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UNIQUENESS OF INFINITE DELOOPINGS FOR K-THEORETIC SPACES

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A functor Φ_p is constructed from spaces to spectra such that, for each spectrum X, $\Phi_p\Omega^\infty X$ is the p-adic completion of the K-theoretic localization of X. This functor is used to obtain uniqueness results for infinite deloopings of K-theoretic spaces and maps, thereby generalizing results of Adams-Priddy and Madsen-Snaith-Tornehave. Non-unique deloopings of K-theoretic maps are shown to involve phantom maps of spectra, and such maps are analyzed.

Introduction. Let K be the spectrum of nonconnective complex K-theory and recall that the associated homology theory K_* determines a localization functor $(-)_K$ on the homotopy category of spaces and of spectra by **[9]**, **[10]**, and **[12]**. In this paper we establish a natural equivalence $\Phi_p\Omega^\infty X \simeq (X_K)^\wedge_p$ for each prime p and spectrum X, where $\Omega^\infty X$ is the 0th space of the associated Ω-spectrum of X, where $(-)^\wedge_p$ is the p-adic completion functor, and where Φ_p is a new functor from spaces to p-adically complete K_*-local spectra. Thus $\Phi_p\Omega^\infty X \simeq X$ when X is a p-adically complete K_*-local spectrum and Ω^∞ therefore embeds the homotopy category of such spectra faithfully into the ordinary pointed homotopy of spaces.

In [7], Adams and Priddy showed by specific calculations that $BSO^\wedge_p$ and $BSU^\wedge_p$ have unique infinite deloopings, i.e., that there are unique homotopy types of connective spectra X and Y such that $\Omega^\infty X \simeq BSO^\wedge_p$ and $\Omega^\infty Y \simeq BSU^\wedge_p$. Using Φ_p we show that this uniqueness phenomenon occurs much more generally: for instance, if E is the $(n-1)$-connected section of a p-adically complete K_*-local spectrum, then the space $\Omega^\infty E$ has a unique infinite delooping when $n \geq 3$ or when $n = 2$ and $\pi_2 E$ is torsion. We obtain unique infinite deloopability results for p-adic completions of various infinite classical groups, their classifying spaces, their homogeneous spaces, and their J-spaces. We likewise generalize the p-local version of the Adams-Priddy theorem by proving unique infinite deloopability for localizations of these spaces at arbitrary finite sets of primes. We also generalize results of Madsen-Snaith-Tornehave **[19]** on the uniqueness of infinite deloopings of maps between various K-theoretic infinite loop spaces and find, under general conditions, that an infinite delooping of the zero map is a phantom map of spectra. We analyze such phantom maps and extend results of Anderson [**8**] and Meier [**22**] on their non-existence. Finally, although non-deloopable maps abound, we show that maps between p-adically complete K-theoretic infinite loop spaces have canonical approximations by infinite loop maps. Some of our main results were announced in [**13**].

The basic idea for constructing the spectrum $\Phi_p Y$ for a space Y is quite simple. For each j and sufficiently large n, the pointed mapping complex from the Moore space $M(Z/p^j, n)$ to Y_K is an infinite loop space with periodicity derived from a K_*-equivalence $\Sigma^q M(Z/p^j, n) \to M(Z/p^j, n)$ of Adams. The spectrum $\Phi_p Y$ is given by the homotopy inverse limit of an associated tower of Ω-spectra indexed by j. However, the details are somewhat complicated since one must cope with various compatibility problems.

This paper is organized as follows. In §2 we state our main result, Theorem 2.1, on the functor Φ_p and derive our general unique deloopability theorems. In §3 we apply the results of §2 to familiar K-theoretic infinite loop spaces and explicitly extend results of Adams-Priddy, Madsen-Snaith-Tornehave, and Anderson. In §4 and §5 we prepare for the proof of Theorem 2.1 by studying K_*-localized Moore spectra and constructing systems of cospectra associated with Moore spaces. Then in §6 we prove Theorem 2.1.

The following notation and terminology are used. *Ho* denotes the pointed homotopy category of CW-complexes, and *HS* denotes the homotopy category of CW-spectra (see [**3**]). A space $Y \in Ho$ or spectrum $Y \in HS$ is called *E_*-local* for $E \in HS$ if each E_*-equivalence $f\colon A \to B$ in Ho or HS induces a bijection $f^*\colon [B, Y] \approx [A, Y]$. Each space $X \in Ho$ or spectrum $X \in HS$ has an *E_*-localization* $u\colon X \to X_E$ which is an E_*-equivalence such that X_E is E_*-local, and these E_*-localizations are functorial. The full subcategories of E_*-local spaces and spectra are denoted by $Ho_E \subset Ho$ and $HS_E \subset HS$.

The p-adic completion $X_p^\wedge$ of a space or spectrum X is defined to be the MZ/p_*-localization $X_{MZ/p}$ using the Moore spectrum MZ/p (see [**9**], [**10**]). This equals the $H_*(-;\ Z/p)$-localization when X is a space or connective spectrum, and equals the p-completion of [**17**] when X is a nilpotent space. Of course, if X is a simple space or spectrum with $\pi_* X$ of finite type, then $\pi_* X_p^\wedge \approx Z_p^\wedge \otimes \pi_* X$ where $Z_p^\wedge$ denotes the p-adic integers. Properties of p-adic completions and cocompletions of spaces and spectra will be discussed in [**15**].

2. The main results. For p prime, let $Ho_p^\wedge$, Ho_K, and $Ho_{K/p}$ denote the full subcategories of Ho given respectively by the p-adically complete, the K_*-local, and the K/p_*-local spaces where $K/p = K \wedge MZ/p$. Also let $HS_p^\wedge$, HS_K, and $HS_{K/p}$ denote the full subcategories of HS given respectively by the p-adically complete, the K_*-local, and the K/p_*-local spectra (see [**10**], [**11**], [**24**]). The K_*-local spectra include all KO-module spectra together with all spectra built from these by taking homotopy inverse limits or homotopy direct limits. The K/p_*-local spectra are precisely the p-adically complete K_*-local spectra, and thus $HS_{K/p} = HS_K \cap HS_p^\wedge$. The following theorem will imply our main results and will be proved in §6.

THEOREM 2.1. *For each prime p there exists a functor* $\Phi_p\colon Ho_K \to HS_{K/p}$ *such that*:

(i) *There is a natural equivalence* $\Phi_p\Omega^\infty X \simeq X_p^\wedge$ *for* $X \in HS_K$.

(ii) *If* $f\colon V \to W$ *is a map in* Ho_K *inducing* mod-p *homotopy isomorphisms above some dimension, then* $\Phi_p f\colon \Phi_p V \simeq \Phi_p W$.

