PACIFIC JOURNAL OF MATHEMATICS
Vol. 129, No. 1, 1987

THE HARMONIC REPRESENTATION OF $U(p,q)$ AND ITS CONNECTION WITH THE GENERALIZED UNIT DISK

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In this paper we study the very close connection between the kth tensor product of the harmonic representation ω of $U(p,q)$ and the generalized unit disk $\mathscr{D}$. We give a global version of ω realized on the Fock space as an integral operator. Each irreducible component of ω is shown to be equivalent in a natural way to a multiplier representation of $U(p,q)$ acting on a Hilbert space $\mathscr{H}(\mathscr{D},\lambda)$ of vector-valued holomorphic functions on $\mathscr{D}$. The intertwining operator between these realizations is then explicitly constructed. We determine necessary and sufficient conditions for square integrability of each component of ω and in this case derive the Hilbert space structure on $\mathscr{H}(\mathscr{D},\lambda)$.

Introduction. Of interest here are the diverse roles the generalized unit disk plays in the constructions mentioned above. Our principal objective is to give a disk picture realization of all $U(p,q)$ highest weight modules. This is done in §3. Further, we are interested in their unitary structure. We will say more on that later.

In the literature various versions of $U(p,q)$ highest weight modules appear. Typical are constructions involving the Siegal upper half plane [**4, 8**] or the open set of positive p-planes in the Grassmannian [**12**]. More recently, Patton and Rossi [**13**] have used cohomological methods to realize these modules and the Penrose transform has related these to other constructions (cf. also [**12, 14**]). Most notable, however, is the paper of Kashiwara and Vergne [**8**]. There they decompose ω (we will use ω to mean the kth tensor product of the standard Segal-Shale-Weil representation of $U(p,q)$) and produce, as they conjectured, all highest weight $U(p,q)$ modules on a Schroedinger-Fock space (cf. [**2, 7**]). In their version ω is constructed by determining its action on certain subgroups whose product is dense in $U(p,q)$. Together these actions lead to a unitary representation of the whole group. Their main results are the decomposition of ω into its irreducible components ω_λ, $\lambda \in \Lambda \subseteq U(k)\,\hat{}$, and an explicit description of Λ in terms of the signature of irreducible representations of the dual group $U(k)$.

In §1, we show how a direct and global version of ω can be realized in a variant of the Fock Space over $\mathbf{C}^{n\times k}$, where $n = p + q$. The generalized unit disk plays an important role here. For $T \in \mathscr{D}$ we introduce a function $q_T \in \mathscr{F}$ invariant under the right action of $U(k)$. In fact, $\{q_T: T \in \mathscr{D}\}$ generates all the invariants. For $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$ we show

$$\omega(g)f(z) = \frac{q_{BD^{-1}}(z)}{\det A^k} \int f(w)\bar{q}_{A^{-1}B}(w)\overline{K}\begin{pmatrix} w, & \begin{matrix} A^{-1}z_1 \\ D^{-1}z_2 \end{matrix} \end{pmatrix} d\mu(w).$$

is a continuous unitary representation (cf 1.12).

The orthogonal complement of the ideal generated by the $U(k)$-invariants is the space of harmonics $\mathscr{H}$. Based on 2.3 the decomposition of ω is reduced to the decomposition of $\omega|_K$ on H, where K is the maximal compact subgroup $U(p) \times U(q)$. In [8] a similar space is defined. Our proof that the λ-isotypic component in $\mathscr{H}$ is irreducible under $K \times U(k)$, $\lambda \in U(k)^\wedge$, differs however. Here, we are able to exploit the role of the generalized unit disk.

In §3 we construct all $U(p,q)$ highest weight modules as Hilbert spaces $\mathscr{H}(\mathscr{D}, \lambda)$ of vector valued holomorphic functions on $\mathscr{D}$. This construction is based on the relation of a kernel function Q on $\mathscr{D}$ to the inner product in the Fock space. Namely, for S, $T \in \mathscr{D}$ and h, $f \in \mathscr{H}$

$$(q_T h \mid q_S f) = (Q(S,T)h \mid f)$$

(cf. 3.1). The positivity of Q follows immediately from this formula. The results of Kunze **[11]** apply to yield the Hilbert spaces desired. We further show that the map $q_T h \to Q(\cdot, T)h$ extends to a unitary operator intertwining T_λ and ω_λ. This extension is expressed globally as an integral operator in 3.7.

In §4 necessary and sufficient conditions are determined on the Kashiwara-Vergne parameter λ for ω_λ to be in the discrete series. We exploit the role of $\mathscr{D}$ to an even greater extent than before. In this case we determine globally the unitary structure of $\mathscr{H}(\mathscr{D}, \lambda)$.

Finally, we mention that Inoue **[6]** has constructed a series of irreducible unitary representations of $U(p,q)$ which generalizes the limits of the discrete series constructed by Knapp and Okamoto. The representation spaces are highest weight modules and are realized as vector-valued holomorphic functions on $\mathscr{D}$. Hence they appear in our constructions. In fact we can describe them in terms of the Kashiwara-Vergne parametrization (cf. 2.9) as follows: Let $k = n - i$, where $1 \le i \le \min(p,q)$ and $n = p + q$. Let $\lambda \in U(n-i)^\wedge$ have signature $(m_1, \dots, m_{p-i}, 0, \dots, 0, -n_{q-i}, \dots, -n_1)$. Then $\mathscr{H}(\mathscr{D}, \lambda)$ is a generalized limit of the discrete series in the sense of Inoue if and only if λ is of the above form. In this case the inner product is given in a form similar to 4.5. However, the integral is over the ith boundary component of $\mathscr{D}$ and $Q(T,T)$ is replaced by a positive operator on the ith boundary component.

This work is in essence my doctoral dissertation. I would like to express my gratitude and respect to Professor Ray A. Kunze for his guidance.

1. Preliminaries. In this section we set down some salient facts about $U(p,q)$ which are used throughout this paper. Our objective is to globally define the harmonic representation ω of $U(p,q)$ on the Fock space. To do this we introduce the Heisenberg group and its essentially only infinite dimensional representation.

Let p, $q > 0$ be integers and let $n = p + q$. For $g \in \mathrm{GL}(n, \mathbf{C})$, where $\mathbf{C}$ denotes the field of complex numbers, we will frequently write g in block form as

$$g = \begin{array}{c} p\{ \\ q\{ \end{array} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Let $I_{p,q} = \left(\begin{smallmatrix} I_p & 0 \\ 0 & -I_q \end{smallmatrix}\right)$. We define

$$U(p,q) = \left\{ g \in \mathrm{GL}(n,C) \colon g I_{p,q} g^* = I_{p,q} \right\},$$

where * denotes the conjugate transpose. Throughout this paper we will denote $U(p,q)$ by G. For $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$, the defining condition of $U(p,q)$ implies the following relations:

$$(1.1)\quad \begin{array}{llcll} (1) & AA^* - BB^* = I_p & & (5) & C = DB^*A^{*-1} \\ (2) & CC^* - DD^* = -I_q & & (6) & B = AC^*D^{*-1} \\ (3) & A^*A - C^*C = I_p & & (7) & C = D^{*-1}B^*A \\ (4) & B^*B - D^*D = -I_q & \text{and} & (8) & B = A^{*-1}C^*D. \end{array}$$

Let $K = G \cap U(n)$. Then K is a maximal compact subgroup of G isomorphic to $U(p) \times U(q)$. Let $\mathscr{D} = \mathscr{D}_{p,q} = \{T \in C^{p\times q}\colon 1 - TT^* > 0\}$, where > 0 denotes positive definite. Then $\mathscr{D}$ is a bounded complex domain open in $\mathbf{C}^{p\times q}$. The map $G/K \to \mathscr{D}$ defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} K \to BD^{-1}$$

is a homeomorphism and the natural action of G on $\mathscr{D}$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot T = (AT + B)(CT + D)^{-1}.$$

The domain $\mathscr{D}$ plays a crucial role in the analysis of the harmonic representation.

Now, let $\mathscr{S} = \mathbf{C}^{n\times k}$, where $n = p + q$ and $k > 0$ is an integer. We will frequently write $z \in \mathscr{S}$ in the form

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

where $z_1 \in \mathbf{C}^{p\times k}$ and $z_2 \in \mathbf{C}^{q\times k}$. We define an inner product $(\cdot\,|\,\cdot)$ on $\mathscr{S}$ by $(z\,|\,w) = \mathrm{tr}(zw^*)$. Let σ be the real form on $\mathscr{S}$ defined by $\sigma(z,w) = \mathrm{Im}(I_{p,q}z\,|\,w)$ for z, $w \in \mathscr{S}$. It is easy to see that σ is nondegenerate and skew-symmetric. Let $H = \mathscr{S} \times \mathbf{R}$, where $\mathbf{R}$ is the set of real numbers. We equip H with the product defined by

$$(z,s)(w,t) = (z + w,\ s + t + \sigma(z,w)).$$

This makes H a group, the Heisenberg group.

