PACIFIC JOURNAL OF MATHEMATICS
Vol. 129, No. 1, 1987

THE BEST MODULUS OF CONTINUITY FOR SOLUTIONS OF THE MINIMAL SURFACE EQUATION

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We consider the Dirichlet problem for the minimal surface equation on a bounded domain in $\mathbf{R}^n$ which has nonnegative mean curvature. We give a modulus of continuity for the solution u in terms of the modulus of continuity of the boundary values ϕ. The modulus obtained is shown to be best possible.

0. Introduction. We consider the Dirichlet problem for the minimal surface equation in a bounded domain $\Omega \subseteq \mathbf{R}^N$ with boundary values ϕ. We shall assume that $\partial\Omega$ has nonnegative mean curvature and so the existence of a solution u for any continuous function ϕ is known **[JS]**. In this paper we consider the way that the regularity of u depends on the regularity of ϕ. Many results have already been obtained for this problem. For example if $\phi \in C^{k,\alpha}(\partial\Omega)$, $k \geq 1$, then $u \in C^{k,\alpha}(\overline{\Omega})$. (See **[GG]**, **[L1]**, **[GT]**.) If $\phi \in C^{0,1}(\partial\Omega)$ then $u \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ which depends on the Lipschitz constant of ϕ and the behaviour of the mean curvature of $\partial\Omega$. (See **[G1]** and also **[W1]** where optimal results are obtained for the value of α.) If $\phi \in C^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$, and the mean curvature of $\partial\Omega$ is strictly positive then $u \in C^{0,\alpha/2}(\overline{\Omega})$ (see **[G1]** and **[L2]**.) More generally if $x_0 \in \partial\Omega$, ϕ satisfies a Hölder condition with exponent α at x_0 and the mean curvature of $\partial\Omega$ is larger than $C|x - x_0|^k$, $C > 0$, near x_0 then u satisfies a Hölder condition at x_0 with exponent $\alpha/(k+2)$. (See **[W1]** and **[W2]**.) However few results, apart from just continuity, have been given in the case $\phi \in C^{0,\alpha}(\partial\Omega)$, $0 < \alpha < 1$, and assuming only nonnegative mean curvature. A modulus of continuity could be found by the method proposed in §13.5 of **[GT]**. The counter-examples of **[W1]** show that in general the solution u will not be Hölder continuous for any exponent β. We shall explicitly find a modulus of continuity for u in this situation and then show that it is the best possible. Actually the results hold, and are presented, for a fairly general modulus of continuity for ϕ rather than just the Hölder condition. Further the results given are local results and so we only need to assume $\partial\Omega$ has nonnegative mean curvature in a neighbourhood of the point under consideration. However in this case a classical solution may not exist so we must consider generalized solutions, that is, functions $u \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$ minimizing

$$\int_\Omega \sqrt{1 + |Dv|^2} + \int_{\partial\Omega} |v - \phi| dH^{n-1}$$

for all $v \in BV(\Omega)$. Such solutions u still satisfy the minimal surface equation

$$\mathscr{M}u = \sum_{i=1}^n D_i\left(D_i u\left(1 + |Du|^2\right)^{-1/2}\right) = 0$$

in Ω but they may not satisfy the boundary conditions, $u = \phi$, everywhere on $\partial\Omega$. The existence and properties of such solutions may be found, for example, in **[G2]**.

The method of proof uses barrier constructions very similar to those in **[W1]** (which had their origins in **[S]**) and essentially reduces the problem to an examination of the behaviour of solution of Laplace's equation in cusped domains.

1. Laplace's equation on cusped domains. The barriers which we construct in later sections are defined in terms of various functions on cusped domains. The nature of the cusp is determined by the modulus of continuity, γ, for the boundary values ϕ, with the required domain given by $D = \{(y', y_n): y' \in \mathbf{R}^{n-1}, \gamma(|y'|) < y_n < \delta\}$. It is convenient to work with polar coordinates so that the required results are given as follows.

PROPOSITION 1. *Suppose $\beta \in C^3(0, \infty)$ with $\beta(r) > 0$ and $\beta'(r) > 0$ for $r > 0$ and*

$$\beta(r) + r\beta'(r) + r^2|\beta''(r)| + r^3|\beta'''(r)| \to 0 \quad \text{as } r \to 0.$$

Let

$$D = \left\{y \in \mathbf{R}^n: |\theta| < \beta(r), r < \delta\right\},$$

where $r = |y|$ and $\theta = \cos^{-1}(y_n/r)$.

For $\alpha \in \mathbf{R}$ let

$$F_\alpha(r) = (r\beta(r))^\alpha \exp\left\{-a\int_r^1 \frac{1}{t\beta(t)}\, dt\right\}, \tag{1.1}$$

where a is the first positive zero of the Bessel function $J_{(n-3)/2}$.

Then for δ sufficiently small,

I *if $\alpha > -(n-2)/2$ there exist a function u defined on D such that*

(i) *$u > 0$ on D and $du = 0$ if $|\theta| = \beta(r)$,*

(ii) *$|u(y)| \le CF_\alpha(r)$, $|Du(y)| \le CF_\alpha(r)/r\beta(r)$, $|D^2u(y)| \le CF_\alpha(r)/r^2\beta^2(r)$,*

(iii) *for* $|\theta| \le \varepsilon\beta(r)$ *with* $0 \le \varepsilon < 1$, $u(y) \ge C(\varepsilon)F_\alpha(r)$,
(iv) $\partial u/\partial y_n > 0$ *in* D,
(v) $\Delta u \ge \alpha_0 F_\alpha(r)(\beta(r) + r\beta'(r))/r^2\beta^2(r)$, *where* $\alpha_0 > 0$,

II *if* $\alpha < -(n-2)/2$ *there exists a function* u *defined on* D *satisfying* (i)–(iv) *above and also*
(v′) $\Delta u \le -\alpha_0 F_\alpha(r)(\beta(r) + r\beta'(r))/r^2\beta^2(r)$, *where* $\alpha_0 > 0$.