(iii) *The functor* Φ_p *preserves homotopy fibre squares.*

(iv) *If M is a pointed finite CW-complex with p-torsion* $\tilde{H}_*(M;\, Z)$ *and with a* K_*-*equivalence* $\alpha\colon \Sigma^t M \to M$ *for some* $t > 0$, *then there is a natural equivalence* $\Omega^\infty(\Phi_p W)^M \simeq W^M$ *for each* $W \in Ho_K$.

2.2. *The functor.* $\Phi_p\colon Ho \to HS_{K/p}$. For p prime and $X \in Ho$, let $\Phi_p X = \Phi_p X_K$ where $\Phi_p X_K$ is as above. One may assume that the K_*-localization functor acts as the identity on Ho_K, and thus the resulting functor $\Phi_p\colon Ho \to HS_{K/p}$ extends the above functor Φ_p. Now 2.1(i) generalizes to give

THEOREM 2.3. *For p prime and* $X \in HS$ *there are natural equivalences* $\Phi_p\Omega^\infty X \simeq (X_K)_p^\wedge \simeq X_{K/p}$.

Proof. By [**11**, §2] the map $\lambda\colon (\Omega^\infty X)_K \to \Omega^\infty X_K$ induces $\pi_i(\Omega^\infty X)_K \approx \pi_i\Omega^\infty X_K$ for $i \geq 3$. Thus there are natural equivalences

$$\Phi_p\Omega^\infty X = \Phi_p(\Omega^\infty X)_K \simeq \Phi_p\Omega^\infty X_K \simeq (X_K)_p^\wedge \simeq X_{K/p}$$

by 2.1 and [**10**, §2].

Theorem 2.3 immediately implies the following faithfulness and uniqueness result for $\Omega^\infty\colon HS \to Ho$.

COROLLARY 2.4. *If f and g are maps of spectra with* $\Omega^\infty f \simeq \Omega^\infty g$ *in Ho, then* $f_{K/p} \simeq g_{K/p}$ *for each prime p. If X and Y are spectra with* $\Omega^\infty X \simeq \Omega^\infty Y$ *in Ho, then* $X_{K/p} \simeq Y_{K/p}$ *for each prime p.*

Let $HS_{K/p}[n, \infty)$ denote the full subcategory of HS given by the $(n-1)$-connected sections of p-adically complete K_*-local spectra for an integer n. Then $(\ )_{K/p}$: $HS_{K/p}[n, \infty) \to HS_{K/p}$ is a categorical equivalence since Eilenberg-MacLane spectra are K/p_*-acyclic, and Theorem 2.3 implies

COROLLARY 2.5. *Up to natural equivalence, the functor* Ω^∞: $HS_{K/p}[n, \infty) \to Ho$ *has a left inverse. Consequently, it is faithful and carries distinct homotopy types of spectra to distinct homotopy types of spaces.*

We now give our main faithfulness theorem for Ω^∞. For a spectrum $X \in HS$, let u: $X \to X_Q$ denote the rationalization map with $X_Q \simeq X \wedge MQ$.

THEOREM 2.6. *For an integer n, let X, $Y \in HS$ be $(n-1)$-connected spectra such that $Y \simeq L[n, \infty)$ for some K_*-local spectrum $L \in HS_K$. Suppose that f: $X \to Y$ is a map such that $\Omega^\infty f = 0$ in Ho. Then $f_p^\wedge = 0$ in HS for each prime p, and consequently f factors as a composition $X \xrightarrow{u} X_Q \xrightarrow{v} Y$ for some map v in HS.*

REMARK 2.7. This theorem and its proof remain valid when the hypothesis $Y \simeq L[n, \infty)$ *for some K_*-local spectrum $L \in HS_K$* is replaced by $Y_p^\wedge$ *is in $HS_{K/p}[n, \infty)$ for each prime p.*

Proof of 2.6. For each prime p, $f_{K/p}$: $X_{K/p} \to Y_{K/p}$ is trivial by 2.4, and thus the composition $X \xrightarrow{f} Y \xrightarrow{\alpha} Y_p^\wedge \xrightarrow{\beta} Y_{K/p}$ is trivial where α and β are the canonical maps. Since β is equivalent to the canonical map $L[n, \infty)_p^\wedge \to L_p^\wedge$, it follows that β_*: $\pi_i Y_p^\wedge \to \pi_i Y_{K/p}$ is an isomorphism for $i > n$ and is a monomorphism for $i = n$ and has $\pi_i Y_p^\wedge = 0$ for $i < n$. Thus the composition αf: $X \to Y_p^\wedge$ is already trivial, and therefore $f_p^\wedge = 0$: $X_p^\wedge \to Y_p^\wedge$. The rest of the theorem follows from 2.8 below.

LEMMA 2.8. *For a map g: $V \to W$ in HS, the following conditions are equivalent*:

(i) *For each prime p, $g_p^\wedge = 0$ in HS.*

(ii) *The map g factors as a composition $V \xrightarrow{u} V_Q \xrightarrow{v} W$ for some map v in HS.*

Proof. Clearly (ii) implies (i). Using the presentation of $\prod_p W_p^\wedge$ as the function spectrum $F(\Sigma^{-1}MQ/Z, W)$, we obtain a cofibre sequence

$$F(MQ, W) \to W \to \prod_p W_p^\wedge$$

with the canonical maps in HS. Now (i) implies that g factors through the rational spectrum $F(MQ, W)$ and this implies (ii).

2.9. *Phantom maps of spectra*. For V, $W \in HS$, there is a natural short exact sequence

$$0 \to \prod_n \operatorname{Ext}(\pi_{n-1}V \otimes Q, \pi_n W) \to [V_Q, W]$$

$$\to \prod_n \operatorname{Hom}(\pi_n V \otimes Q, \pi_n W) \to 0$$

which is easily derived using the splitability of V_Q into a wedge of rational Moore spectra. Thus it is usually straightforward to construct the possible maps g: $V \to W$ satisfying conditions (i) and (ii). Moreover, these maps are usually the same as the *phantom maps* from V to W, i.e., the maps θ: $V \to W$ such that $\theta\gamma = 0$ for each map γ from a finite CW-spectrum to V (see [**22**]). In more detail, suppose that W has finitely generated homotopy groups, or more generally suppose that the groups $\operatorname{Hom}(Q, \pi_i W)$ vanish for all i and that the groups $\pi_i W_p^\wedge$ are finitely generated over the p-adic integers for each prime p. Then conditions (i) and (ii) are equivalent to the condition that g is a phantom map. This follows easily since each phantom map into $W_p^\wedge$ is zero, and since each element of

$$\prod_n \operatorname{Ext}(\pi_{n-1}V \otimes Q, \pi_n W)$$

determines a phantom map from V_Q to W. Finally, we remark that although the possible phantom maps between spectra are usually easy to construct, they are also usually difficult to detect.