The essentially only infinite dimensional irreducible representation ρ of H can be realized in the following way. Let f be a complex-valued function on $\mathscr{S}$. We say f is (p,q) holomorphic if $z_1 \to f\binom{z_1}{z_2}$ is holomorphic for $z_2 \in \mathbf{C}^{q\times k}$, and $z_2 \to f\binom{z_1}{z_2}$ is conjugate holomorphic for $z_1 \in \mathbf{C}^{p\times k}$. Let $\mathscr{F} = \mathscr{F}_{p,q} = \{f\colon\ \mathscr{S} \to \mathbf{C}\colon\ f$ is (p,q) holomorphic and $\int_{\mathscr{S}}|f(z)|^2\,d\mu(z) < \infty\}$, where $d\mu(z) = \mu(z)\,dz$, $\mu(z) = e^{-\pi|z|^2}$, is normalized so that $\int_{\mathscr{S}} e^{-\pi|z|^2}\,dz = 1$. Then $\mathscr{F}$ is a Hilbert space, known as the Fock space and the reproducing kernel K is given by

$$K(z,w) = e^{\pi(z_1|w_1)}e^{\pi(w_2|z_2)}.$$

The representation ρ of H defined by

$$\rho(w,t)f(z) = e^{-\pi it}K(z,w)\mu^{1/2}(w)f(z - w), \tag{1.2}$$

$(w,t) \in H$, $z \in \mathscr{S}$ and $f \in \mathscr{F}$, defines a continuous unitary representation of H on $\mathscr{F}$, such that $\rho(0,t) = e^{-\pi it}I$, for all $t \in \mathbf{R}$. Furthermore, it is well known that ρ is irreducible and has square integrable matrix entries over $\mathscr{S}$.

LEMMA. *Let $A \in \mathrm{GL}(m,\mathbf{C})$ be such that $A + A^* > 0$. Then*

$$\int_{\mathbf{C}^{m\times n}} e^{-\pi(Az|z)}\,dz = \frac{1}{(\det A)^n}.$$

Proof. The lemma is clear for $A > 0$ by making the change of variable $z \to (A^{1/2})^{-1}z$. Then proceed by analytic continuation to the set $\{A \in \mathrm{GL}(m,\mathbf{C})\colon\ A + A^* > 0\}$. $\square$

Let $T \in \mathscr{D}$. We define $q_T \in \mathscr{F}$ by the formula

$$q_T(z) = e^{\pi(z_1 | T z_2)}.$$

It is clear that q_T is (p,q) holomorphic. Furthermore, the following proposition shows that $\|q_T\| < \infty$.

1.4. PROPOSITION. *Let $T \in \mathscr{D}$. Then*

$$\|q_T\|^2 = \frac{1}{\det(1 - TT^*)^k}.$$

Proof. Let $T \in \mathscr{D}$. Then

$$\|q_T\|^2 = \int_{\mathscr{S}} e^{\pi(z_1 | T z_2)} e^{\pi(T z_2 | z_1)}\, d\mu(z) = \int_{\mathbf{C}^{q \times k}} e^{\pi(T^* T z_2 | z_2)}\, d\mu(z_2)$$

$$= \int_{\mathbf{C}^{q \times k}} e^{-\pi(1 - T^* T z_2 | z_2)}\, dz_2 = \frac{1}{\det(1 - T^* T)^k} \quad \text{by Lemma 1.3.} \quad \square$$

This function q_T plays a very important role in the rest of this paper.

Let $\mathrm{Sp}(\sigma)$ be the group of all real linear operators on $\mathscr{S}$ which preserve σ. In other words, $a \in \mathrm{Sp}(\sigma)$ if and only if $\sigma(az, aw) = \sigma(z, w)$, for all z, $w \in \mathscr{S}$. Clearly, $G = U(p,q)$ is a subgroup of $\mathrm{Sp}(\sigma)$. Let $a \in \mathrm{Sp}(\sigma)$. The map $(w,t) \to \rho(aw,t)$ defines an irreducible unitary representation of H on $\mathscr{F}$, which is identical to ρ on the center $\mathbf{R}$ of H. By the Stone-von Neumann theorem they are unitarily equivalent. Hence there is an operator $\omega(a)$, unique up to a unitary constant, so that

$$\omega(a)\rho(w,t) = \rho(aw,t)\omega(a). \tag{1.5}$$

For $g \in G$, we can choose $\omega(g)$ so that $g \to \omega(g)$ is a continuous unitary representation called the harmonic representation. We seek to explicitly determine $\omega(g)$, $g \in G$. Its construction comes from the proof of the Stone-von Neumann theorem which we now review.

1.6. THE STONE-VON NEUMANN THEOREM. *Let τ be a unitary representation of H such that*

$$\tau(0,t) = e^{-\pi i t} I, \qquad t \in \mathbf{R}.$$

Then τ is a multiple of ρ.

Proof. Let S_τ be the representation space of τ. Define a map T on S_τ by

$$(T\phi | \psi) = \int_{\mathscr{S}} (\tau(z,0)\phi | \psi) \mu^{1/2}(z)\, dz,$$

for ϕ, $\psi \in S_\tau$. The matrix entry $(z,0) \to (\tau(z,0)\phi \mid \psi)$ is bounded, so the above integral converges. It's not hard to see that T is a non-zero bounded operator and $T = T^* = T^2$. Further, if $(z,t) \in H$ then

$$T\tau(z,t)T = e^{-\pi it}\mu^{1/2}(z)T. \tag{1}$$

One can further show that the H-invariant subspace generated by the range of T is dense in S_τ. Let ϕ_1, $\phi_2 \in$ range of T. For $h = (z,t) \in H$, let $p(h) = e^{-\pi it}\mu^{1/2}(z)$. By (1) above

$$(\tau(h)\phi_1 \mid \phi_2) = p(h)(\phi_1 \mid \phi_2)$$

and hence

$$(\tau(h_1)\phi_1 \mid \tau(h_2)\phi_2) = p(h_2^{-1}h_1)(\phi_1 \mid \phi_2). \tag{2}$$

Let $\{\phi_\nu\}$ be an orthonormal base for the range of T. Let H_ν be the closed H-invariant subspace generated by ϕ_ν. It follows from (2) that $\{H_\nu\}$ is a set of mutually orthogonal subspaces and $S_\tau = \oplus H_\nu$.

Let τ_ν be the restriction of τ to H_ν. By (1)

$$(\tau_\nu(h)\phi_\nu \mid \phi_\nu) = p(h) = (\rho(h)1 \mid 1).$$

So τ_ν and ρ share a common matrix entry. This is enough to show that ρ is equivalent to τ_ν. In fact, the map Φ of $\operatorname{span}\{\tau(h)\phi_\nu \colon h \in H\}$ into $\mathscr{F}$ defined by

$$\Phi\Big(\sum_j c_j\tau(h_j)\phi_\nu\Big) = \sum_j c_j\rho(h_j)1$$

extends to a unitary intertwining operator of τ_ν and ρ. $\square$

1.7. COROLLARY. *Let $\psi \in H_\nu$ and $z \in \mathscr{S}$. Define*

$$Q_\nu\psi(z) = (\psi \mid \tau(z,0)\phi_\nu)\mu^{-1/2}(z).$$

Then $Q_\nu = \Phi$.