Proof. We let

$$G(t) = t^{(3-n)/2}J_{(n-3)/2}(at), \tag{1.2}$$

where $J_{(n-3)/2}$ is the Bessel function and a is its first zero. Thus

$$\begin{gathered} G(1) = 0, \qquad G(t) > 0 \quad \text{if } |t| < 1, \\ G''(t) + \frac{(n-2)}{t}G'(t) + a^2G(t) = 0, \\ |G| + |G'| + |G''| \le C \quad \text{for } |t| \le 1. \end{gathered} \tag{1.3}$$

We now let $F = F_\alpha(r)$ be given by (1.1) and define

$$g_1(r) = \beta(r) + r\beta'(r),$$

$$g(r) = \beta(r) + r\beta'(r) + r^2|\beta''(r)| + r^3|\beta'''(r)| \to 0.$$

Then

$$F'(r) = \frac{F}{r\beta^2}[a\beta + g_1\alpha\beta], \tag{1.4}$$

$$F''(r) = \frac{F}{(r\beta)^2}[a^2 + a(2\alpha - 1)(\beta + r\beta') + g_1O(g)],$$

$$\begin{aligned} &r^2F'' + (n-1)rF' \\ &\quad = \frac{F}{\beta^2}[a^2 + a(2\alpha - 1)(\beta + r\beta') + a(n-1)\beta + g_1O(g)]. \end{aligned}$$

Next we define

$$H(r,\theta) = \exp\left\{\frac{b}{2}\frac{r\beta'}{\beta^2}\theta^2 + \frac{d}{2}\frac{\theta^2}{\beta}\right\} \tag{1.5}$$

where b and d are constants to be chosen. Then

$$H_\theta = H\left(b\theta\frac{r\beta'}{\beta^2} + d\frac{\theta}{\beta}\right), \qquad H_r = \frac{H}{r\beta}g_1O(g), \tag{1.6}$$

$$H_{\theta\theta} = \frac{H}{\beta^2}[br\beta' + d\beta + g_1O(g)],$$

$$H_{rr} = \frac{H}{r^2\beta^2}g_1O(g), \qquad H_{r\theta} = \frac{H}{r\beta^2}g_1O(g).$$

Finally we set

$$u(r,\theta) = F_\alpha(r)G\left(\frac{\theta}{\beta}\right)H(r,\theta) \tag{1.7}$$

so that

$$\begin{aligned}\Delta u &= u_{rr} + \frac{(n-1)}{r}u_r + \frac{(n-2)}{r^2}\operatorname{Cot}(\theta)u_\theta + \frac{1}{r^2}u_{\theta\theta} \\ &= \frac{1}{r^2\beta^2}\Big\{GH\beta^2[r^2F'' + (n-1)rF'] \\ &\qquad + FH\left[G'' + (n-2)\frac{\beta}{\theta}G'\right] \\ &\qquad + FHG'\left[r\beta'\frac{\theta}{\beta}2(b-a) + 2d\theta\right] \\ &\qquad + FHG(n-1)(br\beta' + d\beta) + FHg_1O(g)\Big\}\end{aligned} \tag{1.8}$$

where we have used the estimates of (1.3), (1.4) and (1.6). Now using (1.3) and (1.4) again we see

(1.9)

$$\begin{aligned}\Delta u &= \frac{FH}{r^2\beta^2}\Big\{[a(2\alpha-1)(\beta + r\beta') + (a+d)(n-1)\beta + br\beta'(n-1)]G \\ &\qquad + G'2d\theta + G'\frac{\theta}{\beta}r\beta'2(b-a) + g_1O(g)\Big\} \\ &= \frac{FH}{r^2\beta^2}\Big\{\beta G(a(2\alpha + n - 2) + d(n-1)) \\ &\qquad + r\beta'G((2\alpha-1)a + b(n-1)) \\ &\qquad + G'\frac{\theta}{\beta}(2d\beta + 2(b-a)r\beta') + g_1O(g)\Big\}\end{aligned}$$

Noting that $G(t) - tG'(t) \geq \alpha_1 > 0$ for $|t| \leq 1$ and $H(r,\theta) \geq 1$ we see that if $2\alpha + n - 2 > 0$ we can choose b and d such that $d < 0$, $b < a$, $(2\alpha - 1)a + b(n-1) > 0$ and $a(2\alpha + n - 2) + d(n-1) > 0$ so that $I(v)$ holds if δ small. Similarly if $2\alpha + n - 2 < 0$ we can ensure II(v′) holds.

Properties (i), (ii) and (iii) hold by our definition of u and estimates (1.3), (1.4) and (1.6). Finally

$$\frac{\partial u}{\partial y_n} = u_r \operatorname{Cos}\theta - \frac{1}{r} u_\theta \operatorname{Sin}\theta \quad \text{and}$$

$$u_r = \frac{FHG}{r\beta}[a + O(g)] - \frac{FHG'}{r\beta}\frac{\theta}{\beta} r\beta',$$

$$u_\theta = \frac{FHG}{\beta}\left[br\beta'\frac{\theta}{\beta} + d\theta\right] + \frac{FHG'}{\beta}$$

so that

$$\frac{\partial u}{\partial y_n} = \frac{FhG}{r\beta}[a \operatorname{Cos}\theta + O(g)] - \frac{FHG'}{r\beta}\left[\operatorname{Sin}\theta + \frac{\theta}{\beta} r\beta'\right]$$
$$> 0 \quad \text{if } \delta \text{ small.}$$

