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WEAK CONVERGENCE AND NON-LINEAR ERGODIC THEOREMS FOR REVERSIBLE SEMIGROUPS OF NONEXPANSIVE MAPPINGS

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Let S be a semitopological semigroup. Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm and $\mathcal{S} = \{T_a;\ a \in S\}$ be a continuous representation of S as nonexpansive mappings of C into C such that the common fixed point set $F(\mathcal{S})$ of $\mathcal{S}$ in C is nonempty. We prove in this paper that if S is right reversible (i.e. S has finite intersection property for closed right ideals), then for each $x \in C$, the closed convex set $W(x) \cap F(\mathcal{S})$ consists of at most one point, where $W(x) = \bigcap\{K_s(x);\ s \in S\}$, $K_s(x)$ is the closed convex hull of $\{T_t x;\ t \geq s\}$ and $t \geq s$ means $t = s$ or $t \in \overline{Ss}$. This result is applied to study the problem of weak convergence of the net $\{T_s x;\ s \in S\}$, with S directed as above, to a common fixed point of $\mathcal{S}$. We also prove that if E is uniformly convex with a uniformly Fréchet differentiable norm, S is reversible and the space of bounded right uniformly continuous functions on S has a right invariant mean, then the intersection $W(x) \cap F(\mathcal{S})$ is nonempty for each $x \in C$ if and only if there exists a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_s = T_s P = P$ for all $s \in S$ and $P(x)$ is in the closed convex hull of $\{T_s(x);\ s \in S\}$, $x \in C$.

1. Introduction. Let S be a semitopological semigroup i.e. S is a semigroup with a Hausdorff topology such that for each $s \in S$ the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from S to S are continuous. S is called *right reversible* if any two closed left ideals of S has non-void intersection. In this case, $(S, \leq)$ is a directed system when the binary relation "$\leq$" on S is defined by $a \leq b$ if and only if $\{a\} \cup \overline{Sa} \supseteq \{b\} \cup \overline{Sb}$, $a, b \in S$. Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups (see [**13**, p. 335]). Left reversibility of S is defined similarly. S is called *reversible* if it is both left and right reversible.

Let E be a uniformly convex Banach space and $\mathcal{S} = \{T_s;\ s \in S\}$ be a continuous representation of S as nonexpansive mappings on a closed convex subset C of E into C i.e. $T_{ab}(x) = T_a T_b(x)$, $a, b \in S$, $x \in C$ and the mapping $(s, x) \to T_s(x)$ from $S \times C$ into C is continuous when $S \times C$ has the product topology. Let $F(\mathcal{S})$ denote the set $\{x \in C;\ T_s(x) = x$ for all $s \in S\}$ of common fixed points of $\mathcal{S}$ in C. Then, as is well known, $F(\mathcal{S})$ (possibly empty) is a closed convex subset of C (see [**2**, Theorem 8]).

Recently Lau [**15**] considers the problem of weak convergence of the net $\{T_s(x);\ s \in S\}$, $x \in C$, to a common fixed point of $\mathcal{S}$ when S is right reversible and C is a closed convex subset of a Hilbert space. When T is a nonexpansive mapping of C into C and $\mathcal{S} = \{T^n;\ n = 1, 2, \dots\}$, this problem is equivalent to that of weak convergence of the sequence $\{T^n(x);\ n = 1, 2, \dots\}$ to a fixed point of T considered by Z. Opial in [**18**] and A. Pazy in [**19**]. However, the proofs employed by Lau [**15**] (Lemma 2.1, Lemma 2.2 and Theorem 2.3) do not extend beyond uniformly convex Banach spaces satisfying Opial's condition (see [**18**, Lemma 1] and [**15**, Lemma 2.1]).

In §3 of this paper, we prove that (Theorem 1) if E is uniformly convex with a Fréchet differentiable norm and S is right reversible, then for each $x \in C$, the closed convex set $W(x) = \bigcap\{K_s(x);\ s \in S\}$, where $K_s(x)$ is the closed convex hull of $\{T_t(x);\ t \geq s\}$, contains at most one common fixed point of $\mathcal{S}$. This result is used to prove that (Theorem 3) if $\|T_{gs}(x) - T_s(x)\| \to 0$ for each fixed g in a generating set of S, then the net $\{T_s(x);\ s \in S\}$ converges weakly to an element in $F(\mathcal{S})$. We also prove that (Theorem 7) if E is uniformly convex with a uniformly Fréchet differentiable norm, S is reversible and the space of bounded right uniformly continuous functions on S has a right invariant mean, then the intersection $W(x) \cap F(\mathcal{S})$ is nonempty for each $x \in C$ if and only if there exists a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $T_s P = P T_s = P$ and $P(x)$ is in the closed convex hull of $\{T_s x;\ s \in S\}$ for all $x \in C$. This improves an ergodic Theorem of Hirano-Takahashi [**12**, Theorem 2] for discrete amenable semigroups. Our proofs employ the methods of Hirano-Takahashi [**12**], Bruck [**3**], [**4**], Lau [**15**], Pazy [**19**], Reich [**21**] and Takahashi [**24**].

If $1 < p < 2$ and $2 < p < +\infty$, then none of the Banach space $L_p[0, 2\pi]$ satisfy Opial's condition (see [**18**, p. 596]). However, they are uniformly convex with Fréchet differentiable norm.

The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [1]: Let C be a closed convex subset of a Hilbert space and T a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{\text{co}}\{T^n x\colon n = 1, 2, \dots\}$ for each $x \in C$, where $\overline{\text{co}}A$ is the closure of the convex hull of A. In [**24**], Takahashi proved the existence of such a retraction for an amenable semigroup of nonexpansive mappings in a Hilbert space. Recently, Hirano-Takahashi [**12**] extended this result to a Banach space.

2. Preliminaries. Throughout this paper, we assume that a Banach space is real. We also denote by **R** the set of all real numbers.

