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PSEUDOGROUPS OF C^1 PIECEWISE PROJECTIVE HOMEOMORPHISMS

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The group $\mathrm{PSL}_2\mathbf{R}$ acts transitively on the circle $S^1 = \mathbf{R} \cup \infty$, by linear fractional transformations. A homeomorphism $g\colon U \to V$ between open subsets of R is called C^1, *piecewise projective* if g is C^1, and if there is some locally finite subset S of U such that, on each component of $U - S$, g agrees with some element of $\mathrm{PSL}_2\mathbf{R}$. Let $\Gamma_{\mathbf{R}}$ be the pseudogroup of such homeomorphisms. We show that the Haefliger classifying space $B\Gamma_{\mathbf{R}}$ is simply connected, and that there is a homology isomorphism $i\colon B\widetilde{\mathrm{PSL}_2\mathbf{R}} \to B\Gamma_{\mathbf{R}}$. ($\widetilde{\mathrm{PSL}_2\mathbf{R}}$ is the universal cover of $\mathrm{PSL}_2\mathbf{R}$, considered as a discrete group.) As a consequence, the classifying space of the discrete group of compactly supported, C^1 piecewise projective homeomorphisms of R is a "homology loop space" of $B\widetilde{\mathrm{PSL}_2\mathbf{R}}$.

1.1. Introduction. More generally, let $F \subset \mathbf{R}$ be a subfield of $\mathbf{R}$. PSL_2F acts on the circle $\mathbf{R} \cup \infty$. The orbit of $1 \in F$ is $F \cup \infty$.

1.2. DEFINITION. Γ_F is the pseudogroup of C^1 homeomorphisms $g\colon U \to V$ between open subsets of $\mathbf{R}$, so that there is some locally finite subset S of $U \cap (F \cup \infty)$ such that, on each connected component of $U - S$, g agrees with some element of PSL_2F.

The set of restrictions of elements of PSL_2F to open subsets of $\mathbf{R}$ forms a subpseudogroup of Γ_F whose classifying space, the total space of the circle bundle over $B\mathrm{PSL}_2F$, is homotopy equivalent to $B\widetilde{\mathrm{PSL}_2F}$, where $\widetilde{\mathrm{PSL}_2F}$ is defined as the pullback

$$\begin{array}{ccc} \widetilde{\mathrm{PSL}_2F} & \to & \widetilde{\mathrm{PSL}_2\mathbf{R}} \\ \downarrow & & \downarrow \\ \mathrm{PSL}_2F & \to & \mathrm{PSL}_2\mathbf{R} \end{array}$$

Therefore, there is an inclusion map $i\colon B\widetilde{\mathrm{PSL}_2F} \to B\Gamma_F$.

1.3. THEOREM. *$\pi_1 B\Gamma_F = 0$, and i is a homology equivalence.*

1.4. DEFINITION. The *group of compactly supported Γ_F homeomorphisms*, denoted K_F, is the group of elements of Γ_F which are compactly supported homeomorphisms of the line $\mathbf{R}$.

Following Segal's proof [**Se2**] of an extension of Mather's theorem [**Ma**] we find:

1.5. Proposition. *There is a homology equivalence* $BK_F \to \Omega B\Gamma_F$.

The proof of 1.5 involves the construction of a homology fibration [**McS**] $BK_F \to M \to B\Gamma_F$ where M is contractible. Pulling this fibration back over $B\widetilde{\mathrm{PSL}_2 F}$ by the inclusion i of 1.3 we obtain:

1.6. Corollary. *There is a homology fibration* $BK_F \to E \to B\widetilde{\mathrm{PSL}_2 F}$ *where* E *is acyclic, and the fundamental group of* $B\widetilde{\mathrm{PSL}_2 F}$ *acts trivially on the homology of the fiber.*

1.7. *Organization.* In §2 Theorem 1.3 is proved, as an application of Corollary 1.10 of [**G2**]. In §3, 1.5 is proved, using a generalization of Segal's proof [**Se2**] of a generalization of Mather's theorem [**Ma**]. The generalization is outlined in §4.