To show how Theorem 2.6 may be applied, we give

EXAMPLE 2.10. Let $su \in HS$ denote the section of $\Sigma^{-1}K$, with $\Omega^\infty su \simeq SU \in Ho$. If $X \in HS$ is any 1-connected spectrum with $Q \otimes \pi_{2i}X = 0$ for all i, then $[X_Q, su] = 0$ and thus Ω^∞: $[X, su] \to [\Omega^\infty X, SU]$ is mono by 2.6. Next, following Anderson [**8**], we suppose that X is the Eilenberg-Mac Lane spectrum $H(Z, 2i)$ for some $i \geq 1$. Since $H(Z, 2i)_K \simeq H(Q, 2i)$, there are isomorphisms

$$[H(Z, 2i), su] \approx [H(Q, 2i), su] \approx \operatorname{Ext}(Q, Z)$$

and we let g: $H(Z,2i) \to su$ correspond to a nonzero element in $\text{Ext}(Q, Z)$. Then g is an essential phantom map satisfying the conditions of 2.8, and Theorem 2.6 does not say whether $\Omega^\infty g$ is essential or trivial. However, on examination we find that $\Omega^\infty g$ is essential for $i > 1$ and is trivial for $i = 1$, because $K(Z,2i)_K \simeq K(Q,2i)$ in Ho for $i > 1$ and because $[CP^\infty, SU] \approx 0$.

We next give our main uniqueness theorem for infinite deloopings of spaces.

THEOREM 2.11. *For an integer n, let X, $Y \in HS$ be $(n-1)$-connected spectra such that $Y \simeq L[n,\infty)$ for some K_*-local spectrum $L \in HS_K$ with* $\text{Hom}(Q/Z, \pi_{n-1}L) = 0$. *Suppose that $\Omega^\infty X \simeq \Omega^\infty Y$ in Ho. If $n \geq 3$ then $\hat{X}_p \simeq \hat{Y}_p$ for each prime p. If $n \leq 2$ then for each prime p there exists a map h: $\hat{X}_p \to \hat{Y}_p$ inducing isomorphisms h_*: $t_p\pi_2\hat{X}_p \approx t_p\pi_2\hat{Y}_p$ and h_*: $\pi_i\hat{X}_p \approx \pi_i\hat{Y}_p$ for $i \geq 3$, where t_p is the p-torsion subgroup functor.*

REMARK 2.12. This theorem and its proof remain valid when the hypothesis *$Y \simeq L[n,\infty)$ for some K_*-local spectrum $L \in HS_K$ with* $\text{Hom}(Q/Z, \pi_{n-1}L) = 0$ is replaced by *$\hat{Y}_p$ is in $HS_{K/p}[n,\infty)$ for each prime p.*

Our proof will depend on the following lemma. Recall that for E, $X \in HS$ the space $\Omega^\infty(X_E)$ is E_*-local and thus there is a natural map λ: $(\Omega^\infty X)_E \to \Omega^\infty(X_E)$ in Ho. In the case $E = K$, the results of **[11]** show that there are isomorphisms λ_*: $t\pi_2(\Omega^\infty X)_K \approx t\pi_2(\Omega^\infty X_K)$ and λ_*: $\pi_i(\Omega^\infty X)_K \approx \pi_i\Omega^\infty(X_K)$ for $i \geq 3$ where t is the torsion subgroup functor. This easily implies

LEMMA 2.13. *For each spectrum $X \in HS$ and prime p, there are isomorphisms λ_*: $t_p\pi_2(\Omega^\infty X)_{K/p} \approx t_p\pi_2\Omega^\infty(X_{K/p})$ and λ_*: $\pi_i(\Omega^\infty X)_{K/p} \approx \pi_i\Omega^\infty(X_{K/p})$ for $i \geq 3$.*

Proof of 2.11. For p prime, there is an equivalence $X_{K/p} \to Y_{K/p}$ by 2.4, and we form the associated diagram

$$\begin{array}{ccc} X_p^\wedge & \overset{h}{\dashrightarrow} & Y_p^\wedge \\ \downarrow \beta & & \downarrow \beta \\ X_{K/p} & \overset{\simeq}{\to} & Y_{K/p} \end{array}$$

using the canonical maps β. Since β: $Y_p^\wedge \to Y_{K/p}$ is equivalent to the canonical map $L[n,\infty)_p^\wedge \to L_p^\wedge$ and since $\text{Hom}(Q/Z, \pi_{n-1}L) = 0$, there are isomorphisms β_*: $\pi_i Y_p^\wedge \approx \pi_i Y_{K/p}$ for $i \geq n$ and $\pi_i Y_p^\wedge = 0$ for $i < n$.

Thus, since $X_p^\wedge$ is $(n-1)$-connected, there is a map $h\colon X_p^\wedge \to Y_p^\wedge$ making the above diagram commute. To prove the case $n \geq 3$ of 2.11, it will now suffice to show that $\beta_*\colon \pi_i X_p^\wedge \approx \pi_i X_{K/p}$ for $i \geq n$, and to prove the case $n \leq 2$ it will suffice to show that $\beta_*\colon \pi_i X_p^\wedge \approx \pi_i X_{K/p}$ for $i \geq 3$ and $\beta_*\colon t_p \pi_2 X_p^\wedge \approx t_p \pi_2 X_{K/p}$. Consider the canonical diagrams

$$\begin{array}{ccccccc} \pi_i(\Omega^\infty X)_p^\wedge & \to & \pi_i\Omega^\infty(X_p^\wedge) & & \pi_i(\Omega^\infty Y)_p^\wedge & \to & \pi_i\Omega^\infty(Y_p^\wedge) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_i(\Omega^\infty X)_{K/p} & \to & \pi_i\Omega^\infty(X_{K/p}) & & \pi_i(\Omega^\infty Y)_{K/p} & \to & \pi_i\Omega^\infty(Y_{K/p}) \end{array}$$

and note that the left maps of the diagrams are equivalent since $\Omega^\infty X \simeq \Omega^\infty Y$. Moreover, the upper maps are isomorphisms for $i \geq 2$ by **[15]**, and the lower maps are isomorphisms for $i > 2$ and induce isomorphisms of p-torsion subgroups for $i = 2$ by 2.13. Now since $\beta_*\colon \pi_i Y_p^\wedge \to \pi_i Y_{K/p}$ is an isomorphism for $i \geq n$, it easily follows that $\beta_*\colon \pi_i X_p^\wedge \to \pi_i X_{K/p}$ has the required properties.

The following theorem generalizes Adams' **[1]** result on the existence of K_*-equivalences $\Sigma^t MZ/p^j \to MZ/p^u$ and shows the generality of 2.1(iv).