Proof. Let $\psi \in H_\nu$ and $z \in \mathscr{S}$. Then

$$\begin{aligned} Q_\nu\psi(z) &= (\psi \mid \tau(z,0)\phi_\nu)\mu^{-1/2}(z) = (\Phi\psi \mid \Phi\tau(z,0)\phi_\nu)\mu^{-1/2}(z) \\ &= (\Phi\psi \mid \rho(z,0)1)\mu^{-1/2}(z) = (\Phi\psi \mid K(\cdot,z)) = \Phi(\psi)(z). \quad \square \end{aligned}$$

For the case we will consider we mention;

1.8. COROLLARY. *If τ is irreducible the range of T is one dimensional.*

Let $g \in G$. Consider the representation τ of H defined by $\tau(z,t) = \rho(g^{-1}z,t)$. Clearly, τ is irreducible and $\tau(0,t) = e^{-\pi it}I$. By the Stone-von Neumann theorem there exists a unitary operatory Φ on $\mathscr{F}$ such that

$\Phi\rho(g^{-1}z, t) = \rho(z, t)\Phi$, for all $(z, t) \in H$. Replace z by gz. We then have $\Phi\rho(z, t) = \rho(gz, t)\Phi$. So $\omega(g)$ as defined by 1.5 is a unitary multiple of Φ. To determine Φ in this case we must determine a vector in the range of T as defined in the proof of 1.6. Let $z \in \mathscr{S}$. Then

$$\begin{aligned}
T1(z) &= (T1 \mid K(\cdot, z)) = \int_{\mathscr{S}} (\rho(g^{-1}w, 0)1 \mid K(\cdot, z))\mu^{1/2}(w)\, dw \\
&= \int_{\mathscr{S}} \rho(g^{-1}w, 0)1(z)\mu^{1/2}(w)\, dw \\
&= \int_{\mathscr{S}} K(z, g^{-1}w)\mu^{1/2}(g^{-1}w)\mu^{1/2}(w)\, dw \\
&= \int_{\mathscr{S}} K(z, g^{-1}w)\mu\left(\begin{pmatrix} A^* & -C^* \\ 0 & 1 \end{pmatrix} w\right) dw \\
&= \frac{1}{|\det A^*|^k} \int_{\mathscr{S}} K\left(z, g^{-1}\begin{pmatrix} A^{*-1} & A^{*-1}C^* \\ 0 & 1 \end{pmatrix} w\right) d\mu(w) \\
&= \frac{1}{|\det A^*|^k} \int e^{\pi(z_1 \mid w_1)} e^{\pi(-B^*A^{*-1}w_1 \mid w_2)} e^{\pi(D^* - B^*A^{*-1}C^*w_2 \mid z_2)}\, d\mu(w) \\
&= \frac{1}{|\det A^*|^k} e^{\pi(z_1 \mid -A^{-1}Bz_2)} = \frac{1}{|\det A^*|^k} q_{-A^{-1}B}(z).
\end{aligned}$$

So $q_{-A^{-1}B} \in$ range of T. By Corollary 1.8 the range of $T = \operatorname{span}\{q_{A^{-1}B}\}$. In order that $g \to \omega(g)$ be a representation we need to judiciously choose $\phi \in \operatorname{span}\{q_{-A^{-1}B}\}$. Let $\phi = (1/\det A^k)q_{-A^{-1}B}$. By 1.1 and Proposition 1.4 $\|\phi\| = 1$. By Corollary 1.7 we have

$$\text{(19a)} \qquad \phi f(w) = \omega(g)f(w) = \left(f \mid \rho(g^{-1}w, 0)q_{-A^{-1}B}\right)\frac{\mu^{-1/2}(w)}{\det A^k}.$$

An easy calculation shows that

$$\text{(1.9b)} \quad \omega(g)f(w) = \frac{q_{BD^{-1}}(w)}{\det A^k}\int_{\mathscr{S}} f(z)\bar{q}_{-A^{-1}B}(z)\overline{K}\left(z, \begin{smallmatrix} A^{-1}w_1 \\ D^{-1}w_2 \end{smallmatrix}\right) d\mu(z).$$

Thus $\omega(g)$ is a unitary operator satisfying 1.5.

We now proceed to show that $g \to \omega(g)$ is a continuous unitary representation of G on $\mathscr{F}$. The following standard lemma will prove useful for that goal and will have frequent use throughout this paper.

1.10. LEMMA. *Let M be a connected complex manifold and h a function on $M \times M$ with the following properties.*

(a) $h(z,z) = 0$ *for all* $z \in M$

(b) $w \to \bar{h}(z,w)$ *and* $z \to h(z,w)$ *are holomorphic for* z, w *fixed, respectively.*

Then $h(z,w) = 0$ *for all* z, $w \in M$.

1.11. LEMMA. *Let* S, $T \in \mathscr{D}$ *and* x, $w \in \mathscr{S}$. *Then*

$$(q_T K(\cdot,x) \,|\, q_S K(\cdot,w)) = \frac{1}{\det(1 - ST^*)^k} q_{T(1-S^*T)^{-1}}(w)\bar{q}_{(1-ST^*)^{-1}S}(x) K\left(\begin{matrix}(1-ST^*)^{-1}w_1 \\ (1-S^*T)^{-1}w_2\end{matrix}, x\right).$$

Proof. Each function given above is holomorphic in S and conjugate holomorphic in T. For $S = T$ it is a straightforward calculation that they agree. By 1.10 the result follows. □

1.12. THEOREM. *The map* $\omega \colon G \to \mathscr{U}(\mathscr{F})$ *defined by* 1.9 *is a continuous unitary representation of* G *on* $\mathscr{F}$.

Proof. It is clear from (1.9a) that $g \to \omega(g)f(w) = (\omega(g)f \,|\, K(\cdot,w))$ is a continuous function of G, for all $w \in \mathscr{S}$. Since $\operatorname{span}\{K(\cdot,w)\colon w \in \mathscr{S}\}$ is dense in $\mathscr{F}$ a standard argument shows $g \to (\omega(g)f \,|\, h)$ is continuous, for f, $h \in \mathscr{F}$. Using 1.1 and Lemma 1.11, it is easy to check that $\omega(g_1)\omega(g_2)K(\cdot,x) = \omega(g_1 g_2)K(\cdot,x)$, for all $x \in \mathscr{S}$. Hence ω is a continuous unitary representation of G on $\mathscr{F}$.

2. The decomposition of the harmonic representation. In this section we give a description of the irreducible components of ω. In the process we will also derive some fundamental formulas necessary for the main results in §3.

The irreducible components of ω are parametrized by a class Λ of irreducible representations of the dual group $U(k)$. Kashiwara and Vergne **[8]** give an explicit description of Λ in terms of the signature of the representation, to which we refer in 2.9. As we observe after Corollary 2.3 the decomposition of ω reduces to a decomposition of the space of harmonics $\mathscr{H}$ under the joint actions of $U(p)$, $U(q)$, and $U(k)$. Our method of proving irreducibility of the isotypic components (Theorem 2.5), is somewhat different from **[8]**. Their proof utilizes arguments involving the relative size of p, q and k. We offer a direct proof for which the generalized unit disk plays an important role.

The dual group $U(k)$ acts on $\mathscr{F}$ by right translation. We may extend this action holomorphically to $\mathrm{GL}(k, \mathbf{C})$ by

$$R(g)f\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = f\begin{pmatrix} z_1 g \\ z_2 g^{*-1} \end{pmatrix}, \qquad g \in \mathrm{GL}(k, \mathbf{C}).$$

Clearly, R commutes with ω. Let $\mathscr{P} = \mathscr{P}_{p,q}$ be the subspace of F of all polynomials holomorphic in z_1 and conjugate holomorphic in z_2. Then $\mathscr{P}$ is dense in $\mathscr{F}$. Let I be the subspace of polynomials invariant under the action of $U(K)$. Then, by a theorem of Weyl, I is generated as an algebra by the constants and the matrix entries of $z \to z_1 z_2^*$. Let $\mathscr{I}$ be the ideal in $\mathscr{P}$ generated by the invariants with zero constant coefficient and let $\mathscr{H} = \mathscr{H}_{p,q}$ be the orthogonal complement of $\mathscr{I}$ in $\mathscr{P}$. We refer to $\mathscr{H}$ as the space of *harmonics*.

For $f \in P$ one can easily prove by induction on $\deg(f)$ that $f \in I\mathscr{H}$. Hence we have

2.1. PROPOSITION. $\mathscr{P} = I\mathscr{H}$.

Clearly $q_T \in \bar{I}$. In fact, one can easily show that $\mathrm{span}\{q_T \colon T \in \mathscr{D}\}$ is dense in $\bar{I}$. The importance of this and the space $\mathscr{H}$ will be clear from the following propositions. Let L denote the left action of $U(p) \times U(q)$ on $\mathscr{F}$. Then L extends holomorphically to $\mathrm{GL}(p, \mathbf{C}) \times \mathrm{GL}(q, \mathbf{C})$ by

$$L(A, D)f\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = f\begin{pmatrix} A^{-1}z_1 \\ D^* z_2 \end{pmatrix}.$$

Since L clearly leaves $\mathscr{I}$ invariant it also leaves $\mathscr{H}$ invariant. This is also true of R.