Let E be a Banach space and E^* its dual. Then, the value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. With each $x \in E$, we associate the set

$$J(x) = \left\{ f \in E^*\colon \langle x, f\rangle = \|x\|^2 = \|f\|^2 \right\}.$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \varnothing$ for each $x \in E$. The multivalued operator $J\colon E \to E^*$ is called the duality mapping of E. Let $B = \{x \in E\colon \|x\| = 1\}$ be the unit sphere of E. Then the norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{r \to 0} \frac{\|x + ry\| - \|x\|}{r}$$

exists for each x and y in B. It is said to be *Fréchet differentiable* if for each x in B, this limit is attained uniformly for y in B. Finally, it is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit is attained uniformly for (x, y) in $B \times B$. It is well known that if E is smooth, then the duality mapping J is single value. It is also known that if E has a Fréchet differentiable norm, then J is norm to norm continuous. (See [**2**] or [**7**] for more details.) Let K be a subset of E. Then we denote by $d(K)$ the diameter of K. A point $x \in K$ is a diametral point of K provided

$$\sup\{\|x - y\|\colon y \in K\} = d(K).$$

A closed convex subset C of a Banach space E is said to have *normal structure*, if for each closed bounded convex subset K of C, which contains at least two points, there exists an element of K which is not a diametral point of K. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure.

If A is a subset of a Banach space E, then $\overline{\text{co}}A$ will denote its closed convex hull in E. When $\{x_\alpha\}$ is a net in E, then $x_\alpha \to x$ (resp. $x_\alpha \rightharpoonup x$) will denote *norm* (resp. *weak*) *convergence* of the net $\{x_\alpha\}$ to x.

3. Weak convergence of $\{T_s x\colon s \in S\}$. Unless other specified, S denotes a semitopological semigroup and $\mathscr{S} = \{T_a\colon a \in S\}$ a continuous representation of S as nonexpansive mappings from a nonempty closed convex subset C of a Banach space E into C. If S is right reversible and S is directed as in §1, then for each $x \in C$, let $\omega(x)$ denote the set of all weak limit points of subnets of the net $\{T_a x\colon a \in S\}$.

LEMMA 1. *Let C be a closed convex subset of a uniformly convex Banach space E and assume that $F(\mathscr{S}) \neq \varnothing$. Let $x \in C$, $f \in F(\mathscr{S})$, $0 < \alpha \leq \beta < 1$ and $r = \inf_{a \in S} \|T_a x - f\|$. Then, for any $\varepsilon > 0$, there is a positive number d such that*

$$\|T_a(\lambda T_b x + (1-\lambda)f) - (\lambda T_a T_b x + (1-\lambda)f)\| < \varepsilon$$

for all $b \in S$ with $\|T_b x - f\| \leq r + d$, $a \in S$ and $\lambda \in R$ with $\alpha \leq \lambda \leq \beta$.

Proof. Let $r > 0$. Then we can choose $d > 0$ so small that

$$(r+d)\left(1 - c\delta\left(\frac{\varepsilon}{r+d}\right)\right) < r,$$

where δ is the modulus of convexity of the norm and

$$c = \min\{2\lambda(1-\lambda)\colon \alpha \leq \lambda \leq \beta\}.$$

Suppose that $\|T_a(\lambda T_b x + (1-\lambda)f) - (\lambda T_a T_b x + (1-\lambda)f)\| \geq \varepsilon$ for some b with $\|T_b x - f\| \leq r + d$, $a \in S$ and $\lambda \in \mathbf{R}$ with $\alpha \leq \lambda \leq \beta$. Put $u = (1-\lambda)(T_a z - f)$ and $v = \lambda(T_a T_b x - T_a z)$, where $z = \lambda T_b x + (1-\lambda)f$. Then $\|u\| \leq (1-\lambda)\|z - f\| = \lambda(1-\lambda)\|T_b x - f\|$ and $\|v\| \leq \lambda\|T_b x - z\| = \lambda(1-\lambda)\|T_b x - f\|$. We also have that $\|u - v\| = \|T_a z - (\lambda T_a T_b x + (1-\lambda)f)\| \geq \varepsilon$ and $\lambda u + (1-\lambda)v = \lambda(1-\lambda)\cdot(T_a T_b x - f)$. So by using the Lemma in **[9]**, we have

$$\begin{aligned}
\lambda(1-\lambda)\|T_a T_b x - f\| &= \|\lambda u + (1-\lambda)v\| \\
&\leq \lambda(1-\lambda)\|T_b x - f\|\left(1 - 2\lambda(1-\lambda)\delta\left(\frac{\varepsilon}{\|T_b x - f\|}\right)\right) \\
&\leq \lambda(1-\lambda)(r+d)\left(1 - c\delta\left(\frac{\varepsilon}{r+d}\right)\right) < \lambda(1-\lambda)r
\end{aligned}$$

and hence $\|T_aT_bx - f\| < r$. This contradicts $r = \inf_{a \in S} \|T_ax - f\|$. In the case when $r = 0$, for any $a, b \in S$, $f \in F(\mathscr{S})$ and $\lambda \in \mathbf{R}$ with $0 \leq \lambda \leq 1$,

$$\begin{aligned}
&\|T_a(\lambda T_bx + (1-\lambda)f) - (\lambda T_aT_bx + (1-\lambda)f)\| \\
&\quad \leq \lambda\|T_a(\lambda T_bx + (1-\lambda)f) - T_aT_bx\| \\
&\qquad + (1-\lambda)\|T_a(\lambda T_bx + (1-\lambda)f) - f\| \\
&\quad \leq \lambda\|\lambda T_bx + (1-\lambda)f - T_bx\| + (1-\lambda)\|\lambda T_bx + (1-\lambda)f - f\| \\
&\quad = 2\lambda(1-\lambda)\|T_bx - f\|.
\end{aligned}$$

So, we obtain the desired result.

LEMMA 2. *Let C be a closed convex subset of a uniformly convex Banach space E, S right reversible and $F(\mathscr{S}) \neq \varnothing$. Let $x \in C$, $f \in F(\mathscr{S})$ and $0 < \alpha \leq \beta < 1$. Then for any $\varepsilon > 0$, there is $b_0 \in S$ such that*

$$\|T_a(\lambda T_bx + (1-\lambda)f) - (\lambda T_aT_bx + (1-\lambda)f)\| < \varepsilon$$

for all $b \in S$ with $b \geq b_0$, $a \in S$ and $\lambda \in \mathbf{R}$ with $\alpha \leq \lambda \leq \beta$.