THEOREM 2.14. *If $Y \in HS$ is a finite CW-spectrum with p-torsion $H_*(Y;\, Z)$, then there exists a K_*-equivalence $\Sigma^t Y \to Y$ with $t > 0$.*

Proof. By induction it suffices to construct a K_*-equivalence $\Sigma^t Y \to Y$ for $t > 0$ when $\Sigma^n MZ/p \to X \to Y$ is a cofibering of finite CW-spectra with a K_*-equivalence $\beta\colon \Sigma^s X \to X$ for $s > 0$. Let θ be the automorphism of the group $[(\Sigma^n MZ/p)_K, X_K]$ determined by the commutative diagram

$$\begin{array}{ccc} \Sigma^r(\Sigma^n MZ/p)_K & \xrightarrow{\Sigma^r f} & \Sigma^r X_K \\ \downarrow \simeq & & \downarrow \simeq \\ (\Sigma^n MZ/p)_K & \xrightarrow{\theta(f)} & X_K \end{array}$$

using an iteration of β and of an Adams map for some $r > 0$. Since X_K has finite mod-p homotopy groups by **[10]**, $\theta^q = 1$ for some $q > 0$. Thus there is an equivalence $\Sigma^{qr} Y_K \simeq Y_K$, and there is a K_*-equivalence $\Sigma^t Y \to Y$ for some $t > 0$ since $[Y, Y]_i \to [Y_K, Y_K]_i$ is onto for sufficiently large i by **[13**, Proposition 1.4].

Let M be any pointed finite CW-complex with p-torsion $\tilde{H}_*(M;\, Z)$ and with a K_*-equivalence $\alpha\colon \Sigma^t M \to M$ for some $t > 0$. For a K_*-local space $W \in Ho_K$, the equivalence $\alpha^\#\colon W^M \simeq \Omega^t W^M$ gives an Ω-spectrum

$B_\alpha^\infty(W^M) \in HS_{K/p}$ with $\Omega^\infty B_\alpha^\infty(W^M) \simeq W^M$, and 2.1(iv) gives $W^M \simeq \Omega^\infty(\Phi_p W)^M$. Applying Φ_p to these equivalences, we obtain

THEOREM 2.15. *There are natural equivalences of spectra* $B_\alpha^\infty(W^M) \simeq \Phi_p(W^M) \simeq (\Phi_p W)^M$ *for* $W \in Ho_K$.

Thus, up to equivalence, the spectrum $B_\alpha^\infty(W^M)$ does not depend on the choice of α. We conclude by introducing

2.16. *Ω^∞-approximations of maps.* For an integer n and spectra $X \in HS$ and $Y \in HS_{K/p}$, each map f: $\Omega^\infty X[n,\infty) \to \Omega^\infty Y[n,\infty)$ in Ho has a Ω^∞-approximation Af: $\Omega^\infty X[n,\infty) \to \Omega^\infty Y[n,\infty)$ obtained by applying Ω^∞ to the composite of $u[n,\infty)$: $X[n,\infty) \to X_{K/p}[n,\infty)$ with $(\Phi_p f)[n,\infty)$: $X_{K/p}[n,\infty) \to Y[n,\infty)$. If f is already an infinite loop map, then $Af = f$. If f': $\Omega^\infty X[n,\infty) \to \Omega^\infty Y[n,\infty)$ and g: $\Omega^\infty Y[n,\infty) \to \Omega^\infty Z[n,\infty)$ are maps in Ho for $Z \in HS_{K/p}$, then $A(ff') = (Af)(Af')$ and $A(g \circ f) = (Ag)\circ(Af)$. If h: $\Omega^\infty W[n,\infty) \to \Omega^\infty X[n,\infty)$ is an infinite loop map with $W \in HS$, then $A(f \circ h) = (Af)\circ h$. Two maps θ, φ: $C \to D$ in Ho are called *p-adically related* if for each $j \geq 1$ there exists n_j such that

$$\theta_\# \simeq \varphi_\#\colon C^{M(Z/p^j,n)} \to D^{M(Z/p^j,n)}$$

for $n \geq n_j$.

THEOREM 2.17. *If f: $\Omega^\infty X[n,\infty) \to \Omega^\infty Y[n,\infty)$ is a map in Ho with $X \in HS$ and $Y \in HS_{K/p}$, then Af is an infinite loop map p-adically related to f. Moreover, Af is the only such map when Y has finite mod-p homotopy groups.*

Proof. For M as in 2.1(iv), $((Af)_K)^M = (f_K)^M$ since $\Phi_p(Af) = \Phi_p f$, and thus $(Af)_K$ is p-adically related to f_K. Hence Af is p-adically related to f since u_*: $\pi_i\Omega^\infty Y[n,\infty) \approx \pi_i(\Omega^\infty Y[n,\infty))_K$ for $i \geq n$ by **[11]**. Now let α, β: $X[n,\infty) \to Y[n,\infty)$ have $\Omega^\infty\alpha$ p-adically related to $\Omega^\infty\beta$. Then for each $j \geq 1$, $\alpha^\#$, $\beta^\#$: $X[n,\infty)^M \to Y^M$ have $\Omega^\infty\alpha^\# = \Omega^\infty\beta^\#$ when $M = M(Z/p^j, n)$ for sufficiently large n. Thus $\alpha^\# = \beta^\#$ by 2.4 since $Y^M \in HS_{K/p}$. Since Y is p-adically complete with finite mod-p homotopy groups,

$$[X[n,\infty), Y] \approx \lim_j [X[n,\infty), MZ/p^j \wedge Y].$$

Hence $\alpha = \beta$: $X[n,\infty) \to Y$, and the theorem follows.

3. Examples. Applying the main results of §2, we now derive examples showing uniqueness of infinite deloopings, faithfulness of Ω^∞, and nonexistence of phantom maps. These examples include results of Adams-Priddy [7], Madsen-Snaith-Tornehave [19], and Anderson [8] as special cases.

As in [18], for an integer q with $|q| > 1$, let $JU(q)$, $JO(q)$, $JSO(q)$, $J(q)$, and $\tilde{J}(q)$ be the spaces given respectively by the homotopy fibres of the maps $\psi^q - 1\colon BU \to BU$, $\psi^q - 1\colon BO \to BSO$, $\psi^q - 1\colon BSO \to BSO$, $\psi^q - 1\colon BO \to B\,\mathrm{Spin}$, and $\psi^q - 1\colon B\,\mathrm{SO} \to B\,\mathrm{Spin}$. Recall that $JU(q) \simeq \Omega^\infty ju(q)$, $JO(q) \simeq \Omega^\infty jo(q)$, $JSO(q) \simeq \Omega^\infty jso(q)$, $J(q) \simeq \Omega^\infty j(q)$, and $\tilde{J}(q) \simeq \Omega^\infty \tilde{j}(q)$, where the indicated spectra are obtained as homotopy fibres of the maps $\psi^q - 1$ on corresponding connective K-theoretic spectra localized away from q. For an $(n-1)$-connected spectrum L and endomorphism $\varphi\colon \pi_n L \to \pi_n L$, let L_ε be the homotopy pull-back spectrum of $H(\pi_n L, n) \xrightarrow{\varphi} H(\pi_n L, n) \xleftarrow{\lambda} L$ where λ is the Postnikov map.