2. Proof of 1.3. One may think of Γ_F as constructed from the action of $\mathrm{PSL}_2 F$ on S^1 by adding C^1 singularities at isolated points of F. As a consequence, 1.10 of [**G2**] says that $B\Gamma_F$ is weakly homotopy equivalent to the direct limit of the diagram

$$\begin{array}{ccccc} BA & \xrightarrow{j} & BG^P & \xrightarrow{l} & BA \\ \downarrow & & \downarrow r & & \\ B\widetilde{\mathrm{PSL}_2 F} & & BA & & \end{array} \tag{2.1}$$

where A is the discrete group of germs of projective maps fixing 0, and G^P is the discrete group of germs of Γ_F maps fixing 0. The map j is inclusion, and l and r arise from the fact that an element of G^P, restricted to the left or right side of 0, can be identified with an element of A. Theorem 1.3 will follow from an analysis of diagram (2.1).

Let F^+ be the positive, nonzero squares of F, considered as a group under multiplication. It is well known that A is a subgroup of the one-dimensional affine group of F, an extension $F \to A \xrightarrow{d} F^+$ where F^+ acts on F by multiplication. Since d: $A \to F^+$ is the derivative map, G^P is the pullback

$$\begin{array}{ccc} G^P & \xrightarrow{l} & A \\ r\downarrow & & \downarrow d \\ A & \xrightarrow{d} & F^+ \end{array}$$

and therefore G^P is an extension $F^2 \to G^P \to F^+$, with F^+ acting on F^2 by multiplication: $f(a,b) = (fa, fb)$.

Let R be the pushout of

$$\begin{array}{ccc} BG^P & \xrightarrow{l} & BA \\ \downarrow r & & \\ BA & & \end{array}$$

2.2. LEMMA. *The inclusion* j: $BA \to BG^P$ *induces a homology equivalence* $BA \to R$.

Assuming 2.2 for now, we prove 1.3.

By 2.2 and 2.1 it is clear that $B\widetilde{\mathrm{PSL}_2F} \to B\Gamma_F$ is a homology equivalence. It remains to show that $\pi_1 B\Gamma_F = 0$.

We first compute $\pi_1 R$. By Van Kampen's theorem, $\pi_1 R = A \times_{G^P} A$. Elements in either A factor with derivative 1 are equal to 1 in $\pi_1 R$. On the other hand, $\pi_1 R \twoheadrightarrow F^+$. It follows that $\pi_1 R$ is isomorphic to F^+.

Now by (2.1), $\pi_1 B\Gamma_F \simeq \widetilde{\mathrm{PSL}_2F} \times_A F^+$, which is isomorphic to $\widetilde{\mathrm{PSL}_2F}$ modulo the normal subgroup $N(F)$ generated by the subgroup F of $\widetilde{\mathrm{PSL}_2F}$. We now show that $N(F)$ is all of $\widetilde{\mathrm{PSL}_2F}$.

Consider PSL_2F acting on $S^1 = \mathbf{R}/\mathbf{Z}$, and $\widetilde{\mathrm{PSL}_2F}$ as acting on $\mathbf{R}$, so that A is the subgroup of $\widetilde{\mathrm{PSL}_2F}$ fixing each integer. Since [**La**] PSL_2F is simple, to show that $N(F) = \widetilde{\mathrm{PSL}_2F}$, it suffices to prove that $N(F)$ contains the translation t: $x \mapsto x + 1$.

In fact, $N(\mathbf{Z})$ contains t. For $\widetilde{\mathrm{PSL}_2F}$ contains $\widetilde{\mathrm{PSL}_2\mathbf{Z}}$ as a subgroup, which contains t. Further, $\widetilde{\mathrm{PSL}_2\mathbf{Z}}$ is generated by a, b with $a^2 = b^3$, and $\mathbf{Z}$ is generated by $a^{-1}b$. Now $a(a^{-1}b)a^{-1} = ba^{-1}$, and $(ba^{-1})(a^{-1}b) = b$, so $N(\mathbf{Z}) \supset \widetilde{\mathrm{PSL}_2\mathbf{Z}}$, and contains t.