Proof. Let $r = \inf_{s \in S} \|T_sx - f\|$. Then, we have

$$r = \inf_a \sup_{a \leq b} \|T_bx - f\|.$$

In fact, for any $\varepsilon > 0$, there is $a_0 \in S$ such that $\|T_{a_0}x - f\| < r + \varepsilon$. Let $b \geq a_0$. Then, since $b \in \{a_0\} \cup \overline{Sa_0}$, we may assume $b \in \overline{Sa_0}$. Let $\{s_\alpha\}$ be a net in S such that $s_\alpha a_0 \to b$. Then, for each α,

$$\|T_{s_\alpha a_0}x - f\| = \|T_{s_\alpha}(T_{a_0}x) - T_{s_\alpha}f\| \leq \|T_{a_0}x - f\|.$$

Hence, $\|T_bx - y\| \leq \|T_{a_0}x - y\|$. So, we have $\sup_{a_0 \leq b} \|T_bx - y\| < r + \varepsilon$ and hence

$$\inf_a \sup_{a \leq b} \|T_bx - y\| < r + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\inf_a \sup_{a \leq b} \|T_bx - f\| \leq r = \inf_a \|T_sx - f\|.$$

The reverse inequality is obvious. Since $r = \inf_a \sup_{a \leq b} \|T_bx - f\|$, for any positive number d, there is $a_0 \in S$ such that

$$\sup_{a_0 \leq b} \|T_bx - f\| < r + d.$$

So, by using Lemma 1, we obtain Lemma 2.

Let x and y be elements of a Banach space E. Then we denote by $[x, y]$ the set $\{\lambda x + (1-\lambda)y: 0 \le \lambda \le 1\}$.

LEMMA 3. *Let C be a closed convex subset of a Banach space E with a Fréchet differentiable norm and $\{x_\alpha\}$ a bounded net in C. Let $z \in \bigcap_\beta \overline{\mathrm{co}}\{x_\alpha: \alpha \ge \beta\}$, $y \in C$ and $\{y_\alpha\}$ a net of elements in C with $y_\alpha \in [y, x_\alpha]$ and*

$$\|y_\alpha - z\| = \min\{\|u - z\|: u \in [y, x_\alpha]\}.$$

If $y_\alpha \to y$, then $y = z$.

Proof. Since J is single-valued, it follows from Theorem 2.5 in [**8**] that $\langle u - y_\alpha, J(y_\alpha - z)\rangle \ge 0$ for all $u \in [y, x_\alpha]$. Putting $u = x_\alpha$, we have

$$\langle x_\alpha - y_\alpha, J(y_\alpha - z)\rangle \ge 0. \tag{1}$$

Since $y_\alpha \to y$ and $\{x_\alpha\}$ is bounded, there exist $K > 0$ and α_0 such that $\|x_\alpha - y\| \le K$ and $\|y_\alpha - z\| \le K$ for all $\alpha \ge \alpha_0$. Let $\varepsilon > 0$ and choose $\delta > 0$ so small that $2\delta K < \varepsilon$. Since the norm of E is Fréchet differentiable, we can choose $\alpha_1 \ge \alpha_0$ such that $\|y_\alpha - y\| \le \delta$ and $\|J(y_\alpha - z) - J(y - z)\| \le \delta$ for all $\alpha \ge \alpha_1$. Since for $\alpha \ge \alpha_1$

$$\begin{aligned}
&|\langle x_\alpha - y_\alpha, J(y_\alpha - z)\rangle - \langle x_\alpha - y, J(y - z)\rangle| \\
&\qquad = |\langle x_\alpha - y_\alpha, J(y_\alpha - z)\rangle - \langle x_\alpha - y, J(y_\alpha - z)\rangle \\
&\qquad\quad + \langle x_\alpha - y, J(y_\alpha - z)\rangle - \langle x_\alpha - y, J(y - z)\rangle| \\
&\qquad \le \|y_\alpha - z\|\,\|y_\alpha - y\| + \|x_\alpha - y\|\,\|J(y_\alpha - z) - J(y - z)\| \\
&\qquad \le 2\delta K < \varepsilon,
\end{aligned}$$

by using (1), we have

$$\langle x_\alpha - y, J(y - z)\rangle \ge \langle x_\alpha - y_\alpha, J(y_\alpha - z)\rangle - \varepsilon \ge 0 - \varepsilon = -\varepsilon.$$

Since $z \in \bigcap_\beta \overline{\mathrm{co}}\{x_\alpha: \alpha \ge \beta\}$, we have $\langle z - y, J(y - z)\rangle \ge -\varepsilon$. This implies $-\|z - y\|^2 \ge 0$ and hence $z = y$.

By using Lemmas 2 and 3, we can prove the following:

LEMMA 4. *Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm, S right reversible, and $F(\mathscr{S}) \ne \varnothing$. Let $x \in C$. Then for any $z \in \bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x: t \ge s\} \cap F(\mathscr{S})$ and $y \in F(\mathscr{S})$, there is $t_0 \in S$ such that*

$$\langle T_t x - y, J(y - z)\rangle \le 0$$

for every $t \ge t_0$.

Proof. Let $z \in \bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x\colon t \geq s\} \cap F(\mathscr{S})$ and $y \in F(\mathscr{S})$. If $y = z$, Lemma 4 is obvious. So, let $y \neq z$. For any $t \in S$, define a unique element y_t such that $y_t \in [y, T_t x]$ and $\|y_t - z\| = \min\{\|u - z\|\colon u \in [y, T_t x]\}$. Then since $y \neq z$, by Lemma 3 we have $y_t \nrightarrow y$. So, we obtain $c > 0$ such that for any $t \in S$, there is $t' \in S$ with $t' \geq t$ and $\|y_{t'} - y\| \geq c$. Setting

$$y_{t'} = a_{t'} T_{t'} x + (1 - a_{t'}) y, \qquad 0 \leq a_{t'} \leq 1,$$

we also obtain $c_0 > 0$ so small that $a_{t'} \geq c_0$. (In fact, since $T_{t'}$ are nonexpansive and $y \in F(\mathscr{S})$, we have

$$c \leq \|y_{t'} - y\| = a_{t'} \|T_{t'} x - y\| \leq a_{t'} \|x - y\|.$$

So, put $c_0 = c/\|x - y\|$.) Since the limit of $\|T_t x - y\|$ exists as in the proof of Lemma 2, putting $k = \lim \|T_t x - y\|$, we have $k > 0$. If not, we have $T_t x \to y$ and hence $y_t \to y$, which contradicts $y_t \nrightarrow y$.