THEOREM 3.1. *Let $X \in HS$ be a connected spectrum and q be an integer with $|q| > 1$.*

(i) *If $\Omega^\infty X \simeq JU(q)$, then $X \simeq ju(q)_\varphi$ where $\varphi\colon \pi_1 ju(q) \to \pi_1 ju(q)$ is multiplication by a divisor of $q-1$.*

(ii) *If $\Omega^\infty X \simeq JO(q)$ with q odd, then X is equivalent to one of the spectra $jo(q)$, $j(q) \times H(Z/2, 1)$, $jso(q) \times H(Z/2, 1)$, and $\tilde{j}(q) \times H(Z/2 \oplus Z/2, 1)$.*

(iii) *If $\Omega^\infty X \simeq JSO(q) \simeq J(q)$ with q odd, then X is equivalent to one of the spectra $jso(q)$, $j(q)$, and $\tilde{j}(q) \times H(Z/2, 1)$.*

(iv) *If $\Omega^\infty X \simeq \tilde{J}(q)$, then $X \simeq j(q)$.*

Proof. These follow from Theorem 2.11 which provides respective maps $X_p^\wedge \to ju(q)_p^\wedge$, $X_p^\wedge \to jo(q)_p^\wedge$, $X_p^\wedge \to jo(q)_p^\wedge$, and $X_p^\wedge \to \tilde{j}(q)_p^\wedge$, inducing π_i-isomorphisms for $i \geq 2$. Part (iii) also requires the equivalence $j_1(q) \simeq j_2(q)$ from [18, p. 14].

Theorem 2.11 also implies the following generalization of Adams-Priddy's result on the uniqueness of infinite deloopings of $bso_p^\wedge$ and $bsu_p^\wedge$.

THEOREM 3.2. *Let $X \in HS$ be a connected spectrum and let p be prime.*

(i) *If $\Omega^\infty X \simeq BU_p^\wedge$, then $X \simeq (bu_p^\wedge)_\varphi$ where $\varphi\colon \pi_2 bu_p^\wedge \to \pi_2 bu_p^\wedge$ is multiplication by 0 or p^r for some $r \geq 0$. If $\Omega^\infty X \simeq BU[n, \infty)_p^\wedge$ for $n \geq 3$, then $X \simeq bu[n, \infty)_p^\wedge$.*

(ii) *If* $\Omega^\infty X \simeq U_p^\wedge$, *then* $X \simeq (u_p^\wedge)_\varphi$ *where* φ: $\pi_1 u_p^\wedge \to \pi_1 u_p^\wedge$ *is multiplication by* 0 *or* p^r *for some* $r \geq 0$. *If* $\Omega^\infty X \simeq U[n, \infty)_p^\wedge$ *for* $n \geq 2$, *then* $X \simeq u[n, \infty)_p^\wedge$.

(iii) *If* $\Omega^\infty X \simeq BO_2^\wedge$, *then either* $X \simeq bo_2^\wedge$ *or* $X \simeq bso_2^\wedge \times H(Z/2, 1)$. *If* $\Omega^\infty X \simeq BO[n, \infty)_p^\wedge$ *with* $n \geq 2$, *then* $X \simeq bo[n, \infty)_p^\wedge$.

(iv) *If* $\Omega^\infty X \simeq SO_2^\wedge$, *then either* $X \simeq so_2^\wedge$ *or* $X \simeq spin_2^\wedge \times H(Z/2, 1)$. *If* $\Omega^\infty X \simeq SO[n, \infty)_p^\wedge$ *with* $n \geq 2$, *then* $X \simeq so[n, \infty)_p^\wedge$.

(v) *If* $\Omega^\infty X \simeq (SO/U)_p^\wedge$, *then* $X \simeq ((so/u)_p^\wedge)_\varphi$ *where* φ: $\pi_2(so/u)_p^\wedge \to \pi_2(so/u)_p^\wedge$ *is multiplication by* 0 *or* p^r *for some* $r \geq 0$. *If* $\Omega^\infty X \simeq (SO/U)[n, \infty)_p^\wedge$ *with* $n \geq 3$, *then* $X \simeq (so/u)[n, \infty)_p^\wedge$.

(vi) *If* $\Omega^\infty X \simeq (U/Sp)_p^\wedge$, *then* $X \simeq ((u/sp)_p^\wedge)_\varphi$ *where* φ: $\pi_1(u/sp)_p^\wedge \to \pi_1(u/sp)_p^\wedge$ *is multiplication by* 0 *or* p^r *for some* $r \geq 0$. *If* $\Omega^\infty X \simeq (U/Sp)[n, \infty)_p^\wedge$ *with* $n \geq 2$, $X \simeq (u/sp)[n, \infty)_p^\wedge$.

(vii) *If* $\Omega^\infty X \simeq BSp[n, \infty)_p^\wedge$ *with* $n \geq 1$, *then* $X \simeq bsp[n, \infty)_p^\wedge$.

(viii) *If* $\Omega^\infty X \simeq Sp[n, \infty)_p^\wedge$ *with* $n \geq 1$, *then* $X \simeq sp[n, \infty)_p^\wedge$.

(ix) *If* $\Omega^\infty X \simeq (Sp/U)_p^\wedge$ *with* p *odd, then* $X \simeq ((sp/u)_p^\wedge)_\varphi$ *where* φ: $\pi_2(sp/u)_p^\wedge \to \pi_2(sp/u)_p^\wedge$ *is* 0 *or* p^r *for some* $r \geq 0$. *If* $\Omega^\infty X \simeq (Sp/U)[n, \infty)_p^\wedge$, *with* $n \geq 3$ *or* $n = 2 = p$, *then* $X \simeq (sp/u)[n, \infty)_p^\wedge$.

(x) *If* $\Omega^\infty X \simeq (U/O)_p^\wedge$, *then* $X \simeq ((u/o)_p^\wedge)_\varphi$ *where* φ: $\pi_1(u/o)_p^\wedge \to \pi_1(u/o)_p^\wedge$ *is multiplication by* 0 *or* p^r *for some* $r \geq 0$. *If* $\Omega^\infty X \simeq (U/O)[n, \infty)_p^\wedge$ *with* $n \geq 2$, *then* $X \simeq (u/o)[n, \infty)_p^\wedge$.