Proof of Lemma 2.2. In fact, we show that the derivative maps $A \to F^+$, $G^P \to F^+$ induce isomorphisms on homology (and, therefore, because $\pi_1 R = F^+$, that

$$\begin{array}{ccc} BG^P & \xrightarrow{l} & BA \\ r\downarrow & & \downarrow \\ BA & \to & BF^+ \end{array}$$

is both a pullback and a pushout). Considering the Serre spectral sequences of the extensions $F \to A \to F^+$ and $F^2 \to G^P \to F^+$, it suffices to prove that the groups $H_p(F^+; H_qF^2)$, $H_p(F^+; H_qF)$ are null for $q > 0$. The proof is essentially that of the "center kills" lemma [**Sa**].

The element $4 \in F^+$ acts on H_qF and H_qF^2 by multiplication by 4^q. Let this isomorphism (H_qF and H_qF^2 are divisible and torsion free) be denoted e_q. Then $e_q - 1$ is also an isomorphism of H_qF and H_qF^2, namely multiplication by $4^q - 1$. Both e_q and $e_q - 1$ induce the identity maps of $H_p(F^+; H_qF)$, $H_p(F^+; H_qF)$. Thus the latter groups must be zero.

3. Proof of 1.5.

In §4 we outline a proof of the following fact:

4.8. PROPOSITION. *Let Γ be a pseudogroup of orientation preserving homeomorphisms of $\mathbf{R}$. Let K be the discrete group of elements of Γ which are compactly supported homeomorphisms of $\mathbf{R}$. Assume that the orbit of any element of $\mathbf{R}$ under Γ is dense in $\mathbf{R}$. Further, assume:*

(3.1) *Suppose g is the germ of an element of Γ with domain $x \in \mathbf{R}$, and let $t \in \mathbf{R}$ such that $t > x, gx$ (or $t < x, gx$). Then there is an element $\bar{g} \in \Gamma$ whose domain is connected and includes t and x, and such that $\bar{g} \equiv \mathrm{id}$ near t, $\bar{g} \equiv g$ near x.*

Then there is a homology equivalence $BK \to \Omega B\Gamma$.

To prove 1.5, therefore, we must verify condition 3.1 for the pseudogroups Γ_F. We rephrase 3.1 as the following lemma, using the fact that F is dense in $\mathbf{R}$.

3.2. LEMMA. *Let $g \in \mathrm{PSL}_2F$, $x \in F$, and assume that $g(x) \neq \infty$.*

(a) *Let $z = \max(x, gx)$. Let $\varepsilon > 0$. Then there is some ε', $0 < \varepsilon' < \varepsilon$, $\delta > 0$, and $s \in \Gamma_F$ with domain $(x - 2\varepsilon', \infty)$ such that $s(t) = gt$, $t \leq x + \delta$, and $s(t) = t$, $t \geq z + \varepsilon'$.*

(b) *Let $z = \min(x, gx)$. Let $\varepsilon > 0$. Then there is some ε', $0 < \varepsilon' < \varepsilon$, $\delta > 0$, and an $s \in \Gamma_F$ with domain $(-\infty, x + 2\varepsilon')$ such that $s(t) = gt$, $t \geq x - \delta$, $s(t) = t$, $t \leq z - \varepsilon'$.*

For the proof we first recall some facts about PSL_2F. A circle in the upper half plane which is tangent to the x-axis is called a horocycle. The action of PSL_2F on $\mathbf{R} \cup \infty$ extends to an action on the upper half plane which takes horocycles to horocycles. Let $f \in F$. The subgroup $T_f \subset \mathrm{PSL}_2F$ of elements which fix f and have unit derivative at f takes each horocycle at f to itself. T_f is isomorphic to the translation group F and acts transitively on $(F \cup \infty)/f$.

We prove 3.2(a); the proof of 3.2(b) follows in parallel.