2. **Continuity estimates.** We now present the continuity estimates for the solution u. These are local in nature so that we assume ϕ satisfies a continuity condition (for example Hölder continuity) only at a particular point $x_0 \in \partial\Omega$ and then derive corresponding continuity conditions for the solution u at x_0. We shall assume that $x_0 \partial\Omega$ and $\mathscr{N}$ is a neighbourhood of x_0 such that $\partial\Omega$ has nonnegative mean curvature in $\mathscr{N}$. Suppose ϕ has an upper modulus of continuity, γ, at x_0 so that

$$\phi(x) - \phi(x_0) \le \gamma(|x - x_0|), \qquad x \in \partial\Omega. \tag{2.1}$$

We may assume, without loss of generality, that $x_0 = 0$, $\phi(x_0) = 0$, the inner unit normal to $\partial\Omega$ at 0 lies on the x_n-axis and that $\partial\Omega \cap \mathscr{N} = \{(x', w(x'))\colon\ x' \in \mathbf{R}^{n-1},\ |x'| < \delta_0\}$ where $w \in C^2(\mathbf{R}^{n-1})$, $w(0) = 0$, $Dw(0) = 0$ and $|w(x')| \le L|x'|^2$ for some constant L. Then from (2.1) we see that, if $x' = (x_1, \ldots, x_{n-1})$,

$$\phi(x) \le \gamma(|x'| + L|x'|^2) = \gamma_1(|x'|), \qquad |x| < \delta_0. \tag{2.2}$$

Fairly complete results have already been obtained in the case of Lipschitz data where $\gamma(t) = Kt$ (see **[W1]**) so we restrict ourselves to the case where $t^{-1}\gamma(t)$ increases to ∞ as t decreases to 0. At the cost of increasing γ by a multiplicative constant it is known (as noted in **[W1]**) in this situation we may assume γ is concave and C^k for any k. Hence we assume γ is C^3 and concave, $\gamma' \to \infty$ as $t \to 0$ and $[\gamma(t)/t]' > 0$. We now wish to write (2.2) in a polar form so that the results of the last section may easily be applied. Hence we set $r^2 = x_{n+1}^2 + |x'|^2$ and $\theta = \cos^{-1}(x_{n+1}/r)$. Then the conditions on γ ensure that there is function β_1 such that

$$r \operatorname{Cos}\beta_1(r) = \gamma_1(r \operatorname{Sin}\beta_1(r)), \qquad r < \delta_1. \tag{2.3}$$

Then (2.2) becomes

$$x_{n+1} = \phi(x), \quad |\theta| \geq \beta_1(r), \quad \text{if } x \in \partial\Omega \cap \mathcal{N}. \tag{2.4}$$

Furthermore by increasing L in (2.2) we may assume the above inequality is strict for $r > 0$. We wish to compare the function β_1, just constructed, with the more easily calculated function, β, which is defined so that $r\beta(r)$ is the inverse of $\gamma(t)$, that is

$$\gamma(r\beta(r)) = r. \tag{2.5}$$

LEMMA 1. *For r sufficiently small,*

$$\beta_1(r) \leq \beta(r) \leq \beta_1(r)\left(1 + Lr\beta_1(r) + \frac{1}{2A}\beta_1(r)\right), \tag{2.6}$$

$$\beta(r) \geq \beta_1 \geq \beta(r)\left(1 + Lr\beta(r) + \frac{1}{2A}\beta(r)\right)^{-1} \tag{2.7}$$

where

$$A = \gamma'(2r\beta_1(r)).$$

Proof. The left hand inequality follows by considering, for fixed r, the derivatives of $\gamma(r\beta)$ and $(\operatorname{Cos}\beta)^{-1}\gamma(r\operatorname{Sin}\beta)$ with respect to β and then using the concavity of γ. For the right hand inequality we have

$$r\left(1 - \frac{1}{2}\beta_1^2\right) \leq r\operatorname{Cos}\beta_1 \leq \gamma\left(r\beta_1 + Lr^2\beta_1^2\right)$$

so that

$$\gamma\left(r\beta_1 + Lr^2\beta_1^2\right) + \frac{1}{2}r\beta_1^1 \geq \gamma(r\beta).$$

If r is small enough then using concavity

$$\gamma\left(r\beta_1 + Lr^2\beta_1^2 + \frac{1}{2A}r\beta_1^2\right) \geq \gamma(r\beta). \qquad \square$$

Now for each δ small enough that Lemma 1 holds, we can set

$$\varepsilon = \varepsilon(\delta) = L\delta + \frac{1}{2\gamma'(2\delta\beta_1(\delta))} \to 0 \quad \text{as } \delta \to 0 \tag{2.8}$$

and

$$\beta_2(r) = \frac{\beta(r)}{1 + \varepsilon\beta(r)} \quad \text{for } r < \delta. \tag{2.9}$$

Then we see that near 0 the graph of $\phi(x', w(x'))$ must lie below the graph of the function given in polar coordinates by $|\theta| = \beta_2(r)$.