Using this theorem we shall prove the following P-local generalization of the Adams-Priddy uniqueness theorem where P is an arbitrary finite set of primes. The original result [7] was for a single prime and applied to BU, BSU, BO, and BSO. The P-localization of a nilpotent space $Y \in Ho$ or spectrum $Y \in HS$ is denoted by $Y_{(P)}$.

THEOREM 3.3. *Let* $X \in HS$ *be a connected spectrum and let* P *be a finite set of primes.*

(i) *If* $\Omega^\infty X \simeq BU_{(P)}$ *and if the generator* $x \in H_2(\Omega^\infty X;\ Z/p)$ *has* $x^p \neq 0$ *for each* $p \in P$, *then* $X \simeq bu_{(P)}$. *If* $\Omega^\infty X \simeq BU[n, \infty)_{(P)}$ *with* $n \geq 3$, *then* $X \simeq bu[n, \infty)_{(P)}$.

(ii) *If* $\Omega^\infty X \simeq U_{(P)}$ *and if the generator* $x \in H_1(\Omega^\infty X;\ Z/p)$ *has the Dyer-Lashof* $Q^1 x \neq 0$ *for each odd* $p \in P$ *and* $Q^2 x \neq 0$ *when* $p = 2 \in P$, *then* $X \simeq u_{(P)}$. *If* $\Omega^\infty X \simeq U[n, \infty)_{(P)}$ *with* $n \geq 2$, *then* $X \simeq u[n, \infty)_{(P)}$.

(iii) *If* $\Omega^\infty X \simeq BO_{(P)}$ *and if the generator* $x \in H_1(\Omega^\infty X;\ Z/2)$ *has* $x^2 \neq 0$ *when* $2 \in P$, *then* $X \simeq bo_{(P)}$. *If* $\Omega^\infty X \simeq BO[n, \infty)_{(P)}$ *with* $n \geq 2$, *then* $X \simeq bo[n, \infty)_{(P)}$.

(iv) *If* $\Omega^\infty X \simeq SO_{(P)}$ *and if the generator* $x \in H_1(\Omega^\infty X;\ Z/2)$ *has* $Q^2x \neq 0$ *when* $2 \in P$ (*or the non-zero primitive element* $y \in H^3(\Omega^\infty X;\ Z/2)$ *has* $Sq^1 y \neq 0$ *when* $2 \in P$), *then* $X \simeq so_{(P)}$. *If* $\Omega^\infty X \simeq SO[n, \infty)_{(P)}$ *with* $n \geq 2$, *then* $X \simeq so[n, \infty)_{(P)}$.

(v) *If* $\Omega^\infty X \simeq (SO/U)_{(P)}$ *and if the generator* $x \in H_2(\Omega^\infty X;\ Z/p)$ *has* $x^p \neq 0$ *for each odd* $p \in P$ *and* $Q^4x \neq 0$ *when* $p = 2 \in P$ (*or the non-zero primitive element* $y \in H^6(\Omega^\infty X;\ Z/2)$ *has* $\mathrm{Sq}^2 y \neq 0$ *when* $p = 2 \in P$), *then* $X \simeq (SO/U)_{(P)}$. *If* $\Omega^\infty X \simeq (SO/U)[n, \infty)_{(P)}$ *with* $n \geq 3$, *then* $X \simeq (so/u)[n, \infty)_{(P)}$.

(vi) *If* $\Omega^\infty X \simeq (U/Sp)_{(P)}$ *and if the generator* $x \in H_1(\Omega^\infty X;\ Z/p)$ *has* $Q^1x \neq 0$ *for each odd* $p \in P$ *and* $Q^4x \neq 0$ *when* $p = 2 \in P$, *then* $X \simeq (u/sp)_{(P)}$. *If* $\Omega^\infty X \simeq (U/Sp)[n, \infty)_{(P)}$ *with* $n \geq 2$, *then* $X \simeq (u/sp)[n, \infty)_{(P)}$.

(vii) *If* $\Omega^\infty X \simeq BSp[n, \infty)_{(P)}$ *with* $n \geq 1$, *then* $X \simeq bsp[n, \infty)_{(P)}$.

(viii) *If* $\Omega^\infty X \simeq Sp[n, \infty)_{(P)}$ *with* $n \geq 1$, *then* $X \simeq sp[n, \infty)_{(P)}$.

(ix) *If* $\Omega^\infty X \simeq (Sp/U)_{(P)}$ *and if the generator* $x \in H_2(\Omega^\infty X;\ Z/p)$ *has* $x^p \neq 0$ *for each odd* $p \in P$, *then* $X \simeq (sp/u)_{(P)}$. *If* $\Omega^\infty X \simeq (Sp/U)[n, \infty)_{(P)}$ *with* $n \geq 3$, *then* $X \simeq (sp/u)[n, \infty)_{(P)}$.

(x) *If* $\Omega^\infty X \simeq (U/O)_{(P)}$ *and if the generator* $x \in H_1(\Omega^\infty X;\ Z/p)$ *has* $Q^1x \neq 0$ *for each odd* $p \in P$ *and* $x^2 \neq 0$ *when* $p = 2 \in P$, *then* $X \simeq (u/o)_{(P)}$. *If* $\Omega^\infty X \simeq (U/O)[n, \infty)_{(P)}$ *with* $n \geq 2$, *then* $X \simeq (u/o)[n, \infty)_{(P)}$.

Proof. In part (i), the condition $\Omega^\infty X \simeq BU_{(P)}$ implies $\Omega^\infty X_p^\wedge \simeq BU_p^\wedge$ for each $p \in P$. Thus $X_p^\wedge \simeq bu_p^\wedge$ by 3.2(i) and the hypothesis on $H_*(\Omega^\infty X;\ Z/p)$. Hence $X \simeq bu_{(P)}$ by the following theorem. The other parts follow similarly.

THEOREM 3.4. *Let* L *be one of the spectra* $KO[n, \infty)$ *or* $K[n, \infty)$ *for an integer* n. *For an* $(n-1)$*-connected spectrum* X *and set* P *of primes, suppose that* $X_p^\wedge \simeq L_p^\wedge$ *for each* $p \in P$ *and that* $\pi_* X \approx \pi_* L_{(P)}$. *If* P *is finite, then* $X \simeq L_{(P)}$. *If* P *is infinite, then there are maps* $X \to L_{(P)}$ *and* $L_{(P)} \to X$ *whose homotopy fibres have finite homotopy groups.*

This will be proved in 3.14.