Assume that $x \geq gx$ so that $z = x$. If this is not true, simply follow the proof for the germ of g^{-1} at gx. Pick $\varepsilon' \in F$, $0 < \varepsilon' < \varepsilon$, so that g is noninfinite on the interval $(x - 2\varepsilon', x + 2\varepsilon')$. Let $y = x + \varepsilon'$. There are three cases.

(i) $y = gy$. In this case pick ε' slightly smaller so as to drop to case (ii) or (iii).

(ii) $y > gy$ (Fig. 3.3). Let H be a horocycle tangent to y, and let gH be its image, tangent to gy. Pick $a_1 \in F$, $gx < a_1 < gy$, close enough to gy, and pick $h \in T_{a_1}$ so that hgy is large enough, so that the base $a_2(a_1, h)$ of the horocycle C tangent to hgH, H and $\mathbf{R}$ (and to the left of H) is between gy and y. Pick h' belonging to the subgroup of $\mathrm{PSL}_2 F$ fixing the horocycles based at a_2, and so that $h'y = hgy$. Then $h'H = hgH$, so that $h'^{-1}hg \in T_y$. Consequently, $a_2 \in F$ and $h' \in \mathrm{PSL}_2 F$.

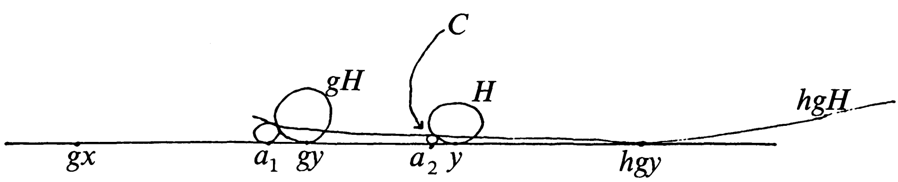

FIGURE 3.3

Now define

$$s(t) = \begin{cases} g(t), & t \leq g^{-1}a_1, \\ hg(t), & g^{-1}a_1 \leq t \leq (hg)^{-1}a_2, \\ h'^{-1}hg(t), & (hg)^{-1}a_2 \leq t \leq y, \\ t, & t \geq y. \end{cases}$$

By construction, $s \in \Gamma_F$.

(iii) $gy > y$ (Fig. 3.4). Let $a_0 = g(x + \delta)$, $\delta = (y - gx)/10$, and let $k \in T_{a_0}$ so that $kgy < y$. Let H be a horocycle tangent to y, and let kgH be its image at kgy. Pick $a_1 \in F$, $a_1 < kgy$ close enough to kgy, and pick

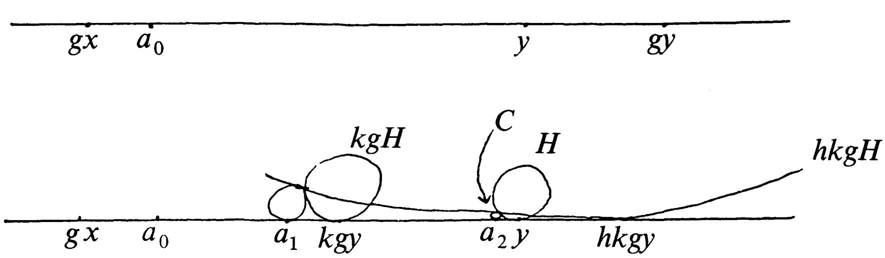

FIGURE 3.4

$h \in T_{a_1}$ so that $hkgy$ is large enough, so that the base $a_2(a_1, h) < y$ of the horocycle C tangent to H, $hkgH$ and $\mathbf{R}$ (and left of H) is between kgy and y. Let $h' \in T_{a_2}$ so that $h' = hkgy$. Note then that $h'H = hkgH$, so that $h'^{-1}hkg \in T_y$. One can show that $a_2 \in F$, $h' \in \mathrm{PSL}_2 F$. Then define

$$s(t) = \begin{cases} g(t), & t \leq x + \delta, \\ kg(t), & x + \delta \leq t \leq (kg)^{-1}a_1, \\ hkg(t), & (kg)^{-1}a_1 \leq t \leq (hkg)^{-1}a_2, \\ h'^{-1}hkg(t), & (hkg)^{-1}a_2 \leq t \leq y, \\ t, & t \geq y. \end{cases}$$

By construction, $s \in \Gamma_F$.