Note that by our assumptions on γ we already have that

$$\beta(r) > 0, \quad \beta'(r) > 0, \quad \text{for } r > 0, \ \lim_{r\to 0} \beta(r) + r\beta'(r) = 0. \tag{2.10}$$

We shall make the further additional assumption that for β (or equivalently β_2)

$$\lim_{r\to 0} |r^2\beta''(r)| + |r^2\beta'''(r)| = 0. \tag{2.11}$$

Thus altogether we have assumed that

(2.12) γ is a C^3 concave modulus of continuity such that

$$\lim_{r\to 0} (\gamma^{-1}(r))' + r|(\gamma^{-1}(r))''| + r^2|(\gamma^{-1}(r))'''| = 0.$$

The new modulus of continuity for u then involves the function $F_\alpha(r)$ given in (1.1). Thus

$$\begin{aligned} F_\alpha(r) &= (r\beta(r))^\alpha \exp\left\{-a\int_r^1 \frac{1}{t\beta(t)}\,dt\right\} \\ &= (\gamma^{-1}(r))^\alpha \exp\left\{-a\int_r^1 \frac{1}{\gamma^{-1}(t)}\,dt\right\} \end{aligned} \tag{2.13}$$

with a the first positive zero of the Bessel function $J_{(n-3)/2}(x)$.

THEOREM 1. *Suppose Ω is a bounded open subset of $\mathbf{R}^n$ with locally Lipschitz boundary $\partial\Omega$, $\phi \in L^1(\partial\Omega)$ and u is a generalized solution of the Dirichlet problem. Suppose $x_0 \in \partial\Omega$ and there is a neighborhood $\mathcal{N}$ of x_0 such that $\partial\Omega$ is C^2 with nonnegative mean curvature in $\mathcal{N}$. Suppose γ: $[0, \infty) \to [0, \infty)$ satisfies* (2.12) *and*

$$\phi(x) - \phi(x_0) \le \gamma(|x - x_0|), \quad x \in \partial\Omega.$$

Then for any $\alpha > -(n-2)/2$ there is a constant c and a neighbourhood G of x_0 such that

$$u(x) - \phi(x_0) \le \max\{\gamma(2|x - x_0|) + c|x - x_0|, F_\alpha^{-1}(cd(x, \partial\Omega))\}$$

for $x \in G \cap \Omega$, where F_α^{-1} is the inverse of the function F_α defined in (2.13).

Proof. We can assume, as above, that $x_0 = 0$, $\phi(x_0) = 0$ and that near 0, $\partial\Omega$ is the graph of w: $\mathbf{R}^{n-1} \to \mathbf{R}$ where $|w(x')| \le L|x'|^2$. Furthermore it is sufficient to consider the case where $\phi(x) = \gamma(|x|)$ so that u is continuous on $\overline{\Omega} \cap \mathcal{N}$ ([M]). We let β_2 be as above and suppose $\alpha > -(n-2)/2$ then define

$$v(y) = w(y) + \lambda f(y)$$

where f is the function given by (I) of Proposition 1 in §1 but with β replaced by β_2 and $\lambda \in (0,1)$ is to be chosen. We have taken v and f to be defined on a set

$$D = \{ y \in \mathbf{R}^n \colon |\theta| < \beta_2(r), r < \delta \}$$

and w is independent of y_n. We shall prove the theorem with F_α replaced by $\tilde{F}_\alpha$ defined as in (2.13) but with β_2 replacing β. But by definition (2.9)

$$\begin{aligned}\tilde{F}_\alpha(r) &= (r\beta_2)^\alpha \exp\left\{ -a \int_r^1 \frac{1}{t\beta}\, dt + a\varepsilon \log r \right\} \\ &\geq c(r\beta)^{\alpha + a\varepsilon} \exp\left\{ -a \int_r^1 \frac{1}{t\beta}\, dt \right\} = cF_{\alpha + a\varepsilon}(r)\end{aligned}$$

and ε can be made as small as we like by taking δ sufficiently small and so the required result follows.

Now by Proposition 1 we have $|Df(y)| \leq c\tilde{F}_\alpha(r)/r\beta_2(r) \to 0$ as $r \to 0$, so that taking $\lambda \leq 1$ and δ sufficiently small we can ensure that $\lambda |Df(y)| \leq 1$ in D. Then Lemma 1 of [S] may be applied to give

$$\mathcal{M}_0 v = \mathcal{M}_0 w + \lambda \Delta f + \lambda E \geq \lambda(\Delta f + E)$$

where

$$\begin{aligned}\mathcal{M}_0 g &= \sqrt{1 + |Dg|^2}\, \mathcal{M} g \quad \text{and} \\ |E| &\leq c\left(|D^2 f| + |Df|\,|D^2 w|\right)\left(\lambda |Df| + |Dw|\right) \\ &\leq c \frac{\tilde{F}_\alpha(r)}{r^2 \beta_2^2(r)} \left(\lambda \frac{F_\alpha(r)}{r\beta_2(r)} + |y'| \right) \\ &\leq \frac{c\tilde{F}_\alpha(r)}{r\beta_2(r)} \quad \text{for } r \text{ sufficiently small.}\end{aligned}$$

Thus we see that

$$\mathcal{M}_0 v \geq \frac{\lambda \tilde{F}_\alpha(r)}{r^2 \beta_2^2(r)} \left(\alpha_0 \beta_2(r) - cr\beta_2(r) \right) \geq 0$$

in D, provided we take δ sufficiently small. Then for D we have $\mathcal{M}_0 v \geq 0$, $D_n v > 0$, $v > w$ and $v = w$ if $|\theta| = \beta_2(r)$. Arguing as in [S] there is then a neighbourhood G of 0 and a function $\tilde{v}$ defined on $\overline{G \cap \Omega}$

such that $\mathscr{M}\tilde{v} \le 0$ in $G \cap \Omega$, graph of $\tilde{v}$ over $G \cap \partial\Omega$ = graph (in polars) of $|\theta| = \beta_2(r)$ which lies above the graph of $\phi(x)$, and $\tilde{v}(x) = \sqrt{\delta^2 - |x'|^2} \ge \delta \operatorname{Cos} \beta_2(\delta)$ on $\partial G \cap \Omega$. Furthermore

$$\partial G \cap \Omega = \{x \colon x_n = w(x') + \lambda f(\delta, \theta), |\theta| \le \beta_2(\delta)\}$$

so that if $x \in \partial G \cap \Omega$, $x_n - w(x') \le c\lambda \tilde{F}_\alpha(\delta)$.