Now, choose $\varepsilon > 0$ so small that

$$(R + \varepsilon)\left(1 - \delta\left(\frac{c_0 k}{R + \varepsilon}\right)\right) < R,$$

where δ is the modulus of convexity of the norm and $R = \|z - y\|$. Then by Lemma 2, there exists $t_0 \in S$ such that

$$\text{(2)} \qquad \|T_s(c_0 T_t x + (1 - c_0) y) - (c_0 T_s T_t x + (1 - c_0) y)\| < \varepsilon$$

for all $s \in S$ and $t \geq t_0$. Fix $t' \in S$ with $t' \geq t_0$ and $\|y_{t'} - y\| \geq c$. Then since $a_{t'} \geq c_0$, we have

$$c_0 T_{t'} x + (1 - c_0) y \in [y, a_{t'} T_{t'} x + (1 - a_{t'}) y] = [y, y_{t'}].$$

Hence

$$\|c_0 T_{t'} x + (1 - c_0) y - z\| \leq \max\{\|z - y\|, \|z - y_{t'}\|\} = \|z - y\| = R.$$

By using (2), we obtain

$$\begin{aligned} \|c_0 T_s T_{t'} x + (1 - c_0) y - z\| &\leq \|T_s(c_0 T_{t'} x + (1 - c_0) y) - z\| + \varepsilon \\ &\leq \|c_0 T_{t'} x + (1 - c_0) y - z\| + \varepsilon \leq R + \varepsilon \end{aligned}$$

for all $s \in S$. On the other hand, since $\|y - z\| = R < R + \varepsilon$ and

$$\|c_0 T_s T_{t'} x + (1 - c_0) y - y\| = c_0 \|T_s T_{t'} x - y\| \geq c_0 k$$

for all $s \in S$, we have, by uniform convexity,

$$\begin{aligned} &\left\|\frac{1}{2}((c_0 T_s T_{t'} x + (1 - c_0) y - z) + (y - z))\right\| \\ &\qquad \leq (R + \varepsilon)\left(1 - \delta\left(\frac{c_0 k}{R + \varepsilon}\right)\right) < R \end{aligned}$$

and hence

$$\left\| \frac{c_0}{2} T_s T_{t'} x + \left(1 - \frac{c_0}{2}\right) y - z \right\| < R$$

for all $s \in S$. This implies that if

$$u_s = \frac{c_0}{2} T_s T_{t'} x + \left(1 - \frac{c_0}{2}\right) y,$$

then

$$\| u_s + \alpha(y - u_s) - z \| \geq \| y - z \|$$

for all $\alpha \geq 1$. By Theorem 2.5 in [**8**], we have

$$\langle u_s + \alpha(y - u_s) - y, J(y - z) \rangle \geq 0$$

and hence $\langle u_s - y, J(y - z) \rangle \leq 0$. Then $\langle c_0 T_s T_{t'} x - c_0 y, J(y - z) \rangle \leq 0$. Therefore

$$\langle T_s T_{t'} x - y, J(y - z) \rangle \leq 0 \quad \text{for all } s \in S.$$

Let $t \geq t'$. Then, since there exists a net $\{s_\alpha\}$ in S with $s_\alpha t' \to t$, we obtain

$$\langle T_t x - y, J(y - z) \rangle \leq 0 \quad \text{for all } t \geq t'.$$

We are now ready to prove one of our main theorems.

THEOREM 1. *Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm, S right reversible, and $F(\mathscr{S}) \neq \varnothing$. Let $x \in C$. Then, the set*

$$\bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x : t \geq s\} \cap F(\mathscr{S})$$

consists of at most one point.

Proof. Let $y, z \in F(\mathscr{S}) \cap \bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x:\ t \geq s\}$. Then, since $(y + z)/2 \in F(\mathscr{S})$, it follows from Lemma 4 that there is $t_0 \in S$ such that

$$\left\langle T_t x - \frac{y + z}{2}, J\left(\frac{y + z}{2} - z\right) \right\rangle \leq 0$$

for every $t \geq t_0$. Since $y \in \overline{\mathrm{co}}\{T_t x : t \geq t_0\}$, we have

$$\left\langle y - \frac{y + z}{2}, J\left(\frac{y + z}{2} - z\right) \right\rangle \leq 0$$

and hence $\langle y - z, J(y - z) \rangle \leq 0$. This implies $y = z$.

By using Theorem 1, we now study the problem of the weak convergence of $\{T_a x\colon a \in S\}$.

THEOREM 2. *Let C be a closed convex subset of a uniformly convex Banach space with Fréchet differentiable norm, S right reversible and $F(\mathscr{S}) \neq \varnothing$. Let $x \in C$. If $\omega(x) \subseteq F(\mathscr{S})$, then the net $\{T_a x\colon a \in S\}$ converges weakly to some $y \in F(\mathscr{S})$.*

Proof. Since $F(\mathscr{S}) \neq \varnothing$, $\{T_a x\colon a \in S\}$ is bounded. So, $\{T_a x\colon a \in S\}$ must contain a subnet $\{T_{a_\alpha} x\}$ which converges weakly to some $z \in C$. Since $\omega(x) \subseteq F(\mathscr{S})$ and $z \in \bigcap_{s \in S} \overline{\text{co}}\{T_t x\colon t \geq s\}$, we obtain

$$z \in F(\mathscr{S}) \cap \bigcap_{s \in S} \overline{\text{co}}\{T_t x\colon t \geq s\}.$$

Therefore, it follows from Theorem 1 that $\{T_a x\colon a \in S\}$ converges weakly to $z \in F(\mathscr{S})$.

A subset G of S is called a *generating set* if elements of the form $g_1 g_2 \cdots g_m$, $g_1, g_2, \ldots, g_n \in G$, is dense in S.