4. Groups of compactly supported homeomorphisms. In this section we specify a condition on a pseudogroup which allows one to mimic Segal's proof [**Se2**] of a generalization of Mather's theorem [**Ma**]. We work in the context of groupoids of homeomorphisms. References for topological categories are [**Se1**], [**Se3**].

4.1. DEFINITION. A *groupoid* Γ *etale over* $\mathbf{R}$ is a topological groupoid Γ whose space of objects is $\mathbf{R}$, in which the domain and range maps D, R: $\Gamma \to \mathbf{R}$ are locally homeomorphisms (abusing notation, we let Γ denote the space of morphisms of the topological groupoid Γ).

Given a pseudogroup Γ *on* $\mathbf{R}$, *one can construct an associated* groupoid Γ etale over $\mathbf{R}$, whose space of morphisms is the sheaf of germs of elements of the pseudogroup. Taking the geometric realization (in the "thick" sense of [**Se1**], App.) of the nerve of the groupoid, we *obtain a* classifying space $B\Gamma$, which is weakly homotopy equivalent to the classifying space of the pseudogroup.

We make the following assumption throughout §4 of the paper. Let Γ be a groupoid of homeomorphisms of $\mathbf{R}$.

4.2. *Assumption.* (a) For any $x \in \mathbf{R}$ the orbit of x under Γ is dense in $\mathbf{R}$.

(b) If $g \in \Gamma$, and $t < Dg, Rg$ (or $t > Dg, Rg$) then there is a section $s: U \to \Gamma$ of the domain map, over an open interval U containing Dg and t, *such that* $s(Dg) = g$, and $s(t) = \mathrm{id}_t$.

The following proposition gives what is needed to mimic Segal's proofs.

4.3. PROPOSITION. (a) *Let $a < b < c < d$, so that a and b, and likewise c and d, are in the same Γ-orbit. Then there is a section $s:[a, d] \to \Gamma$ of D so that $Rs(a) = b$, $Rs(d) = c$.*

(b) *If $a < b < c < d$, $\varepsilon > 0$ there is a section $s:[a, d] \to \Gamma$ of D so that $s(a) = \mathrm{id}_a$, $s(d) = \mathrm{id}_d$, and $|Rs(b) - a| < \varepsilon$, $|Rs(c) - d| < \varepsilon$.*

Proof. (a) Let $s_1 \in \Gamma$, with $Ds_1 = a$ and $Rs_1 = b$, and $s_2 \in \Gamma$ so that $Ds_2 = d$, $Rs_2 = c$. Then 4.2 guarantees a section s of D, over some interval containing $[a, d]$, so that $s(a) = s_1$, $s(d) = s_2$, and $s|_{(b+\varepsilon, c-\varepsilon)} \equiv \mathrm{id}$.

(b) Let $s_1 \in \Gamma$ so that $Ds_1 = b$, $Rs_1 \in (a, a + \varepsilon)$, and $Rs_1 < b$, and let $s_2 \in \Gamma$ with $Ds_2 = c$, $Rs_2 \in (d - \varepsilon, d)$ and $Rs_2 > c$. Then 4.2 guarantees a section s of D, over some interval containing $[a, d]$, so that $s(a) = \mathrm{id}_a$, $s(d) = \mathrm{id}_d$, $s(b) = s_1$, $s(c) = s_2$, and $s|_{(b+\varepsilon, c-\varepsilon)} \equiv \mathrm{id}$.

Let $X \subset Y$ be open intervals such that $\partial\overline{X} \cap \partial\overline{Y} = \varnothing$, and such that $\partial\overline{X} \cup \partial\overline{Y}$ is contained in a single Γ orbit.