2.2. PROPOSITION. *Let $h \in \mathscr{H}$ and $g \in \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$. Then*

$$\omega(g)h = \frac{q_{BD^{-1}}}{\det A^k} L(A, D^{*-1})h.$$

Proof. By 1.9b

$$\omega(g)h(z) = \frac{q_{BD^{-1}}(z)}{\det A^k} \int_{\mathscr{S}} h(w)\bar{q}_{-A^{-1}B}(w) K\left(w, \begin{matrix} A^{-1}z_1 \\ D^{-1}z_2 \end{matrix}\right) d\mu(z).$$

Since $q_{-A^{-1}B} = 1 + \phi$, where $\phi \in \bar{I}$, and h is harmonic

$$\omega(g)h(z) = \frac{q_{BD^{-1}}(z)}{\det A^k} \int_{\mathscr{S}} h(w) K\left(w, \begin{matrix} A^{-1}z_1 \\ D^{-1}z_2 \end{matrix}\right) d\mu(z)$$

$$= \frac{q_{BD^{-1}}(z)}{\det A^k} h\begin{pmatrix} A^{-1}z_1 \\ D^{-1}z_2 \end{pmatrix}. \qquad \square$$

2.3. COROLLARY. *Let $T \in \mathscr{D}$ and $h \in \mathscr{H}$. Let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$. Then*

$$\omega(g)(q_T h) = \frac{q_{g\cdot T}}{\det(A + BT^*)^k} L\big((A + BT^*), (CT + D)^{*-1}\big)h.$$

Proof. Let $g_1 = \left(\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right) \in G$ be such that $g_1 \cdot 0 = B_1 D_1^{-1} = T$. Then by 2.2 $q_T h = \det A_1^k\, \omega(g_1) L(A_1^{-1}, D_1^*)h$. Thus

$$\omega(g) q_T h = \det A_1^k\, \omega(g g_1) L\big(A_1^{-1}, D_1^*\big)h.$$

The result now follows by applying Proposition 2.2 and the properties listed in 1.1. □

The formula given in Corollary 2.3 suggests that to decompose ω one only need to decompose the action of L on $\mathscr{H}$. This is indeed the case. Since R commutes with L we can use its representations to pick out the isotypic components.

Let $U(k)^\wedge$ be the equivalence classes of irreducible unitary representations of $U(k)$, and let $\lambda \in U(k)^\wedge$. We will also use λ to denote a representation in that class. Let $P_\lambda \colon \mathscr{P} \to \mathscr{P}$ be defined by

$$P_\lambda f(z) = \deg\lambda \int_{U(k)} f(zu)\overline{\chi_\lambda(u)}\, du.$$

Then P_λ is a projection. Let $\mathscr{P}_\lambda$ be the range of P_λ and let $\Lambda = \{\lambda \in U(k)^\wedge \colon P_\lambda \neq 0\}$. Then $\mathscr{P} = \dotplus_{\lambda\in\Lambda} \mathscr{P}_\lambda$. Let $\mathscr{H}_\lambda = P_\lambda(\mathscr{H})$. Since P_λ fixes each invariant we have by Proposition 2.1 that $\mathscr{P}_\lambda = I\mathscr{H}_\lambda$. It's easy to see that $\mathscr{H}_\lambda$ and thus $\mathscr{P}_\lambda$ are invariant under L. We will show that $\mathscr{H}_\lambda$ is in fact irreducible under $L \times R$.

Let V_λ be the representation space for λ and $(V_\lambda)' \cong V_{\lambda'}$ be the dual space where λ' is the contragredient of λ. Let $\mathscr{F}(\lambda) = \{f\colon \mathscr{S} \to V_\lambda\colon f(zu) = \lambda(u)^{-1} f(z)$ for all $u \in U(k)$ and $\gamma \circ f \in \mathscr{F}$ for all $\gamma \in (V_\lambda)'\}$ and $\mathscr{H}(\lambda) = \{f \in \mathscr{F}(\lambda)\colon \gamma \circ f \in \mathscr{H}$, for all $\gamma \in (V_\lambda)'\}$. We may define an action $\tau = \tau(\lambda)$ of $\mathrm{GL}(p, \mathbf{C}) \times \mathrm{GL}(q, \mathbf{C})$ on $\mathscr{H}(\lambda)$ by

$$\tau(A, D)h\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = h\begin{pmatrix} A^{-1}z_1 \\ D^* z_2 \end{pmatrix}.$$

It is easy to see that τ is unitary when restricted to $U(p) \times U(q)$, with respect to the inner product

$$(*) \qquad (f \mid g) = \int_{\mathscr{S}} (f(z) \mid g(z))\, d\mu(z).$$

We can also define a representation $\omega(\lambda)$ of G on $\mathscr{F}(\lambda)$ by the rule $\gamma\circ(\omega(\lambda)(g)f)=\omega(g)(\gamma\circ f)$. It is easy to see that ω_λ is unitary with respect to the inner product given by $(*)$.

The following theorem due to **[8]** reduces the question of irreducibility of $L\times R$ on $\mathscr{H}_\lambda$ to the irreducibility of τ on $\mathscr{H}(\lambda)$.

2.4. THEOREM. *There is an isomorphism of $\mathscr{H}_{\lambda'}$ onto $\mathscr{H}(\lambda)\times V_{\lambda'}$ intertwining the representations $L\times R|_{H_{\lambda'}}$ and $\tau\times\lambda'$.*

2.5. THEOREM. *The representation τ of* $\mathrm{GL}(p,\mathbf{C})\times\mathrm{GL}(q,\mathbf{C})$ *on $\mathscr{H}(\lambda)$ is irreducible.*

Proof. Suppose $V\subset\mathscr{H}(\lambda)$ is a non-zero invariant subspace. Let $V^\perp$ be the orthogonal complement. Let $f\in V$ and $g\in V^\perp$. We first show that the condition $(f\,|\,g)=0$ implies $(f(z)\,|\,g(z))=0$, for all $z\in S$. Let $a=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in G$. Then

$$(\omega(\lambda)(a)f\,|\,g)=\frac{1}{\det A^k}(q_{BD^{-1}}\tau(A,D)f\,|\,g),$$

by Proposition 2.2. Since g is harmonic and $q_{BD^{-1}}$ is an invariant with constant 1 coefficient we have $(q_{BD^{-1}}\tau(A,D)f\,|\,g)=(\tau(A,D)f\,|\,g)=0$. By unitarity of $\omega(\lambda)$ it follows that $(\omega(\lambda)(a_1)f\,|\,\omega(\lambda)(a_2)g)=0$ for all a_1, $a_2\in G$. In particular this says that

$$\int_{\mathscr{S}}(f(z)\,|\,g(z))\,q_T(z)\overline{q_S(z)}\,d\mu(z)=0,$$

for all T, $S\in\mathscr{D}$. Since $\operatorname{span}\{q_T\colon T\in\mathscr{D}\}$ is dense in $\bar{I}$ it follows that

$$\int_S(f(z)\,|\,g(z))\,\phi_1(z)\overline{\phi_2(z)}\,d\mu(z)=0$$

for all ϕ_1, $\phi_2\in I$. By the invariance of V, $V^\perp$, and I by the action of $\mathrm{GL}(p,\mathbf{C})\times\mathrm{GL}(q,\mathbf{C})$, we have

$$\int_{\mathscr{S}}(f(z)\,|\,g(z))\,\phi_1(z)\overline{\phi_2(z)}e^{-\pi(aa^*z_1|z_1)}e^{-\pi(bb^*z_2|z_2)}\,dz=0,$$

for all $(a,b)\in\mathrm{GL}(p,\mathbf{C})\times\mathrm{GL}(q,\mathbf{C})$. Let $\mathscr{A}$ be the span of

$$\Big\{z\to\phi_1(z)\overline{\phi_2(z)}e^{-\pi(aa^*z_1|z_1)}e^{-\pi(bb^*z_2|z_2)}\colon$$
$$\phi_1,\ \phi_2\in I\text{ and }(a,b)\in GL(p,\mathbf{C})\times GL(q,\mathbf{C})\Big\}.$$

Clearly $\mathscr{A}$ is an algebra closed under complex conjugation. An easy argument shows that $\mathscr{A}$ separates $U(k)$ orbits of $\mathscr{S}$. Hence the uniform closure of $\mathscr{A}$ is the set of all continuous functions on $\mathscr{S}$ which vanish at infinity and are $U(k)$ invariant. This implies that $(f(z) \mid g(z)) = 0$, for all $z \in \mathscr{S}$. By Lemma 1.10 $(f(z) \mid g(w)) = 0$, for all z, $w \in \mathscr{S}$. Now assume f is nonzero. Then the span of the range of f is a nonzero $U(k)$ invariant subspace and hence is all of V_λ. This implies that g must be identically zero and hence $V^\perp$ is the null space. □