REMARK 3.5. Applying Theorem 3.3 to the spectrum $bso_\otimes$ which satisfies $\Omega^\infty bso_\otimes \simeq BSO$, we deduce that $bso_{\otimes(P)} \simeq bso_{(P)}$ for any finite set P of primes, generalizing the result of Adams-Priddy for a single prime. However, $bso_\otimes \not\simeq bso$ by [4, p. 146], so our finiteness assumption on P

cannot be omitted. Theorem 3.3 clearly remains valid when the P-localization $Y_{(P)}$ is replaced by the P-adic completion $Y_p^\wedge = \prod_{p \in P} Y_P^\wedge$, and the finiteness assumption on P can then be omitted.

Let C and D be among the spaces considered in 3.3, but with P possibly infinite. Then C is equivalent to $\Omega^\infty(\Sigma^i K)[m, \infty)_{(P)}$ or $\Omega^\infty(\Sigma^i KO)[m, \infty)_{(P)}$, and D is equivalent to $\Omega^\infty(\Sigma^j K)[n, \infty)_{(P)}$ or $\Omega^\infty(\Sigma^j KO)[n, \infty)_{(P)}$. We call C *out of phase* to D if C or D involves K and $i - j$ is odd, or if C and D involve KO and $i - j \equiv 3, 5, 6$, or $7 \bmod 8$. We call C *in weak phase* to D if C and D involve KO and $i - j \equiv 1$ or $2 \bmod 8$. The following theorem generalizes faithfulness results of Madsen-Snaith-Tornehave **[19]** for Ω^∞ and non-existence results of Anderson **[8]** for phantom maps.

THEOREM 3.6. *For an arbitrary set P of primes and $n \geq 1$, let X, $Y \in HS$ be connected spectra such that, up to equivalence, $\Omega^\infty X$ and $\Omega^\infty Y$ are among the spaces: $BU[n, \infty)_{(P)}$, $U[n, \infty)_{(P)}$, $BO[n, \infty)_{(P)}$, $SO[n, \infty)_{(P)}$, $(SO/U)[n, \infty)_{(P)}$, $(U/Sp)[n, \infty)_{(P)}$, $BSp[n, \infty)_{(P)}$, $Sp[n, \infty)_{(P)}$, $(Sp/U)[n, \infty)_{(P)}$, and $(U/O)[n, \infty)_{(P)}$. Suppose that the* mod-p *(co)homology of $\Omega^\infty X$ and $\Omega^\infty Y$ satisfies the conditions in* 3.3 *for each prime $p \in P$. Then:*

(i) Ω^∞: $[X, Y] \to [\Omega^\infty X, \Omega^\infty Y]$ *is mono.*

(ii) *If $\Omega^\infty X$ is out of phase to $\Omega^\infty Y$, then $[X, Y] = 0$.*

(iii) *If $\Omega^\infty X$ is not in weak phase to $\Omega^\infty Y$, then π_*: $[X, Y] \to [Q \otimes \pi_* X, Q \otimes \pi_* Y]$ is mono.*

(iv) *There are no non-zero phantom maps in $[X, Y]$.*

This will be proved in 3.11.

REMARK 3.7. Theorem 3.6 remains valid when the P-localization is replaced by the P-adic completion, or when $\Omega^\infty X$ is allowed to have higher connectivity than $\Omega^\infty Y$. However, parts (i)–(iii) can fail under the reverse connectivity assumption. For instance, consider the standard fibration of spectra

$$bu_{(P)} \to H(Z_{(P)}, 2) \to u[5, \infty)_{(P)}$$

for $2 \in P$ and let $(so/u)_{(P)} \to H(Z_{(P)}, 2)$ be the Postnikov map. Then the composite map f: $(so/u)_{(P)} \to u[5, \infty)_{(P)}$ is essential although $\Omega^\infty f \simeq 0$, $\Omega^\infty(so/u)_{(P)}$ is out of phase to $\Omega^\infty u[5, \infty)_{(P)}$, and f is not detected by homotopy groups.

To prove 3.6 we need the following result, due largely to Adams [**4**] but covering additional cases. Let $XG = X \wedge MG$ for a spectrum X and abelian group G.

THEOREM 3.8. *For a torsion free abelian group Λ and integers n and i, the groups $[K[n,\infty), K\Lambda]_i$, $[KO[n,\infty), K\Lambda]_i$, and $[K[n,\infty), KO\Lambda]_i$ are naturally isomorphic to countable products of Λ's for even i and to* 0 *for odd i. The group $[KO[n,\infty), KO\Lambda]_i$ is naturally isomorphic to a countable product of Λ's for $i \equiv 0$ or* 4 mod 8, *to a countable product of $\Lambda/2\Lambda$'s for $i \equiv 1$ or* 2 mod 8, *and to* 0 *otherwise.*

Proof. For any $X \in HS$ there are natural exact sequences

$$0 \to \operatorname{Ext}(K_{j-1}X, \Lambda) \to (K\Lambda)^j X \to \operatorname{Hom}(K_j X, \Lambda) \to 0$$

$$0 \to \operatorname{Ext}(KO_{j+3}X, \Lambda) \to (KO\Lambda)^j X \to \operatorname{Hom}(KO_{j+4}X, \Lambda) \to 0$$

of Anderson which may be obtained from the universal coefficient theorem of [**3**] using the $\pi_* K$-injectivity of $\pi_* KG$ and the $\pi_* KO$-injectivity of $\pi_* KOG$ when G is divisible. Since the K_*-localizations ($= KO_*$-localizations) of $K(-\infty, n-1]$ and $KO(-\infty, n-1]$ are given by $K(-\infty, n-1]_Q$ and $KO(-\infty, n-1]_Q$, the canonical map $r\colon K \to KO$ induces epimorphisms $K_*K(-\infty, n-1] \to K_*KO(-\infty, n-1]$, $K_*K(-\infty, n-1] \to KO_*K(-\infty, n-1]$, and $KO_*K(-\infty, n-1] \to KO_*KO(-\infty, n-1]$. By [**4**, p. 162], the map $K_*K[n,\infty) \to K_*K$ is mono, so $K_*K \to K_*(-\infty, n-1]$ is epi. Hence the maps $K_*KO[n,\infty) \to K_*KO$, $KO_*K[n,\infty) \to KO_*K$, and $KO_*KO[n,\infty) \to KO_*KO$ are all mono with rational cokernels. By [**5**], K_*K is countably free over π_*K on generators of degree 0, and thus KO_*KO is countably free over π_*KO on generators of degree 0, since $KO_0KO \subset K_0K$ and $KO_*KO \approx \pi_*KO \otimes KO_0KO$ by [**6**]. This gives sufficient information on the groups $K_iK[n,\infty)$, $K_iKO[n,\infty)$, $KO_iK[n,\infty)$, and $KO_iKO[n,\infty)$ to deduce the theorem from Anderson's exact sequences.