Now as noted above we may assume u is continuous on $\overline{\Omega} \cap \mathscr{N}$ so there is a function $p(t)$ such that $p(t) \to 0$ as $t \to 0$ and $|u(x) - u(y)| \le p(|x-y|)$ for $x, y \in \overline{G \cap \Omega}$. Then, in particular, if $x \in \partial G \cap \Omega$,

$$\begin{aligned} u(x) &\le u(x', w(x')) + p(|x_n - w(x')|) \\ &\le \gamma(|x|) + p(c\lambda \tilde{F}_\alpha(\delta)) \\ &< \gamma_1(\delta \operatorname{Sin} \beta_2(\delta)) + p(c\lambda \tilde{F}_\alpha(\delta)) \\ &\le \delta \operatorname{Cos} \beta_2(\delta) + p(c\lambda \tilde{F}_\alpha(\delta)). \end{aligned}$$

Hence for a fixed $\delta > 0$ we can choose λ sufficiently small so that $u(x) \le \tilde{v}(x)$ on $\partial G \cap \Omega$. The comparison principle now gives that $u(x) \le \tilde{v}(x)$ in $G \cap \Omega$ and so it only remains to show that $\tilde{v}(x)$ has the required growth. By the definition of $\tilde{v}$ we have for $x = (x', x_n) \in G$,

$$(x', x_n, \tilde{v}(x)) = (x', v(x', \tilde{v}(x)), \tilde{v}(x)).$$

If

$$\theta = \tan^{-1}\left(\frac{|x'|}{\tilde{v}(x)}\right) \le \frac{1}{2}\beta_2(r)$$

then

$$x_n = v(x', \tilde{v}(x)) \ge w(x') + c\tilde{F}_\alpha(r)$$

where $r^2 = |x'|^2 + (\tilde{v}(x))^2$. So that

$$\tilde{v}(x) \le r \le \tilde{F}_\alpha^{-1}(c(x_n - w(x'))).$$

On the other hand if $|\theta| \ge \frac{1}{2}\beta_2(r)$ then

$$|x'| = |r \operatorname{Sin} \theta| \ge r \operatorname{Sin} \tfrac{1}{2}\beta_2(r) \ge r\left(\tfrac{1}{2}\beta_2 - C\beta_2^3\right)$$

and so

$$r\beta(1 - \varepsilon\beta) \le r\beta_2 \le 2|x'| + \varepsilon r \beta^2.$$

Hence for r small,

$$r\beta \le 2|x'|\left(1 + 8\varepsilon\frac{|x'|}{r}\right)$$

and, using $[\gamma(t)/t]' \ge 0$,

$$r = \gamma(r\beta) \le \gamma\left(2|x'|\left(1 + 8\frac{\varepsilon|x'|}{r}\right)\right) \le \gamma(2|x'|)\left(1 + 8\frac{\varepsilon|x'|}{r}\right)$$

and so

$$\tilde{v}(x) \leq r \leq \gamma(2|x'|) + 8\varepsilon|x'|. \qquad \square$$

The exponential term in F_α is of course far more important than $(r\beta)^\alpha$ and so in most cases the next result is all that is required.

COROLLARY 1. *If $n > 2$ and the conditions for Theorem 1 hold then*

$$u(x) - \phi(x_0) \leq \max\left\{\gamma(2|x - x_0|) + c|x - x_0|, F^{-1}\left(\frac{-\ln(d(x, \partial\Omega) + c}{a}\right)\right\}$$

where

$$F(r) = \int_r^1 \frac{1}{\gamma^{-1}(t)}\, dt.$$

If $n = 2$ the same statement holds provided we replace a $(= \pi/2)$ by a' where a' is any number greater than $\pi/2$.

The main case of interest is when the given data is Hölder continuous in which case $\gamma(t) = Kt^\lambda$ with $0 < \lambda < 1$. Conditions (2.12) are easily checked and F_α and F are readily calculated.

COROLLARY 2. *Suppose Ω, ϕ, u, x_0 and $\mathcal{N}$ are as in Theorem 1. Suppose there are constants K, M and $\lambda \in (0, 1)$ such that*

$$\phi(x) - \phi(x_0) \leq K|x - x_0|^\lambda, \qquad x \in \partial\Omega \cap \mathcal{N},$$
$$\phi(x) - \phi(x_0) \leq \mathcal{M}, \qquad x \in \partial\Omega.$$

Then for any $\alpha > -(n-2)/2$ there is a constant c and a neighborhood G of x_0 such that $u(x) - \phi(x_0) \leq \max\{2^{\lambda+1}K|x - x_0|^\lambda, F_\alpha^{-1}(cd(x, \partial\Omega))\}$ for $x \in \Omega \cap G$, where

$$F_\alpha(r) = r^{\alpha/\lambda}\exp\left\{-aK^{1/\lambda}\frac{\lambda}{1-\lambda}r^{-(1-\lambda)/\lambda}\right\}.$$

COROLLARY 3. (i) *If $n > 2$ and the conditions of Corollary 2 hold then for $|x - x_0|$ sufficiently small*

$$(2.14) \qquad u(x) - \phi(x_0) \leq \max\left\{2^{\lambda+1}K|x - x_0|^\lambda, \left(aK^{1/\lambda}\frac{\lambda}{1-\lambda}\right)^{\lambda/(1-\lambda)}(-\ln(d(x, \partial\Omega)) + c)^{-\lambda/(1-\lambda)}\right\},$$

where a is the first positive zero of $J_{(n-3)/2}(t)$.