COROLLARY. *Let C be a closed convex subset of a Hilbert space, S right reversible, and $F(\mathscr{S}) \neq \varnothing$. Let $x \in C$. Then $T_a x \rightharpoonup y \in C$ if and only if $T_{ga} x - T_a x \rightharpoonup 0$ for all g in a generating set G of S.*

Proof. We need only prove the "if" part. Let $\{T_{a_\alpha} x\}$ be a subnet of $\{T_a x;\ a \in S\}$ with $T_{a_\alpha} x \rightharpoonup z$. If $u \in F(\mathscr{S})$, then we have

$$\begin{aligned}
0 &\leq \|T_{a_\alpha} x - z\|^2 - \|T_{ga_\alpha} x - T_g z\|^2 \\
&= \|T_{a_\alpha} x - u\|^2 + 2\langle T_{a_\alpha} x - u, u - z\rangle + \|u - z\|^2 - \|T_{ga_\alpha} x - u\|^2 \\
&\quad -2\langle T_{ga_\alpha} x - u, u - T_g z\rangle - \|u - T_g z\|^2 \\
&= \|T_{a_\alpha} x - u\|^2 - \|T_{ga_\alpha} x - u\|^2 + 2\langle T_{a_\alpha} x - u, T_g z - z\rangle \\
&\quad +2\langle T_{a_\alpha} x - T_{ga_\alpha} x, u - T_g z\rangle + \|u - z\|^2 - \|u - T_g z\|^2,
\end{aligned}$$

and hence by letting α tend to infinity

$$0 \leq 2\langle z - u, T_g z - z\rangle + \|u - z\|^2 - \|u - T_g z\|^2 = -\|z - T_g z\|^2$$

(note that $\|T_a x - u\|^2$ is a decreasing net and hence

$$\lim_\alpha \|T_{a_\alpha} x - u\|^2 = \lim_\alpha \|T_{ga_\alpha} x - u\|^2 = \lim \|T_a x - u\|^2).$$

Consequently $z \in F(\mathcal{S})$ and $\omega(x) \subseteq F(\mathcal{S})$. By Theorem 2, the net $\{T_a x: a \in S\}$ converges weakly to some $y \in F(\mathcal{S})$.

The following theorem is a generalization of Lau's result ([**15**, Theorem 2.3]), which has been proved in the case when E is a Hilbert space. Note that Lau's proof does not apply beyond Banach spaces for which Opial's condition is valid (e.g. $L_p[0,1]$, $1 < p < 2$ and $2 < p < \infty$). See [**18**, p. 596].

THEOREM 3. *Let C be a closed convex subset of a uniformly convex Banach space with a Fréchet differentiable norm, S right reversible and $F(\mathcal{S}) \neq \varnothing$. Let $x \in C$. If $\lim_a \|T_{ga}x - T_a x\| = 0$ for all g in a generating set G of S, then the net $\{T_a x: a \in S\}$ converges weakly to some $y \in F(\mathcal{S})$.*

Proof. By Theorem 2, it suffices to show that $\omega(x) \subseteq F(\mathcal{S})$. Let $\{T_{a_\alpha} x\}$ be a subnet of $\{T_a x;\ a \in S\}$ converging weakly to some $y \in C$. Let $g \in G$ and $T = T_g$. Write $x_\alpha = T_{a_\alpha} x$. Then $\|Tx_\alpha - x_\alpha\| \to 0$. For each n, choose α_n such that $\|Tx_\alpha - x_\alpha\| \leq 1/n$ for all $\alpha \geq \alpha_n$. Since $y \in \bigcap_\alpha \overline{\mathrm{co}}\{x_\beta;\ \alpha \leq \beta\}$, there is $x_n \in \mathrm{co}\{x_\beta;\ \alpha_n \leq \beta\}$ such that $\|y - x_n\| \leq 1/n$. Let $x_n = \sum_{i=1}^m a_i x_{\beta_i}$, $\beta_i \geq \alpha_n$. Then we have

$$\begin{aligned}\|Tx_n - x_n\| &\leq \left\| Tx_n - \sum_{i=1}^m a_i T x_{\beta_i} \right\| + \left\| \sum_{i=1}^m a_i T x_{\beta_i} - x_n \right\| \\ &\leq r^{-1}\left(\frac{2}{n}\right) + \frac{1}{n}\end{aligned}$$

where $r: \mathbf{R}^+ \to \mathbf{R}$ is a continuous, strictly increasing, convex function with $r(0) = 0$ such that for any $\{u_1, \ldots, u_k\} \subseteq C$ and $\lambda_1, \ldots, \lambda_k \geq 0$ with $\sum_{i=1}^k \lambda_i = 1$,

$$r\left(\left\| T\left(\sum_{i=1}^k \lambda_i u_i\right) - \sum_{i=1}^k \lambda_i T u_i \right\|\right) \leq \max_{1 \leq i, j \leq k} \left(\|u_i - u_j\| - \|Tu_i - Tu_j\|\right)$$

(the existence of such an r follows from Theorem 2.1 of Bruck [**6**].) In fact

$$\begin{aligned}\left\| \sum_{i=1}^m a_i T x_{\beta_i} - x_n \right\| &= \left\| \sum_{i=1}^m a_i T x_{\beta_i} - \sum_{i=1}^m a_i x_{\beta_i} \right\| \\ &\leq \sum_{i=1}^m a_i \|T x_{\beta_i} - x_{\beta_i}\| \leq \frac{1}{n}\end{aligned}$$

and

$$\left\| Tx_n - \sum_{i=1}^{m} a_i Tx_{\beta_i} \right\| \le r^{-1}\left(\max_{1 \le i, j \le m} \left(\|x_{\beta_i} - x_{\beta_j}\| - \|Tx_{\beta_i} - Tx_{\beta_j}\| \right) \right)$$

$$\le r^{-1}\left(\max_{1 \le i, j \le m} \left(\|x_{\beta_i} - Tx_{\beta_i}\| + \|Tx_{\beta_j} - x_{\beta_j}\| \right) \right)$$

$$\le r^{-1}\left(\frac{2}{n} \right).$$

Since r^{-1} is continuous and $r^{-1}(0) = 0$, we have $r^{-1}(2/n) + 1/n \to 0$ as $n \to \infty$. Therefore, $\|Tx_n - x_n\| \to 0$ as $n \to \infty$. Since $\|x_n - y\| \to 0$, we have $y = Ty$. Since G is a generating set of S and $g \in G$ is arbitrary, $y \in F(\mathcal{S})$. This implies $\omega(x) \subseteq F(\mathcal{S})$.

