if and only if R is a semilocal ring such that $Q[x]$ is primitive.*

Proof. Since an HNP ring satisfies the restricted minimum condition, an HNP ring is bounded if and only if it is FB, and is a G-ring if and only if it is semilocal. Thus, again, the conditions are necessary by Lemma 2 and Theorem 4. Conversely, if R is a semilocal HNP ring, then R is equivalent to a Dedekind prime ring by [**13**, Theorem 6.3]. Thus the result follows from Theorem 12.

We remark that the hypothesis in Theorem 13 that R should be an FB ring equivalent to a Dedekind prime ring S is considerably weaker than demanding that R be a (semilocal) HNP ring. For example, let

$$V = k(x_1, \dots, x_n)[y]_{(y)}$$

and

$$R = \begin{pmatrix} k[x_1, \dots, x_n] + yV & yV \\ V & V \end{pmatrix} \subseteq S = M_2(V).$$

Then R is equivalent to the Dedekind prime ring S, yet is an FB ring of (classical) Krull dimension $n + 1$. Note that R is, in fact, a G-ring, since the intersection of the non-zero prime ideals is just

$$\begin{pmatrix} yV & yV \\ V & yV \end{pmatrix}.$$

As we remarked at the end of §1, we know very little about the structure of FBN G-rings. Of course, such rings need not be equivalent to Dedekind prime rings, since this fails even in the commutative case (see [**5**, page 38]). However, it follows from [**3**, Prop. 2.5] that a bounded Noetherian maximal order of Krull dimension one is a Dedekind prime ring. Thus if every FBN G-ring does have Krull dimension one, it might be possible to prove Corollary 13 for an arbitrary FBN ring R by pulling the result down from some sort of integral closure of R.

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Received May 24, 1985.

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