(ii) *If $n = 2$ then* (2.14) *holds with any $a > \pi/2$.*

3. Optimal growth. In this section we show that the results of the previous section cannot be improved. Thus we show that, in general, Theorem 1 and Corollary 2 do not hold for $\alpha < -(n-2)/2$ and Corollaries 1 and 3 do not hold for any a less than the first positive zero of the Bessel function $J_{(n-3)/2}(t)$. Since Corollaries 1 and 3 are obtained by weakening Theorem 1 it is enough to give counter examples for $\alpha < -(n-2)/2$. It should be noted that in the case where $\partial\Omega$ has strictly positive mean curvature near x_0 (or mean curvature growing like a power of $|x - x_0|$) better regularity results are known (see **[W1]** and **[W2]**) thus any counter examples must be for the case of zero mean curvature near x_0. A typical application of Theorem 2 would show that if Ω has zero mean curvature near $0 \in \partial\Omega$ (and has interior normal at 0 the x_n-axis) and we prescribe boundary values $\phi(x) = \gamma(|x|)$ on $\partial\Omega$ then the solution u must grow faster than $F_\alpha^{-1}(x_n)$ along the x_n-axis near 0 for any $\alpha < -(n-2)/2$.

For technical reasons we introduce one further restriction on the types of modulus of continuity we consider. Namely, we assume in addition to (2.12) that there are constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$(1+\varepsilon_1)t\gamma''(t) + (2n-3)\gamma'(t) \geq 0 \quad \text{for } 0 < t < \varepsilon_2. \tag{3.1}$$

This additional restriction could probably be removed but in any case it allows Hölder continuous data when $\gamma(t) = Kt^\alpha$, $0 < \alpha < 1$ and also logarithmic growth $\gamma(t) = -A/(\log t + B)$ when $n \geq 3$.

THEOREM 2. *Suppose Ω is a bounded open subset of $\mathbf{R}^n$ with locally Lipschitz boundary $\partial\Omega$, $\phi \in L^1(\partial\Omega)$ and u is the generalized solution of the Dirichlet problem. Suppose $x_0 \in \partial\Omega$ and there is a neighbourhood $\mathcal{N}$ of x_0 such that $\partial\Omega$ is C^2 and has non-positive mean curvature in $\mathcal{N}$. Suppose γ: $[0,\infty) \to [0,\infty)$ satisfies* (2.12) *and* (3.1) *and*

$$\phi(x) - \phi(x_0) \geq \gamma(|x - x_0|), \qquad x \in \partial\Omega \cap \mathcal{N},$$
$$\phi(x) \geq \phi(x_0) + \varepsilon_0, \qquad x \in \partial\Omega \sim \mathcal{N},$$

for some $\varepsilon_0 > 0$.

Then for any $\alpha < -(n-2)/2$ and any $c > 0$ there is $t_0 > 0$ such that

$$u(x_0 + t\nu) \geq \phi(x_0) + F_\alpha^{-1}(ct), \qquad 0 < t \leq t_0$$

where ν is the unit inner normal to $\partial\Omega$ at x_0.

Proof. We take $x_0 = 0$, $\nu = e_n = (0,\dots,0,1)$, $\phi(x_0) = 0$ and suppose $\partial\Omega$ is the graph of w: $\mathbf{R}^{n-1} \to \mathbf{R}$ near 0. It is sufficent to consider the case $\phi(x) = \gamma(|x|)$ near 0. We first prove a preliminary result about the growth of u in such a situation. This result is similar to, but slightly more precise than, results presented in the last section of **[W1]**.

LEMMA 2. *Suppose u, Ω and ϕ are as above. Then there are constants $C_1 > 0$ and $C_2 > 0$ such that for $|x|$ sufficiently small and $x \in \Omega$*

$$u(x) \geq \gamma\left(\sqrt{|x'|^2(1 - C_1|x'|) + C_2(x_n - w(x'))}\right). \tag{3.2}$$

Proof. Suppose $\Omega \subseteq B_R(0)$.

For each $\varepsilon > 0$ we can construct a function $W(x')$ depending on ε such that

(i) $W(x') = (w')$ if $|x'| < \varepsilon^2$, $W(x') = 0$ if $|x'| > 2\varepsilon^2$,
(ii) $|W(x')| \leq L|x'|^2$,
(iii) $|DW(x')| \leq C_0 L|x'|$,
(iv) $|D^2W(x')| \leq C_0 L$,

where C_0 is a constant not depending on ε. Furthermore if ε is sufficiently small we may assume

(v) if $x \in \partial\Omega$, $|x'| < \varepsilon^2$, $x_n \neq W(x')$ then $|x_n| > 2\varepsilon^2$,
(vi) $\varepsilon < 1$ and $4C_0L\sqrt{\varepsilon} < 1$.

Let A be a constant with $0 < A < \min\{\varepsilon^3, 1/2L\}$ and consider the functions

$$\delta(x) = \left|x + \frac{A}{2}e_n\right| - \frac{A}{2} - W(x'), \tag{3.3}$$

$$v(x) = \eta(A\delta(x)) \quad \text{where } \eta(t) = \gamma(\sqrt{t}). \tag{3.4}$$

We aim to show that, for suitable choices of ε and A, v is a lower barrier for the solution u and it has the required growth.