The next result is also a generalization of Lau's result [**15**, Proposition 2.4].

THEOREM 4. *Let C be a closed convex subset of a uniformly convex Banach space E, S right reversible, and $F(\mathcal{S}) \neq \varnothing$. Let P be the metric projection on E onto $F(\mathcal{S})$. Then, for each $x \in C$, the net $\{PT_a x;\ a \in S\}$ converges in norm to some $z \in F(\mathcal{S})$.*

Proof. Let $x \in C$. Observe that

$$\|PT_a x - T_a x\| \le \|PT_b x - T_a x\|$$

for any $a, b \in S$. If $a \ge b$ and $a \ne b$, let $\{s_\alpha b\}$ be a net converging to a. Then for each α,

$$\|PT_b x - T_{s_\alpha b} x\| = \|T_{s_\alpha} PT_b x - T_{s_\alpha} T_b x\| \le \|PT_b x - T_b x\|.$$

So, if $a \ge b$, we have

$$\|PT_b x - T_a x\| \le \|PT_b x - T_b x\|. \tag{3}$$

Hence, if $a \ge b$, then $\|PT_a x - T_a x\| \le \|PT_b x - T_b x\|$. This implies that the limit $\|PT_a x - T_a x\|$ exists. Now, we show that $\{PT_a x\colon a \in S\}$ is a Cauchy net in C. Let $r = \lim_a \|PT_a x - T_a x\|$. If $r = 0$, then for $\varepsilon > 0$, there is $c \in S$ such that $\|PT_a x - T_a x\| < \varepsilon/4$ for $a \ge c$. So, if $a, b \ge c$, then by (3)

$$\|PT_a x - PT_b x\| \le \|PT_a x - PT_c x\| + \|PT_c x - PT_b x\|$$

$$\le \|PT_a x - T_a x\| + \|T_a x - PT_c x\|$$

$$+ \|PT_b x - T_b x\| + \|T_b x - PT_c x\|$$

$$\le \|PT_a x - T_a x\| + \|T_c x - PT_c x\|$$

$$+ \|PT_b x - T_b x\| + \|T_c x - PT_c x\| < \varepsilon.$$

This implies that $\{PT_a x: a \in S\}$ is a Cauchy net in the case when $r = 0$. Let $r > 0$. Then $\{PT_a x: a \in S\}$ is also a Cauchy net. If not, there exists $\varepsilon > 0$ such that for any $s \in S$, there are $a, b \in S$ with $\|PT_a x - PT_b x\| \geq \varepsilon$ and $a, b \geq s$. Choose $d > 0$ so small that

$$(r + d)\left(1 - \delta\left(\frac{\varepsilon}{r + d}\right)\right) < r$$

and $s_0 \in S$ so large that

$$r \leq \|PT_t x - T_t x\| < r + d$$

for all $t \geq s_0$. For this $s_0 \in S$, there are $a, b \in S$ with $\|P_a x - PT_b x\| \geq \varepsilon$ and $a, b \geq s_0$. Since $(S, \geq)$ is a directed system, there is $c \in S$ with $c \geq a$ and $c \geq b$. For this $c \in S$, we have by (3)

$$\|PT_a x - T_c x\| \leq \|PT_a x - T_a x\| < r + d$$

and

$$\|PT_b x - T_c x\| \leq \|PT_b x - T_b x\| < r + d.$$

Since E is uniformly convex, we have

$$r \leq \|PT_c x - T_c x\| \leq \left\|\frac{PT_a x + PT_b x}{2} - T_c x\right\|$$

$$\leq (r + d)\left(1 - \delta\left(\frac{\varepsilon}{r + d}\right)\right) < r,$$

which is a contradiction.

4. Nonexpansive retraction. Let $\mathscr{S} = \{T_a;\ a \in S\}$ be a continuous representation of a semitopological semigroup S as nonexpansive mappings from a nonempty closed convex subset C of a Banach space E into C. We study in this section the existence of a nonexpansive "ergodic" retraction of C onto the common fixed point set $F(\mathscr{S})$ of $\mathscr{S}$ in C. We begin with the following simple observation:

LEMMA 5. *Let C be a nonempty closed convex subset of a reflexive Banach space E. Let $\{W_\alpha: \alpha \in I\}$ be a decreasing net of subsets contained in a bounded set of E. Let A be the asymptotic center of $\{W_\alpha: \alpha \in I\}$ with respect to C, i.e., $A = \{x \in C: r(x) = r\}$, where $r(x) = \inf\{r_\alpha(x): \alpha \in I\}$, $r_\alpha(x) = \sup\{\|y - x\|: y \in W_\alpha\}$ and $r = \inf\{r(x): x \in C\}$. Then A is nonempty, bounded, convex and closed.*

Proof. That A is closed and convex follows from Lim [**16**]. To see that A is nonempty, we observe that

$$A_n = \left\{x \in C: r(x) \leq r + \frac{1}{n}\right\}$$

is a nonempty weakly compact convex subset of E. Indeed, it suffices to show that A_n is bounded. Let $x \in A_n$, then for some α_0, $r_{\alpha_0}(x) \leq r + 2/n$. Hence $\|y - x\| \leq r + 2/n$ for each $y \in W_{\alpha_0}$, i.e., $\|x\| \leq r + 2/n + \|y\|$ for each $y \in W_{\alpha_0}$. It is obvious that $A = \bigcap_{n=1}^{\infty} A_n$.