In view of this theorem and Theorem 2.4 we immediately get:

2.6. COROLLARY. *The representation* $L \times R$ *of* $\mathrm{GL}(p, \mathbf{C}) \times \mathrm{GL}(q, \mathbf{C}) \times \mathrm{GL}(k, \mathbf{C})$ *on* $\mathscr{H}_\lambda$ *is irreducible.*

Let $\mathscr{H}^\lambda$ be a subspace of $\mathscr{H}_\lambda$ irreducible under the action L. Then $L|_{\mathscr{H}_\lambda}$ is equivalent to $\deg(\lambda)$ copies of $L|_{\mathscr{H}^\lambda}$. Let $\mathscr{F}^\lambda = \overline{I\mathscr{H}^\lambda}$. Then if $\mathscr{F}_\lambda = P_\lambda(\mathscr{F})$, $\mathscr{F}_\lambda = \deg \lambda(\mathscr{F}^\lambda)$. Since the span of $\{q_T\colon T \in \mathscr{D}\}$ is dense in $\bar{I}$ the span of $\{q_T h\colon T \in \mathscr{D},\ h \in \mathscr{H}^\lambda\}$ is dense in $\mathscr{F}^\lambda$. By Corollary 2.3 $\mathscr{F}^\lambda$ is invariant under ω. Further, we have:

2.7. THEOREM. *The restriction* ω_λ *of* ω *to* $\mathscr{F}^\lambda$ *is an irreducible representation of* $U(p, q)$.

Proof. Let $s \in U(1)$. Define $A(s) = \left(\begin{smallmatrix} I & 0 \\ 0 & sI \end{smallmatrix}\right) \in U(p, q)$. Since $L|_{H^\lambda}$ is irreducible $\omega(A(s)) = L(I, sI) = a(s)I$ on $\mathscr{H}^\lambda$, where $s \to a(s)$ is as character of $U(1)$. Define an operator P on $\mathscr{F}^\lambda$ by

$$Pf = \int_{U(1)} a^{-1}(s)\,\omega(A(s)) f\, ds.$$

An easy calculation shows that $P = P^* = P^2$. Further, if $h \in \mathscr{H}^\lambda$ then $Ph = h$. Let $\phi \in I$ with zero constant coefficient. Then

$$P(\phi h) = \int_{U(1)} a^{-1}(s) L(I, sq) \phi a(s) h\, ds = \int_{U(1)} L(I, sI) \phi\, ds\, h = 0$$

(cf. Hua [5], p. 97). It follows that P is the orthogonal projection of $\mathscr{F}^\lambda$ onto $\mathscr{H}^\lambda$. Let V be a closed subspace of $\mathscr{F}^\lambda$ invariant under ω. Then P leaves V invariant. We may assume there is an $f \in V$ such that $Pf \neq 0$, for otherwise $V^\perp$ will contain such a vector. Since

$$P\left(\omega\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} f\right) = \frac{1}{\det A^k} L(A, D) Pf$$

if follows that $\mathscr{H}^\lambda \subset V$. By Proposition 2.2, $q_T h \in V$ for all $T \in \mathscr{D}$ and $h \in \mathscr{H}^\lambda$. This implies $V = \mathscr{F}^\lambda$ and ω_λ is irreducible. □

We therefore obtain the complete decomposition of ω. We summarize it as:

2.8. THEOREM. *The representation ω decomposes as follows*:

$$\omega = \bigoplus_{\lambda \in \Lambda} \deg(\lambda)\omega_\lambda$$

We conclude with an explicit description of Λ as given in [8]. Let $z_1 \in \mathbf{C}^{p\times k}$ be partitioned as follows:

$$z_1 = \begin{array}{cc} \begin{pmatrix} z_{11} & z_{12} \\ z_{13} & z_{14} \end{pmatrix} & \begin{array}{c} p-i \\ i \end{array} \\ \begin{array}{cc} i & \quad k-i \end{array} & \end{array}$$

Let $\Delta_i(z_1) = \det z_{13}$. Similarly, let $z_2 \in \mathbf{C}^{q\times k}$ be partitioned as follows:

$$z_2 = \begin{array}{cc} \begin{pmatrix} z_{21} & z_{22} \\ z_{23} & z_{24} \end{pmatrix} & \begin{array}{c} j \\ q-j \end{array} \\ \begin{array}{cc} k-j & \quad j \end{array} & \end{array}$$

Let $M_j(z_2) = \det z_{22}$. Suppose $\lambda \in U(k)^\wedge$ has signature

$$(*) \qquad (m_1, \ldots, m_r, 0, 0, \ldots, 0, -n_s, \ldots, -n_1),$$

where $m_1 \geq \cdots \geq m_r \geq 0$ and $n_1 \geq \cdots \geq n_s \geq 0$, $r \leq p$, and $s \leq q$. Let $h_\lambda(z) = \Delta_1^{\alpha_1}(z_1) \cdots \Delta_r^{\alpha_r}(z_1)\overline{M}_1^{\beta_1}(z_2) \cdots \overline{M}_s^{\beta_s}(z_2)$, where $\alpha_i = m_i - m_{i+1}$, $i = 1, \ldots, r-1$ and $\alpha_r = m_r$ and $\beta_i = n_i - n_{i+1}$, $i = 1, \ldots, s-1$ and $\beta_s = n_s$. By [8] we get:

2.9. THEOREM. (1) *$\lambda \in \Lambda$ if and only if the signature of λ satisfies* $(*)$.

(2) *If $\lambda \in \Lambda$ then $h_\lambda \in \mathscr{H}_\lambda$ is the highest weight vector for $L \times R|_{H_\lambda}$ with respect to the lower triangular subgroups of* $\mathrm{GL}(p, \mathbf{C})$, $\mathrm{GL}(q, \mathbf{C})$, *and* $\mathrm{GL}(k, \mathbf{C})$.

(3) *If $\lambda \in \Lambda$ then the signature of $L \times R|_{\mathscr{H}_\lambda}$ is*

$$(0, \ldots, 0, -m_r, \ldots, -m_1) \times (n_1, \ldots, n_s, 0, \ldots, 0)$$
$$\times (m_1, \ldots, m_r, 0, \ldots, 0, -n_s, \ldots, -n_1).$$

3. The connection with the disk $\mathcal{D}$. In the previous section the invariant q_T, $T \in \mathcal{D}$, played a key role in the decomposition of ω. In this section we exploit this function further to derive an operator valued kernel function Q on $\mathcal{D}$. Our key result, Theorem 3.2, shows Q is positive definite. We can therefore construct Hilbert spaces and irreducible representations of G which we show are equivalent to those in the decomposition of ω. The following result is the key to these constructions.

3.1. PROPOSITION. *Let $h, f \in \mathscr{H}$ and $S, T \in \mathscr{D}$. Then*
$$(q_T h \mid q_S f) = \frac{1}{\det(1 - ST^*)^k}\left(L\left(1 - ST^*, (1 - S^*T)^{*-1} h \mid f\right)\right).$$

Proof. Let $g_1, g_2 \in G$ be such that $g_1 \cdot 0 = T$ and $g_2 \cdot 0 = S$. By 2.3
$$q_T h = \det A_1^k \omega(g_1) L\left(A_1^{-1}, D_1^*\right) h$$
and
$$q_S f = \det A_2^k \omega(g_2) L\left(A_2^{-1}, D_2^*\right) f,$$
where $g_i = \left(\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right)$, $i = 1, 2$. By the unicity of ω and 2.3 we get
$$\begin{aligned}
(q_T h \mid q_S f) &= \det A_1^k \overline{A_2^k}\left(\omega\left(g_2^{-1} g_1\right) L\left(A_1^{-1}, D_1^*\right) h \mid L\left(A_2^{-1}, D_2^*\right) f\right) \\
&= \frac{1}{\det\left(1 - A_2^{*-1} C_2^* C_1 A_1^{-1}\right)^k} \\
&\quad \times \left(L\left(1 - A_2^{*-1} C_2 C_1 A_1^{-1}, \left(1 - D_2^{*-1} B_2 B_1 D_1^{-1}\right)^{*-1}\right) h \mid f\right) \\
&= \frac{1}{\det(1 - ST^*)^k}\left(L\left(1 - ST^*, (1 - S^*T)^{*-1}\right) h \mid f\right). \qquad \square
\end{aligned}$$

Let
$$Q(S, T) = \frac{1}{\det(1 - ST^*)^k} L\left(1 - ST^*, (1 - S^*T)^{*-1}\right).$$
Then the formula in Proposition 3.2 can be written
$$(q_T h \mid q_S f) = (Q(S, T) h \mid f).$$