First note that by (v) above and the choice $A < \varepsilon^3$ we have $0 \leq \delta(x) \leq R + 1$ in Ω and so v is defined on Ω. Since $\eta' \geq 0$ and $\eta'' \leq 0$ we can calculate

$$\begin{aligned}
\left(1 + |Dv|^2\right)\mathscr{M}_0 v &= \left(1 + |Dv|^2\right)\Delta v - D_i v D_j v D_{ij} v \\
&= A\eta'\Delta\delta + A^2\eta''|D\delta|^2 + A^3(\eta')^3\left(|D\delta|^2\Delta\delta - D_i\delta D_j\delta D_{ij}\delta\right) \\
&\geq \frac{A}{|s + Ae_n/2|}\left\{(n - 1 - c\varepsilon)\eta' + A\left|x + \frac{A}{2}e_n\right|\eta''(1 + c\varepsilon)\right\} \\
&\quad + \frac{A^3(\eta')^3}{|x + Ae_n/2|}(n - 1 - c\varepsilon).
\end{aligned}$$

Now if $t = A\delta(x) \leq A(R+1)$ then

$$A\left|x + \frac{A}{2}e_n\right| = t + \frac{A^2}{2} + AW(x') \leq t + \frac{A^2}{2} + A\varepsilon|x'|$$

so that

$$A\left|x + \frac{A}{2}e_n\right| \leq \frac{A^2}{2(1-\varepsilon)} + \frac{t}{1-\varepsilon}.$$

Thus

$$\begin{aligned}
&\left(1 + |Dv|^2\right)\mathcal{M}_0 v \\
&\quad \geq \frac{A(1-c\varepsilon)}{|x + Ae_n/2|}\{(n-1)\eta'(t) + (1+c\varepsilon)t\eta''(t)\} \\
&\quad\quad + \frac{A^3(1-c\varepsilon)}{2|x + Ae_n/2|}\left\{(\eta'(t))^3 2(n-1) + (1+c\varepsilon)\eta''(t)\right\} \\
&\quad \geq \frac{A(1-c\varepsilon)}{|x + Ae_n/2|}\frac{1}{4s}\{(2n-3)\gamma'(s) + (1+c\varepsilon)s\gamma''(s)\} \\
&\quad\quad + \frac{A^3(1-c\varepsilon)}{|x + Ae_n/2|}\frac{1}{8s^3}\Big\{(1+c\varepsilon)s\gamma''(s) \\
&\quad\quad\quad + \gamma'(s)\big((n-1)(\gamma'(s))^2 - (1+c\varepsilon)\big)\Big\}
\end{aligned}$$

where $s = \sqrt{t} = \sqrt{A\delta(x)} \leq \sqrt{A(R+1)}$. Then using (3.1) and the assumption that $\gamma'(s) \to \infty$ as $s \to 0$ we see that $\mathcal{M}_0 v \geq 0$ provided ε and $\sqrt{A(R+1)}$ are sufficiently small. We now examine the behaviour of v on $\partial\Omega$ near 0. If $x_n = W(x')$ then

$$\delta(x) = \frac{|x'|^2}{|x + Ae_n/2| + A/2 + W(x')}. \tag{3.5}$$

On the other hand, as $2AL \leq 1$, we have

$$\left(\frac{A}{2} - W(x')\right)^2 - \left(\frac{A}{2} + W(x')\right)^2 \leq |x'|^2$$

and so

$$\frac{A}{2} - W(x') \leq \left|x + \frac{A}{2}e_n\right|$$

and

$$A \leq \left|x + \frac{A}{2}e_n\right| + \frac{A}{2} + W(x').$$

Thus, if $x_n = W(x')$, $A\delta(x) \leq |x'|^2$. Consequently, if $x \in \partial\Omega$, $|x'| < \varepsilon^2$ and $x_n = w(x')$ then

$$v(x) \leq \gamma(|x'|) \leq \phi(x).$$

If $x \in \partial\Omega$ but $|x'| \geq \varepsilon^2$ or $x_n \neq w(x')$ then $\phi(x) \geq \min\{\varepsilon_0, \gamma(\varepsilon^2)\}$ while $v(x) \leq \gamma(\sqrt{A(R+1)})$. Hence, if we first choose ε sufficiently small and then A such that $\gamma(\sqrt{A(R+1)}) \leq \min\{\varepsilon_0, \gamma(\varepsilon^2)\}$, we have $\mathscr{M}_0 v \geq 0$ in Ω and $v \leq \phi$ on $\partial\Omega$. The comparison principle gives $v \leq u$ in Ω.

Finally we examine the growth of $v(x)$ near 0. We have, for $x \in \Omega$ and $|x| < A/4$,

$$\begin{aligned}\delta(x) &= \frac{|x'|^2 + (x_n - W(x')(A + x_n + W(x')))}{|x + Ae_n/2| + A/2 + W(x')}\\ &\geq \frac{|x'|^2 + (A/2)(x_n - (x'))}{|x'| + x_n + A + W(x')}\\ &\geq \frac{|x'|^2 + (A/2)(x_n - W(x'))}{A + 2|x'| + (x_n - W(x'))}.\end{aligned}$$

Thus

$$A\delta(x) \geq |x'|^2\left(1 - 2\frac{|x'|}{A}\right) + \frac{A}{8}(x_n - W(x')).$$

Proof of Theorem 2. Suppose $\varepsilon > 0$. We assume the situation as described before Lemma 2 and define β_1 and β by $r\operatorname{Cos}\beta_1(r) = \gamma(r\operatorname{Sin}\beta_1(r))$ and $r = \gamma(r\beta(r))$ so the graph of $\phi(x', w(x'))$ is above the graph given in polars by $|\theta| = \beta_1(r)$ near 0.