THEOREM 5. *Let C be a closed convex subset of a reflexive Banach space with normal structure and S left reversible. If there exists $x_0 \in C$ such that $\{T_a x_0 \colon a \in S\}$ is bounded, then*

(a) *C contains a common fixed point of $\mathcal{S}$.*

(b) *There is a nonexpansive retraction r of C onto $F(\mathcal{S})$ for which any $\mathcal{S}$-invariant closed convex subset of C is r-invariant.*

Proof. (a) For each $s \in S$, let $W_s = \overline{T_s \mathcal{S} x_0}$. Then $\{W_s \colon s \in S\}$ is a directed set with $s \leq t$ meaning $\overline{sS} \supseteq tS$ and each W_s, $s \in S$ is bounded. Let A be the asymptotic center of $\{W_s \colon s \in S\}$ with respect to C. Then by Lemma 5 A is bounded, closed, convex and nonempty. Also A is $\mathcal{S}$-invariant. Indeed, if $x \in A$, $s \in S$, given $\varepsilon > 0$, there exists $t \in S$ such that $T_t \mathcal{S} x_0 \subset W_t \subset B(x, r + \varepsilon)$, where $B(z, r) = \{x \in E;\ \|z - x\| \leq r\}$. So, $W_{st} \subset B(T_s x, r + \varepsilon)$. It follows that $r(T_s x) \leq r_{st}(T_s x) \leq r + \varepsilon$. So $T_s x \in A$. Since A has normal structure, it follows from Theorem 3 in [16] that A contains a common fixed point of $\mathcal{S}$.

(b) We follow an idea of Bruck in [5]. Let $G = \{s \colon s$ is a nonexpansive mapping of C into itself, $F(s) \supseteq F(\mathcal{S})$ and every $\mathcal{S}$-invariant closed convex subset of C is s-invariant$\}$. Then, G is a semigroup and compact in the topology of pointwise weak convergence on C. We shall show that $Gx \cap F(G) \neq \varnothing$ for $x \in C$. In fact, since Gx is an $\mathcal{S}$-invariant bounded closed convex subset of C and has normal structure, by Theorem 3 in [16] Gx contains a common fixed point of $\mathcal{S}$ and hence a common fixed point of G. By Theorem 3(a) in [5], there exists a retraction $r \in G$ of C onto $F(G) = F(\mathcal{S})$.

Let S be a semitopological semigroup. Let $C(S)$ be the Banach algebra of all continuous bounded real valued functions on S with the supremum norm. Then, for each $s \in S$ and $f \in C(S)$, we can define $r_s f$ in $C(S)$ by $r_s f(t) = f(ts)$ for all $t \in S$. Let RUC(S) be the space of bounded right uniformly continuous functions on S, i.e., RUC(S) is the set of all $f \in C(S)$ such that the mapping: $s \to r_s f$ is continuous. Then RUC(S) is a closed translation invariant subalgebra of $C(S)$ containing constants; see [17] for more details.

A linear functional m on $\mathrm{RUC}(S)$ is called a *right invariant mean* if $\|m\| = m(1) = 1$ and $m(r_s f) = m(f)$ for all $f \in \mathrm{RUC}(S)$, $s \in S$. In general, S need not be right reversible even when the space of bounded continuous functions on S has a right invariant mean unless S is normal. See [**13**, p. 335] for details.

LEMMA 6. *Let C be a closed convex subset of a reflexive Banach space E and S be a semitopological semigroup for which* $\mathrm{RUC}(S)$ *has a right invariant mean. Suppose that there is an element in C with bounded orbit. Then there exists a nonexpansive mapping Q of C into itself such that $Qx \in \overline{\mathrm{co}}\mathscr{S}x$ for each $x \in C$ and $QT_s = Q$ for all $s \in S$.*

Proof. Let $x \in C$ and observe that if $f \in E^*$, then $h(t) = \langle T_t x, f\rangle$ is in $\mathrm{RUC}(S)$. In fact, if $s_\alpha \to s$,

$$|h(ts_\alpha) - h(ts)| \le |\langle T_{ts_\alpha}x - T_{ts}x, f\rangle|$$

$$\le \|T_tT_{s_\alpha}x - T_tT_sx\|\,\|f\| \le \|T_{s_\alpha}x - T_sx\|\,\|f\| \to 0$$

uniformly in t. So, let μ be a right invariant mean on $\mathrm{RUC}(S)$ and consider a functional F on E^* given by

$$F(f) = \mu_t\langle T_t x, f\rangle$$

for every $f \in E^*$. Then F is bounded and linear. Since E is reflexive, there is an $x_0 \in E$ such that

$$\mu_t\langle T_t x, f\rangle = \langle x_0, f\rangle$$

for every $f \in E^*$. Put $Qx = x_0$. We shall show that Q has the desired properties. That $QT_s = Q$ follows from the right invariance of μ. Let $u_\alpha = \sum_{i=1}^n \lambda_i \delta_{t_i}$ be a net of convex combinations of point evaluations converging to μ in the weak*-topology of $\mathrm{RUC}(S)^*$, then for each $f \in E^*$, $\langle Qx, f\rangle = \lim_\alpha\langle \sum_{i=1}^n \lambda_i T_{t_i}x, f\rangle$ i.e. $Qx \in \overline{\mathrm{co}}\mathscr{S}(x)$. Also if $x, y \in C$, $f \in E^*$, $\|f\| \le 1$, then

$$|\langle Qx - Qy, f\rangle| = \lim_\alpha\left|\left\langle \sum_{i=1}^n \lambda_i T_{t_i}x - \sum_{i=1}^n \lambda_i T_{t_i}y, f\right\rangle\right| \le \|x - y\|.$$

Hence $\|Qx - Qy\| \le \|x - y\|$.

The following Theorem improves a result of Hirano-Takahashi [**12**, Theorem 1].

THEOREM 6. *Let C be a closed convex subset of a reflexive Banach space with normal structure and S left reversible. If* $\mathrm{RUC}(S)$ *has a right invariant mean and there exists an element in C with bounded orbit, then there exists a* *nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for every $t \in S$ and every $\mathcal{S}$-invariant closed convex subset of C is P-invariant.*

Proof. Let r be a nonexpansive retraction obtained in Theorem 5 and Q a nonexpansive mapping obtained in Lemma 6. Then $P = rQ$ is a nonexpansive retraction satisfying the conclusion of Theorem 6.

Similarly, we can prove the following theorem which generalizes Theorem 2 in [**12**].