3.2. THEOREM. *The function Q on $\mathscr{D} \times \mathscr{D}$ is a positive definite operator-valued kernel.*

Proof. Let $h_1, \ldots, h_n \in \mathscr{H}$ and $T_1, \ldots, T_n \in \mathscr{D}$. Then by Proposition 3.1
$$\sum_{i,j}\left(Q(T_i, T_j) h_j \mid h_i\right) = \sum_{i,j}\left(q_{T_j} h_j \mid q_{T_i} h_i\right) = \left\| \sum q_{T_i} h_i \right\|^2 \geq 0.$$
Clearly $Q(S, S) > 0$, $S \in \mathscr{D}$. So Q is positive definite. $\square$

3.3. Let $Q_\lambda(\cdot,\cdot) = Q(\cdot,\cdot)|_{\mathscr{H}^\lambda}$. By Kunze [**11**]. there is a unique Hilbert space, $\mathscr{H}(\mathscr{D},\lambda)$, of continuous functions $f\colon \mathscr{D} \to \mathscr{H}^\lambda$ with the following properties:

(1) The span of the set $\{S \to Q_\lambda(S,T)h\colon T \in \mathscr{D},\ h \in \mathscr{H}^\lambda\}$ is dense in $\mathscr{H}(\mathscr{D},\lambda)$,

(2) For $S \in \mathscr{D}$, $E_S\colon f \to f(S)$ is a continuous map from $\mathscr{H}(\mathscr{D},\lambda)$ to $\mathscr{H}^\lambda$,

(3) $Q_\lambda(S,T) = E_S E_T^*$ for all S, $T \in \mathscr{D}$, and

(4) $(Q_\lambda(\cdot,T)h \mid Q_\lambda(\cdot,S)f) = (Q_\lambda(S,T)h \mid f)$.

Since $S \to Q_\lambda(S,T)$ is holomorphic, $\mathscr{H}(\mathscr{D},\lambda)$ consists of holomorphic functions on $\mathscr{D}$.

We can construct a multiplier representation of G on $\mathscr{H}(\mathscr{D},\lambda)$ as follows: Let

$$L_k^\lambda(A,D) = \frac{1}{\det A^k} L(A,D)$$

restricted to $\mathscr{H}^\lambda$. Define J_λ on $G \times \mathscr{D}$ by $J_\lambda(g,T) = L_k^\lambda((A + BT^*)^{*-1}, (CT + D))$. Then J_λ satisfies

$$(1) \qquad J_\lambda(1,T) = I, \quad \text{for all } T \in \mathscr{D}$$

$$(2) \qquad J_\lambda(g_1 g_2, T) = J_\lambda(g_1, g_2 T) J_\lambda(g_2, T) \quad \text{for all } g_1, g_2 \in G,\ T \in \mathscr{D}.$$

Hence J_λ is a multiplier. We further have

$$(3) \qquad J_\lambda(g,T)^{-1} = J_\lambda(g^{-1}, gT)$$

and

$$(4) \qquad J_\lambda\left(\begin{matrix} u & 0 \\ 0 & v \end{matrix}, T\right) = L_k^\lambda(u,v).$$

For $h \in \mathscr{H}^\lambda$ we can rewrite 2.3 as $\omega(g) q_T h = q_{g\cdot T} J_\lambda(g,T)^{*-1} h$.

The relationship between Q_λ and J_λ can be expressed by the following proposition.

3.4. Proposition. *Let S, $T \in \mathscr{D}$ and $g \in G$. Then*

$$Q_\lambda(gS, gT) = J_\lambda(g,S) Q_\lambda(S,T) J_\lambda(g,T)^*$$

Proof. The result follows from the easily verified formulas:

$$(1) \quad 1 - gS(gT)^* = (SB^* + A^*)^{-1}(1 - ST^*)(BT^* + A)^{-1} \quad \text{and}$$

$$(2) \quad 1 - (gS)^* gT = (S^*C^* + D^*)^{-1}(1 - S^*T)(CT + D)^{-1}. \qquad \square$$

3.5. THEOREM. *The formula*

$$T_\lambda(g)f(S) = J_\lambda(g^{-1}, S)^{-1} f(g^{-1}S), \qquad f \in \mathscr{H}(\mathscr{D}, \lambda),$$

defines a strongly continuous unitary representation of G on $\mathscr{H}(\mathscr{D}, \lambda)$.

Proof. This is easily verified. For details see **[11]**. □

3.6. THEOREM. *The representations* T_λ *and* ω_λ *are unitarily equivalent and the map defined by*

$$\Phi\colon \sum q_{T_i} h_i \to \sum Q_\lambda(\cdot, T_i) h_i$$

extends to a unitary intertwining map of F^λ *onto* $\mathscr{H}(\mathscr{D}, \lambda)$.

Proof. Let $h_i \in \mathscr{H}^\lambda$ and $T_i \in \mathscr{D}$. By 3.1 and 3.3

$$\left\| \sum_i Q_\lambda(\cdot, T_i) h_i \right\| = \sum_{i,j} \left(Q_\lambda(T_j, T_i) h_i \mid h_j \right)$$

$$= \sum_{i,j} \left(q_{T_i} h_i \mid q_{T_j} h_j \right) = \left\| \sum_i q_{T_i} h_i \right\|.$$

It follows that the above map is well defined and unitary. It extends uniquely to a unitary map Φ of $\mathscr{F}^\lambda$ onto $\mathscr{H}(\mathscr{D}, \lambda)$. Let $g \in G$ and $S \in \mathscr{D}$. Then

$$\Phi(\omega(g) q_T h) = \phi\left(q_{g\cdot T} J_\lambda(g, T)^{*-1} h\right) = Q_\lambda(\cdot, gT) J_\lambda(g, T)^{*-1} h$$

$$= J_\lambda(g^{-1}, \cdot)^{-1} Q_\lambda(g^{-1}(\cdot), T) h = T_\lambda(g)(Q_\lambda(\cdot, T) h) = T_\lambda(g)\Phi(q_T h),$$

by Proposition 3.4. It follows that Φ is an intertwining map and T_λ is unitarily equivalent to ω_λ □

A global version of Φ may be defined in terms of the reproducing kernel of $\mathscr{H}^\lambda$. Since evaluation is a continuous linear functional on H^λ there is a function $K^\lambda(\cdot, w) \in \mathscr{H}^\lambda$, $w \in \mathscr{S}$, such that $(f \mid K^\lambda(\cdot, w)) = f(w)$, for all $f \in \mathscr{H}^\lambda$ and $w \in \mathscr{S}$.

3.7. COROLLARY. *Let* $f \in \mathscr{F}^\lambda$ *and* $S \in \mathscr{D}$. *Then*

$$\Phi f(S)(w) = \left(f \mid q_S K^\lambda(\cdot, w)\right) = \int_{\mathscr{S}} f(z) \overline{q_S(z)}\, \overline{K^\lambda(z, w)}\, d\mu(z).$$

Proof. Let $f \in \mathscr{F}^{\lambda}$ and $S \in \mathscr{D}$ then

$$\begin{aligned}(f \mid q_S K^{\lambda}(\cdot, w)) &= (\Phi f \mid \Phi q_S K^{\lambda}(\cdot, w)) = (\Phi f \mid Q(\cdot, S) K^{\lambda}(\cdot, w)) \\ &= (\Phi f(S) \mid K^{\lambda}(\cdot, w)) \quad \text{(by 3.3.3)} \\ &= \Phi f(S)(w).\end{aligned}$$

□

4. The square integrable representations. In **[8]**, it is mentioned that for $k \geq n$ all irreducible components of ω are in the discrete series. While, for $k < \min(p, q)$ there are no such components. In this section we give necessary and sufficient conditions on the signature of λ for ω_{λ} to be square integrable. Of course, one could trace the Harish-Chandra condition on the weight corresponding to L_k^{λ}. However, our techniques are more indigenous to the situation at hand. Our methods underscore the importance of the role of the generalized unit disk. We conclude the section with an explicit description of the unitary structure for $\mathscr{H}(\mathscr{D}, \lambda)$ for the square integrable case.