By Lemma 1 we have

$$\beta_1 \leq \beta. \tag{3.6}$$

We now define $\beta_2 = \beta/(1 - \varepsilon\beta)$ and $\beta_3 = \beta/(1 - 2\varepsilon\beta)$ then

$$\beta_2 \leq (1 - \varepsilon\beta)\beta_3. \tag{3.7}$$

Now suppose $\alpha < -(n-2)/2$ and define

$$v(y) = w(y) + \lambda f(y)$$

where f is the function given by part II of Proposition 1 with β replaced by β_3 and $\lambda > 0$. We consider v on the set $D = \{y \in \mathbf{R}^n\colon |\theta| < \beta_3(r), r < \delta\}$. We note that as in Theorem 1 the effect of changing β to β_3 is to change α but by an amount proportional to ε which can be taken as small as we like. Also as in Theorem 1 we see that provided

$$\lambda c \frac{\tilde{F}_\alpha(\delta)}{\delta \beta_3(\delta)} \leq 1, \qquad r < \delta, \tag{3.8}$$

and δ is sufficiently small we have $\mathscr{M}_0 v \leq 0$, $D_n v > 0$, $v > w$ and $v = w$ if $|\theta| = \beta_3(r)$. Thus there is a neighbourhood G of 0 and a function $\tilde{v}$ defined on $\overline{G \cap \Omega}$ such that $\mathscr{M}\tilde{v} \geq 0$ in $G \cap \Omega$, graph of $\tilde{v}$ over $G \cap \partial\Omega$ = graph (in polars) of $|\theta| = \beta_3(r)$ which lies below the graph of $\phi(x)$ and $\tilde{v}(x) = \sqrt{\delta^2 - |x'|^2} \leq \delta$ on $\partial G \cap \Omega$. Also

$$\partial G \cap \Omega = \{x \colon x_n = w(x') + \lambda f(\delta, \theta), |\theta| \leq \beta_3(\delta)\}.$$

In order to apply the comparison principle in $G \cap \Omega$ we want to ensure that $u \geq \tilde{v}$ on $\partial G \cap \Omega$. But $\tilde{v} \leq \delta = \gamma(\delta\beta(\delta))$ so that by Lemma 2 we require, on $\partial G \cap \Omega$,

$$|x'|^2(1 - C_1|x'|) + C_2(x_n - w(x')) \geq \delta^2\beta^2(\delta).$$

That is we require

$$\delta^2 \operatorname{Sin}^2\theta(1 - C_1\delta|\operatorname{Sin}\theta|) + C_2\lambda f(\delta, \theta) \geq \delta^2\beta^2(\delta) \tag{3.9}$$

for $|\theta| \leq \beta_3(\delta)$.

Now, if δ sufficiently small,

$$\delta^2 \operatorname{Sin}^2\theta(1 - C_1\delta|\operatorname{Sin}\theta|) \geq \frac{\delta^2\theta^2}{(1 + \varepsilon\theta)^2}$$

and so (3.9) holds for

$$|\theta| \geq \frac{\beta(\delta)}{1 - \varepsilon\beta(\delta)} = \beta_2(\delta).$$

On the other hand if $|\theta| \leq \beta_2(\delta) \leq (1 - \varepsilon\beta(\delta))\beta_3(\delta)$, by (3.7), then by the definition of f given in (1.7) we have $f(\delta, \theta) \geq C_3\varepsilon\beta(\delta)\tilde{F}_\alpha(\delta)$ for some constant $C_3 > 0$. Consequently (3.9) holds provided

$$C\lambda\beta(\delta)\tilde{F}_\alpha(\delta) \geq \delta^2\beta^2(\delta). \tag{3.10}$$

It is clear that by taking δ small and then making a suitable choice of λ we can satisfy both (3.8) and (3.10) and so obtain that $u \geq \tilde{v}$ in $G \cap \Omega$. Finally we note that if $|y'| = 0$, and so $\theta = 0$, we have $f(r, 0) = c\tilde{F}_\alpha(r)$. That is $v(re_n) \leq c\tilde{F}_\alpha$ and so by the definition of $\tilde{v}$, $\tilde{v}(te_n) \geq \tilde{F}_\alpha^{-1}(ct)$.

References

[GG] M. Giaquinta and E. Giusti, *Global $C^{1,\alpha}$-regularity for second order quasilinear elliptic equations in divergence form*, J. Reine Angew. Math., **351** (1984), 55–65.

[GT] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Heidelberg-New York 1977.

[G1] E. Giusti, *Boundary behaviour of non-parametric minimal surfaces*, Indiana Univ. Math. J., **22** (1972), 435–444.

[G2] ______, *Minimal Surfaces and Functions of Bounded Variation*, Birkhaüser-Boston. Inc. 1984.

[JS] H. Jenkins and J. Serrin, *The Dirichlet problem for the minimal surface equation in higher dimensions*, J. Reine Angew. Math., **229** (1968), 170–187.

[L1] G. Lieberman, *The quasilinear Dirichlet problem with decreased regularity at the boundary*, Comm. Partial Differential Equations, **6** (1981), 437–497.

[L2] ______, *The Dirichlet problem for quasilinear elliptic equations with Hölder continuous boundary values*, Arch. Rat. Mech. Anal., **79** (1982), 305–323.

[M] M. Miranda, *Un principio di massimo forte per le frontiere minimali e sua appliazione alle superfici di area minima*, Rend. Sem. Mat. Univ. Padova, **45** (1971), 255–36.

[S] L. Simon, *Boundary behaviour of solutions of the nonparametric least area problem*, Bull. Austral. Math. Soc., **26** (1982), 17–27.

[W1] G. Williams, *Global regularity for solutions of the minimal surface equation with continuous boundary values*, Ann. de L'Inst. Henri Poincaré. Non. Anal., to appear.

[W2] ______, *Solutions of the minimal surface equation continuous and discontinuous at the boundary*, preprint.

Received December 12, 1985.

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