4.4. DEFINITION.

$M(Y) = \{m: Y \to \Gamma: m \text{ continuous}, Dm = \mathrm{id}, RmY \subseteq Y\}$

$M(Y, X) = \{m \in M(Y): RmX \subseteq X\}$

$M(\overline{Y}) = \{m: \overline{Y} \to \Gamma: Dm = \mathrm{id}, Rm\overline{Y} \subseteq \overline{Y}, m \text{ continuous}\}$

$M(\overline{Y}, X) = \{m \in M(\overline{Y}): RmX \subseteq X\}$

These four sets are monoids of embeddings of Y; give them the discrete topology. Notice that $M(\overline{Y})$ is the monoid of embeddings of $\overline{Y}$, with a germ of an extension to a neighborhood of $\overline{Y}$. As a consequence of 4.3(a) and **[G1]**, 2.8 there is a weak homotopy equivalence $BM(Y) \to B\Gamma$.

There are extension and restriction homomorphisms

$$M(Y) \overset{i}{\leftarrow} M(Y, X) \overset{r}{\rightarrow} M(X)$$

$$M(\overline{Y}) \overset{\bar{i}}{\leftarrow} M(\overline{Y}, X) \overset{\bar{r}}{\rightarrow} M(\overline{X})$$

4.5. PROPOSITION. *The homomorphisms i, $\bar{i}$, r, $\bar{r}$ induce homotopy equivalences of classifying spaces.*

Proof. Follow **[Se2]**, 2.7.

4.6. PROPOSITION. *The restrictions $M(\overline{Y}, X) \to M(Y, X)$ and $M(\overline{X}) \to M(X)$ induce homotopy equivalences of classifying spaces.*

Proof. Following Segal, consider the sequence of homomorphisms $M(\overline{Y}, X) \to M(Y, X) \to M(\overline{X}) \to M(X)$. Note that the composition of any two arrows induces a homotopy equivalence of classifying spaces, by 4.5. The result follows.

4.7. Definition. $K(X) = \{g \in M(\overline{X}) : Rg\overline{X} = \overline{X}$, and $g|_{\partial\overline{X}} = \mathrm{id}\}$. $K(X)$ is the group of Γ-homeomorphisms with compact support in X.

4.8. Proposition. *There is a homology equivalence* $BK(X) \to \Omega B\Gamma$.

Proof. Follow 2.11 in **[Se2]**, where, in fact, a homology fibration $K(X) \to M \to B\Gamma$ is constructed, with M contractible.

4.9. Corollary. *There is a homology equivalence* $BK(\mathbf{R}) \to \Omega B\Gamma$.

Proof. We construct a continuous section of the domain map $s : \mathbf{R} \to \Gamma$ so that Rs is a Γ-homeomorphism from $\mathbf{R}$ onto X, conjugating $K(\mathbf{R})$ to $K(X)$. Let $x_n, y_n, n \in \mathbf{Z}$, be members of a single Γ-orbit such that (i) $x_n < x_{n+1}$, $y_n < y_{n+1}$, $n \in \mathbf{Z}$, and (ii) $\bigcup_n(y_{-n}, y_n) = X$, $\bigcup_n(x_{-n}, x_n) = \mathbf{R}$. Further, we assume that $x_0 = y_0$, that $x_n > y_n$ for $n > 0$, and that $x_n < y_n$ for $n < 0$.

Because the x_n and y_n belong to a single orbit, there are $s_n \in \Gamma$ with $Ds_n = x_n$, $Rs_n = y_n$; we take $s_0 = \mathrm{id}$. Define s so that $s(x_n) = s_n$, as follows. Suppose $n \geq 0$. By 4.2 there is a continuous section $f : [x_n, x_{n+1}] \to \Gamma$ of the domain map such that $f(x_n) = s_n$, $f(x_{n+1}) = \mathrm{id}$. Also, there is a continuous section of the domain map $g : [y_n, x_{n+1}] \to \Gamma$ such that $g(y_n) = \mathrm{id}$, $g(x_{n+1}) = s_{n+1}$. Define s to be $g \circ f$ on $[x_n, x_{n+1}]$; note that $s(x_n) = s_n$ and $s(x_{n+1}) = s_{n+1}$. Similarly, define s on the intervals $[x_n, x_{n+1}]$ for $n < 0$.

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