Suppose $f \in L^1(G)$ and $f(gk) = f(g)$ for all $k \in K$. Let $T \in \mathscr{D}$ and let $g \in G$ such that $g \cdot 0 = T$. Define $f^{\#}\colon \mathscr{D} \to \mathbf{C}$ by $f^{\#}(T) = f(g)$. Then $f^{\#}$ is well defined and we can normalize measures in such a way that

$$\int_G f(g)\, dg = \int_{\mathscr{D}} f^{\#}(T) \frac{dT}{\det(1 - TT^*)^n}.$$

4.1. Proposition. *The representation ω_{λ} is square integrable if and only if*

$$\int_{\mathscr{D}} \chi_{L_k^{\lambda}}\big((1 - TT^*)^{-1},\ (1 - T^*T)\big) \frac{dT}{\det(1 - TT^*)^n} < \infty,$$

where $\chi_{L_k^{\lambda}}$ is the character for L_k^{λ}.

Proof. Let $\{e_1, \ldots, e_d\}$ be an orthonormal base of $\mathscr{H}^{\lambda}$. By Godemonts theorem **[15]**, ω_{λ} is square integrable if and only if

$$\sum_{i,j} \int_G |\omega(g^{-1}) e_i | e_j|^2\, dg < \infty.$$

If $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ then

$$\begin{aligned}(\omega(g^{-1}) e_i \mid e_j) &= (e_i \mid \omega(g) e_j) \\ &= (e_i \mid q_{BD^{-1}} L_k^{\lambda}(A, D^{*-1}) e_j) = (L_k^{\lambda}(A^*, D^{-1}) e_i \mid e_j).\end{aligned}$$

Hence

$$\sum_{i,j}\left|\left(\omega(g^{-1})e_i\,|\,e_j\right)\right|^2 = \sum_{i,j}\left|\left(L_k^\lambda(A^*, D^{-1})e_i\,|\,e_j\right)\right|^2 = \sum_i\left|L_k^\lambda(A^*, D^{-1})e_i\right|^2$$

$$= \sum_i\left(L_k^\lambda(AA^*, D^{*-1}D^{-1})e_i\,|\,e_i\right) = \chi_{L_k^\lambda}(AA^*, D^{*-1}D^{-1}).$$

The function $g \to \chi_{L_k^\lambda}(AA^*, D^{*-1}D^{-1})$ is invariant under K. If $g \in G$ and $g\cdot 0 = T$. Then $AA^* = (1 - TT^*)^{-1}$ and $D^{*-1}D^{-1} = 1 - T^*T$. Hence ω_λ is square integrable if and only if

$$\int_{\mathscr{D}} \chi_{L_k^\lambda}\left((1 - TT^*)^{-1}, 1 - T^*T\right)\frac{dT}{\det(1 - TT^*)^n} < \infty. \qquad \square$$

4.2. Lemma. *Let $a = (a_1, \dots, a_p)$ where $a_1 \ge a_2 \ge \cdots \ge a_p \ge 0$ and let $b = (b_1, \dots, b_p)$ where $b_1 \ge b_2 \ge \cdots \ge b_p \ge 0$. Let k be an integer and assume $p \le q$. Then*

$$\Omega = \int_{\mathscr{D}} \chi_a(1 - TT^*)\chi_b(1 - TT^*)\det(1 - TT^*)^k\, dT < \infty$$

if and only if $a_p + b_p + k \ge 0$, where χ_a and χ_b are the characters for the representations of $\mathrm{GL}(p, \mathbf{C})$ *with signature a and b, respectively.*

Proof. We will utilize the notation and some results of Hua [**5**]. By formula 5.2.13 of Hua [**5**],

$$\Omega = C\int_{\mathscr{D}_{p,p}} \chi_a(1 - ZZ^*)\chi_b(1 - ZZ^*)\det(ZZ^*)^{q-p}\det{}^k(1 - ZZ^*)\, dZ,$$

where C is a constant. Let $r = q - p$. By formula 5.2.3 of Hua [**5**],

$$\Omega = C_pC\int_0^1 \cdots \int_0^1 \chi_a(1 - \lambda_1, \dots, 1 - \lambda_p)\chi_b(1 - \lambda_1, \dots, 1 - \lambda_p)$$
$$\cdot\det{}^r(\lambda_1, \dots, \lambda_p)D^2(\lambda_1, \dots, \lambda_p)$$
$$\cdot\det{}^k(1 - \lambda_1, \dots, 1 - \lambda_p)\, d\lambda_1 \cdots d\lambda_p,$$

where C_p is a constant, and the arguments of χ_a, χ_b, and det are diagonal matrices. Now, it is easy to see that $D^2(1 - \lambda_1, \dots, 1 - \lambda_p) = D^2(\lambda_1, \dots, \lambda_p)$. We apply Weyls character formula and make the change of variable $\lambda_i \to 1 - \lambda_i$, $i = 1, \dots, p$, to get

$$\Omega = C_pC\int_0^1 \cdots \int_0^1 M_a(\lambda_1, \dots, \lambda_p)M_b(\lambda_1, \dots, \lambda_p)$$
$$\cdot\det{}^r(1 - \lambda_1, \dots, 1 - \lambda_p)$$
$$\cdot\det{}^k(\lambda_1, \dots, \lambda_p)\, d\lambda_1 \cdots d\lambda_p.$$

Let $l_i = a_i + p - i$ and $m_i = b_i + p - i$. Then $M_a(\lambda_1, \ldots, \lambda_p) = \det|\lambda_j^{l_i}|_{i,j=1}^p$ and $M_b(\lambda_1, \ldots, \lambda_p) = \det|\lambda_j^{m_i}|_{i,j=1}^p$. Expanding the above integrand gives

$$\begin{aligned}\Omega &= C_p \sum_{\sigma,\tau \in S_p} \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)\prod_i \left[\int_0^1 (1-\lambda_i)^r \lambda_i^{l_{\sigma(i)}+m_{\tau(i)}+k}\, d\lambda_i\right] \\ &= C_p C \sum_{\sigma,\tau \in S_p} \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)\prod_i B\big(l_{\sigma(i)} + m_{\tau(i)} + k + 1, r+1\big) \\ &= C_p C p! \det\left|B\big(l_i + m_j + k + 1, r+1\big)\right|_{i,j=1}^p,\end{aligned}$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ is the Beta function which is finite if and only if $x,\ y > 0$ (cf. Ryzhik **[3]** p. 948). It follows that Ω is finite if and only if $r + 1 > 0$ and $l_i + m_j + k + 1 > 0$, for all $i,\ j = 1, \ldots, p$. This is only true if and only if $a_p + b_p + k = l_p + m_p + k > 0$. $\square$

4.3. REMARK. In the following theorem we will use the following observation regarding the branching theorem (cf. Boerner **[1]** page 175). If $A \in \mathrm{GL}(m, \mathbf{C})$ we can regard $\mathrm{GL}(m, \mathbf{C})$ a subgroup of $\mathrm{GL}(n, \mathbf{C})$, $n > m$, by the injection $A \to \left[\begin{smallmatrix} A & 0 \\ 0 & 1\end{smallmatrix}\right]$. If $(a) = (a_1, \ldots, a_n)$ is the signature of an irreducible representation $T_{(a)}$ of $\mathrm{GL}(n, \mathbf{C})$, then its restriction to $\mathrm{GL}(m, \mathbf{C})$ decomposes with multiplicities $m(a_1', \ldots, a_m')$ as:

$$T(a)\,|_{\mathrm{GL}(m)} = \sum_{a_1' \geq \cdots \geq a_m'} m(a_1', \ldots, a_m') T_{(a_1', \ldots, a_m')}.$$

One crucial observation for our purpose is that $a_m' \geq a_n$ whenever $m(a_1', \ldots, a_m') \neq 0$ and there is a nonzero multiplicity for which $a_m' = a_n$.