THEOREM 7. *Let C be a closed convex subset of a uniformly convex Banach space with a uniformly Fréchet differentiable norm and S a reversible semitopological semigroup. If* RUC(S) *has a right invariant mean, then the following are equivalent*:

(a) $\bigcap_{s \in S} \overline{\text{co}}\{T_t x\colon t \geq s\} \cap F(\mathcal{S}) \neq \varnothing$, *for each $x \in C$;*

(b) *$F(\mathcal{S})$ is nonempty and there is a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{\text{co}}\{T_t x\colon t \in S\}$ for every $x \in C$.*

Proof. (b) $\Rightarrow$ (a). Let $x \in C$. Then $Px \in F(\mathcal{S})$. Also $Px \in \bigcap_s \overline{\text{co}}\{T_t x\colon t \geq s\}$. In fact,

$$Px = PT_s x \in \overline{\text{co}}\{T_t T_s x\colon t \in S\} \subset \overline{\text{co}}\{T_t x\colon t \geq s\}$$

for every $s \in S$.

(a) $\Rightarrow$ (b). By Theorem 5, there exists a nonexpansive retraction of C onto $F(\mathcal{S})$. Then from [**23**, Theorem 4.1] or [**26**, Theorem 1], there is a sunny nonexpansive retraction r of C onto $F(\mathcal{S})$. Let Q be as in Lemma 6 and $P = rQ$. Then P is a nonexpansive retraction of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for all $t \in S$. Let $x \in C$. Then since r is sunny, we have by [**22**, Lemma 2.7]

$$\langle Qx - Px, J(Px - v)\rangle \geq 0 \tag{4}$$

for every $v \in F(\mathcal{S})$. On the other hand, if

$$z \in \bigcap_{s \in S} \overline{\text{co}}\{T_t x\colon t \geq s\} \cap F(\mathcal{S}),$$

from Lemma 4, there is $t_0 \in S$ such that

$$\left\langle T_{tt_0} x - \frac{Px + z}{2}, J\left(\frac{Px + z}{2} - z\right)\right\rangle \leq 0$$

for every $t \in S$. Hence

$$\left\langle Qx - \frac{Px+z}{2}, J\left(\frac{Px+z}{2} - z\right)\right\rangle$$
$$= \mu_t\left\langle T_t x - \frac{Px+z}{2}, J\left(\frac{Px+z}{2} - z\right)\right\rangle$$
$$= \mu_t\left\langle T_{tt_0} x - \frac{Px+z}{2}, J\left(\frac{Px+z}{2} - z\right)\right\rangle$$
$$\leq \sup_t \left\langle T_{tt_0} x - \frac{Px+z}{2}, J\left(\frac{Px+z}{2} - z\right)\right\rangle \leq 0.$$

Therefore by using (4) we have

$$\langle z - Px, J(Px - z)\rangle \geq 0$$

and hence $z = Px$. This completes the proof.

THEOREM 8. *Let S be right reversible and C be a closed convex subset of a uniformly convex Banach space with Fréchet differentiable norm. The following are equivalent*:

(a) $\bigcap_{s \in S} \overline{\text{co}}\{T_t x\colon t \geq s\} \cap F(\mathscr{S}) \neq \varnothing$ *for each* $x \in C$.

(b) *There exists a retraction P of C onto $F(\mathscr{S})$ such that $PT_t = T_t P = P$ for every $t \in S$ and $Px \in \overline{\text{co}}\{T_t x\colon t \in S\}$ for every $x \in C$.*

Proof. (b) $\Rightarrow$ (a). Same as Theorem 8.

(a) $\Rightarrow$ (b). In this case, by Theorem 1, for each $x \in C$, $\bigcap_{s \in S} \overline{\text{co}}\{T_t x\colon t \geq s\} \cap F(\mathscr{S})$ contains exactly one point $P(x)$. Clearly $T_t P = P$ for each $t \in S$. Let $t_0 \in S$ be fixed. We shall show that

$$(5) \qquad \bigcap_{s \in S} \overline{\text{co}}\{T_{tt_0} x;\ t \geq s\} \supseteq \bigcap_{s \in S} \overline{\text{co}}\{T_t x;\ t \geq s\}.$$

When this is proved, then

$$\bigcap_{s \in S} \overline{\text{co}}\{T_{tt_0} x;\ t \geq s\} \cap F(\mathscr{S}) = \bigcap_{s \in S} \overline{\text{co}}\{T_t x;\ t \geq s\} \cap F(\mathscr{S}).$$

In particular $P(T_{t_0} x) = P(x)$.

Let $s \in S$ be fixed. Then $\{T_u x;\ u \geq st_0\} \supseteq \overline{\{T_{tt_0} x;\ t \geq s\}}$ (since if $t \geq s$, $t \in \{s\} \cup \overline{Ss}$; hence $tt_0 \in \{st_0\} \cup \overline{Sst_0}$ i.e. $tt_0 \geq st_0$) i.e. $\{T_u x;\ u \geq st_0\} \supseteq \{T_{tt_0} x;\ t \geq s\}$. On the other hand, if $u \geq st_0$, then $u \in \{st_0\} \cup \overline{Sst_0}$. If $u = st_0$, then $T_u(x) = T_{st_0}(x) \in \overline{\{T_{tt_0}(x);\ t \geq s\}}$. If $u \in \overline{Sst_0}$, $u = \lim_\alpha a_\alpha st_0$ for some net $\{a_\alpha\} \subseteq S$. So $T_u(x) = \lim_\alpha T_{a_\alpha st_0}(x)$ i.e. $T_u(x) \in \overline{\{T_{tt_0}(x);\ t \geq s\}}$. Hence $T_u(x) \in \overline{\{T_{tt_0}(x);\ t \geq s\}}$ also. Consequently

$$\overline{\text{co}}\{T_u x;\ u \geq st_0\} = \overline{\text{co}}\{T_{tt_0} x;\ t \geq s\}.$$

Now if $y \in \bigcap_{s \in S} \overline{\text{co}}\{T_t x;\ t \geq s\}$, then $y \in \bigcap_{s \in S} \overline{\text{co}}\{T_u x;\ u \geq st_0\} = \bigcap_{s \in S} \overline{\text{co}}\{T_{tt_0} x;\ t \geq s\}$ i.e. (5) holds.

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