4.4. THEOREM. *Suppose $\lambda \in \Lambda$ has signature $(m_1, \ldots, m_r, 0, \ldots, 0, -n_s, \ldots, -n_1)$, $r \leq p$, $s \leq q$, $r + s \leq k$. Then ω_λ is square integrable if and only if $k - n + m_p + n_q \geq 0$.*

Proof. Let $(m) = (0, \ldots, 0, -m_r, \ldots, -m_1)$ and $(n) = (n_1, \ldots, n_s, 0, \ldots, 0)$. By 2.9 the signature of L^λ is $(m) \times (n)$. Without loss of generality we may assume $p \leq q$. Let $T \in \mathscr{D}$. Then there exists $u \in U(p)$ and $v \in U(q)$ such that $T = udv$, where $d = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_p \end{smallmatrix}\ \ 0\right)$, where $0 \leq \lambda_i < 1$ (cf. Hua **[5]** page 33). Now

$$\begin{aligned}1 - T^*T = v^*(1 - d^*d)v &= v^*\begin{pmatrix} 1 - dd^* & 0 \\ 0 & 1\end{pmatrix} v \\ &= v^*\begin{pmatrix} u^* & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 - TT^* & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix} v.\end{aligned}$$

Therefore

$$\chi_{(n)}(1 - T^*T) = \chi_{(n)}\begin{pmatrix} 1 - TT^* & 0 \\ 0 & 1 \end{pmatrix}.$$

By 4.1 ω_λ is square integrable if and only if

$$\int_{\mathscr{D}} \det(1 - TT^*)^{k-n}\chi_{(m)}(1 - TT^*)^{-1} \cdot \chi_{(n)}(1 - T^*T)\,dT$$

$$= \int_{\mathscr{D}} \det(1 - TT^*)^{k-n}\chi_{(m')}(1 - TT^*)\chi_{(n)}\begin{pmatrix} 1 - TT^* & 0 \\ 0 & 1 \end{pmatrix} dT < \infty,$$

where $(m') = (m_1, \dots, m_r, 0, \dots, 0)$. Applying the branching the theorem we see that ω_λ is square integrable if and only if

$$\int_{\mathscr{D}} \det(1 - TT^*)^{k-n}\chi_{(m')}(1 - TT^*)\chi_{(n'_1, \dots, n'_p)}(1 - TT^*)\,dT < \infty$$

for all signatures $(n'_1, \dots, n'_p)$ such that $m(n'_1, \dots, n'_p) \neq 0$. By Lemma 4.2 and Remark 4.3 ω_λ is square integrable if and only if $k - n + m_p + n_q \leq 0$. □

For the remainder of this section we will assume $\lambda \in \Lambda$ is such that ω_λ is square integrable. Let $L^2(G, \lambda)$ be the space of $\mathscr{H}^\lambda$ valued functions f on G such that

(1) $f(g(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix})) = L_k^\lambda(u, v)f(g)$ for $u \in U(p)$ and $v \in U(q)$ and

(2) $\int_G |f(g)|^2\,dg < \infty$.

Define a map Θ: $\mathscr{H}(\mathscr{D}, \lambda) \to L^2(G, \lambda)$ by

$$\Theta F(g) = C_\lambda^{1/2} E_0\big(T(g^{-1})F\big)$$

where C_λ is a constant defined below and where E_0 is evaluation at $0 \in \mathscr{D}$. Since $E_0(T_\lambda(g^{-1})F) = J_\lambda^{-1}(g, 0)F(g \cdot 0)$, its easy to see that ΘF satisfies (1). To verify (2) we proceed as follows: Let $h \in \mathscr{H}^\lambda$. From 3.3.3 $E_0^* h = 1_h$. Let $\{e_1, \dots, e_d\}$ be an orthonormal basis for $\mathscr{H}^\lambda$. Then

$$\left\| E_0 T_\lambda(g^{-1})F \right\|^2 = \sum_{i=1}^{d} \left|\big(E_0 T_\lambda(g^{-1})F \mid e_i\big)\right|^2 = \sum_{i=1}^{d} \left|\big(T_\lambda(g^{-1})F \mid 1_{e_i}\big)\right|^2.$$

Since T_λ is unitarily equivalent to ω_λ, T_λ is square integrable. Therefore, we have

$$\int_G \left\| E_0 T_\lambda(g^{-1})F \right\|^2 dg = \sum_{i=1}^{d} \int \big(T_\lambda(g^{-1})F \mid 1_{e_i}\big)\big(T_\lambda(g^{-1})F \mid 1_{e_i}\big)\,dg$$

$$= \sum_{i=1}^{d} \frac{1}{C}(F \mid F)\big(1_{e_i} \mid 1_{e_i}\big) = \frac{\dim \mathscr{H}^\lambda}{C} \|F\|^2,$$

where C is the formal degree of T_λ. If we let $C_\lambda = C/\dim H^\lambda$ then Θ is a unitary map of $\mathscr{H}(\mathscr{D},\lambda)$ into $L^2(G,\lambda)$.

4.5. Theorem. *The inner product on $\mathscr{H}(\mathscr{D},\lambda)$ may be written*

$$(F_1 \mid F_2) = C_\lambda \int_{\mathscr{D}} \left(Q^{-1}(T,T)F_1(T) \mid F_2(T)\right) \frac{dT}{\det(1-TT^*)^n}.$$

Proof. Since Θ is unitary

$$\begin{aligned}(F_1 \mid F_2) &= (\Theta F_1 \mid \Theta F_2)\\ &= C_\lambda \int_G \left(J_\lambda^{-1}(g,0)F_1(g\cdot 0) \mid J_\lambda^{-1}(g,0)F_2(g\cdot 0)\right) dg\\ &= C_\lambda \int_G \left(J_\lambda^{*-1}(g,0)J_\lambda^{-1}(g,0)F_1(g\cdot 0) \mid F_2(g\cdot 0)\right) dg.\end{aligned}$$

Now $J_\lambda^{*-1}(g,0)J_\lambda^{-1}(g,0) = Q_\lambda^{-1}(g\cdot 0, g\cdot 0)$. Clearly the integrand is invariant under $g \to gk$. Hence

$$(F_1 \mid F_2) = C_\lambda \int_{\mathscr{D}} \left(Q_\lambda^{-1}(T,T)F_1(T) \mid F_2(T)\right) \frac{dT}{\det(1-TT^*)^n}. \qquad \square$$

4.6. Corollary. *The reproducing property can be written*

$$F(S) = C_\lambda \int_{\mathscr{D}} Q(S,T)Q^{-1}(T,T)F(T)\frac{dT}{\det(1-TT^*)^n},$$

for all $F \in \mathscr{H}(\mathscr{D},\lambda)$.

Proof. Let $h \in \mathscr{H}^\lambda$. By Theorem 4.5 we have

$$\begin{aligned}(F(S) \mid h) &= (F \mid E_s^* h)\\ &= C_\lambda \int_{\mathscr{D}} \left(Q^{-1}(T,T)f(T) \mid Q(T,S)h\right)\frac{dT}{\det(1-TT^*)^n}\\ &= C_\lambda \int_{\mathscr{D}} \left(Q(S,T)Q^{-1}(T,T)f(T) \mid h\right)\frac{dT}{\det(1-TT^*)^n}.\end{aligned}$$

Since h is arbitrary the corollary follows. $\square$

4.7. Corollary. *The map Ψ_λ: $\mathscr{H}(\mathscr{D},\lambda) \to \mathscr{F}^\lambda$ defined by*

$$\Psi_\lambda F = C_\lambda \int_{\mathscr{D}} q_T Q^{-1}(T,T)F(T)\frac{dT}{\det(1-TT^*)^n}$$

is a unitary map intertwining ω_λ and T_λ.

Proof. Let $F \in \mathscr{H}(\mathscr{D}, \lambda)$. Then by 4.5 and 4.6

$$
\begin{aligned}
(\Psi_\lambda F \mid \Psi_\lambda F) &= C_\lambda^2 \int_{\mathscr{D}} \int_{\mathscr{D}} \big(q_T Q^{-1}(T,T) F(T) \mid q_S Q^{-1}(S,S) F(S)\big)\, dT\, dS \\
&= C_\lambda^2 \int_{\mathscr{D}} \int_{\mathscr{D}} (Q(S,T) Q^{-1}(T,T) F(T) \mid Q^{-1}(S,S) F(S))\, dT\, dS \\
&= C_\lambda \int_{\mathscr{D}} (F(S) \mid Q^{-1}(S,S) F(S))\, dS = \|F\|^2.
\end{aligned}
$$

Let $R \in \mathscr{D}$ and $h \in \mathscr{H}^\lambda$. Then, by Corollary 4.6, $\Psi_\lambda(Q(\cdot, R)h) = q_R h$ for

$$
\begin{aligned}
&(\Psi_\lambda(Q(\cdot, R)h) \mid q_S f) \\
&\quad = C_\lambda \int_{\mathscr{D}} \big(q_T Q^{-1}(T,T) Q(T,R) h \mid q_S f\big) \frac{dT}{\det(1 - TT^*)^n} \\
&\quad = C_\lambda \int_{\mathscr{D}} (Q(S,T) Q^{-1}(T,T) Q(T,R) h \mid f) \frac{dT}{\det(1 - TT^*)^n} \\
&\quad = (Q(S,R) h \mid f) = (q_R h \mid q_S f),
\end{aligned}
$$

for all $S \in \mathscr{D}$ and $f \in \mathscr{H}^\lambda$. It now follows that Ψ_λ is the inverse of Φ as defined in 3.10. Therefore Ψ_λ is a unitary map intertwining ω_λ and T_λ. □

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Received February 17, 1986.

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