This proof also shows for any n that $KO_*KO[n,\infty)$ is countably free over π_*KO on generators of degree 0, since KO_*KO has this property and $KO_*KO[n,\infty) \to KO_*KO$ is mono with rational cokernel.

If $\Lambda/2\Lambda \neq 0$ and $i \equiv 1$ or 2 mod 8, then there are uncountably many $f \in [KO[n,\infty), KO\Lambda]_i$ with $f_*\colon \pi_*KO[n,\infty) \to \pi_{*+i}KO\Lambda$ zero. This follows when $\Lambda = Z$ by Theorem 3.8 since there is only one non-zero f_* compatible with the action of η and the $[MZ/2, -]_*$-periodicity, and it follows in general by naturality under $Z \to \Lambda$. However, the other homotopy classes in 3.8 are detected by homotopy groups.

COROLLARY 3.9. *If X equals $K[n,\infty)$ or $KO[n,\infty)$ and Y equals K or KO, then*
$$\pi_*\colon [X, Y\Lambda]_i \to \mathrm{Hom}(\pi_* X, Q \otimes \pi_{*+i} Y\Lambda)$$
is mono, except in the above-mentioned case.

Proof. By Theorem 3.8 the map $[X, Y\Lambda]_i \to [X, Y\Lambda Q]_i$ is mono in the required cases.

Theorem 3.8 also permits a very short proof of Anderson's result **[8]** on the non-existence of phantom cohomology operations in connective K-theory.

COROLLARY 3.10. *Let X equal $\Sigma^i K$ or $\Sigma^i KO$ for some i and let Y equal K or KO. If $f\colon X[m,\infty)_{(P)} \to Y[n,\infty)_{(P)}$ is a phantom map for a set P of primes and integers m, n, then $f \simeq 0$.*

Proof. Let $k = \min\{m, n\}$ and consider the exact sequence
$$\begin{array}{ccccc} \cdots & \to & [\Sigma X[m,\infty)_{(P)}, Y[k,n-1]_{(P)}] & \to & [X[m,\infty)_{(P)}, Y[n,\infty)_{(P)}] \\ & \to & [X[m,\infty)_{(P)}, Y[k,\infty)_{(P)}] & \to & \cdots. \end{array}$$
Since the first group is finitely generated over $Z_{(P)}$ and the third group is a product of $Z_{(P)}$'s or $Z/2$'s, the second group is reduced, i.e., has no non-zero divisible subgroup. Thus it contains no non-trivial phantom map by 2.9.

3.11. *Proof of Theorem* 3.6. Let L and M be the spectra of form $(\Sigma^i K)[n,\infty)$ or $(\Sigma^i KO)[n,\infty)$ such that $\Omega^\infty X \simeq \Omega^\infty L_{(P)}$ and $\Omega^\infty Y \simeq \Omega^\infty M_{(P)}$. Then for each $p \in P$, $\Omega^\infty X_p^\wedge \simeq \Omega^\infty L_p^\wedge$ and $\Omega^\infty Y_p^\wedge \simeq \Omega^\infty M_p^\wedge$, and thus $X_p^\wedge \simeq L_p^\wedge$ and $Y_p^\wedge \simeq M_p^\wedge$ by 3.4. Let $f\colon X \to Y$ be a map satisfying one of the conditions: (i) $\Omega^\infty f = 0$; (ii) $\Omega^\infty X$ is out of phase to $\Omega^\infty Y$; (iii) $\Omega^\infty X$ is not in weak phase to $\Omega^\infty Y$ and $f_*\colon Q \otimes \pi_* X \to Q \otimes \pi_* Y$ is zero; (iv) f is phantom. Then $f_p^\wedge\colon X_p^\wedge \to Y_p^\wedge$ is zero for each $p \in P$ by 2.5, 3.8 with $\Lambda = Z_p^\wedge$, 3.9 with $\Lambda = Z_p^\wedge$, and 2.9. Thus f factors as a composition $X \xrightarrow{u} X_Q \to Y$ by 2.8. There are maps $X \to L_{(P)}$ and $M_{(P)} \to Y$ whose homotopy fibres have finite homotopy groups by 3.4, and there is an associated factorization $X \to L_{(P)} \to X_Q \to M_{(P)} \to Y$ of f. The resulting map $L_{(P)} \to M_{(P)}$ is phantom by 2.9 and thus zero by 3.10. Hence $f = 0$.

To prove Theorem 3.4, we need results on the self-equivalences of $L/p^\infty = LZ/p^\infty$ for p prime where L is one of the spectra $K[n,\infty)$ or $KO[n,\infty)$ and where Z/p^∞ is the p-torsion subgroup of Q/Z. Consider the set of degrees d such that $\pi_d L/p^\infty \approx Z/p^\infty$ and let $d(i)$ denote the ith such degree in increasing order. A set $\{\varphi_i\}_{1 \le i < k}$ of endomorphisms φ_i: $\pi_{d(i)} L/p^\infty \to \pi_{d(i)} L/p^\infty$ is called *realizable* if there exists a map f: $L/p^\infty \to L/p^\infty$ inducing φ_i for $1 \le i < k$. An endomorphism φ_k is called *compatible* with $\{\varphi_i\}_{1 \le i < k}$ if the set $\{\varphi_i\}_{1 \le i \le k}$ is realizable. The following lemma will give sufficient control over the self-maps of L/p^∞ or $L_p^\wedge$. Some related results are given by Adams in [**4**].

LEMMA 3.12. *There is a sequence of finite subgroups $S_i \subset \pi_{d(i)} L/p^\infty$ such that each realizable set $\{\varphi_i\}_{1 \le i < k}$ determines an endomorphism $\tilde{\varphi}_k$ of S_k whose extensions to $\pi_{d(k)} L/p^\infty$ are precisely the endomorphisms compatible with $\{\varphi_i\}_{1 \le i < k}$. If $\{\varphi_i\}_{1 \le i < \infty}$ is a set of endormphisms with $\{\varphi_i\}_{1 \le i < k}$ realizable for each $k < \infty$, then there exists a unique map f: $L/p^\infty \to L/p^\infty$ inducing φ_i for $1 \le i < \infty$.*

Proof. We suppose that $L = KO[n, \infty)$, but our proof can be adapted to $K[n, \infty)$. The canonical map $L/p^\infty \to KO/p^\infty$ induces isomorphisms

$$[L/p^\infty, L/p^\infty] \approx [L/p^\infty, KO/p^\infty] \approx [KO/p^\infty, KO/p^\infty],$$

and there is a universal coefficient isomorphism

$$[KO/p^\infty, KO/p^\infty] \approx \operatorname{Hom}(KO_4 KO/p^\infty, \pi_4 KO/p^\infty)$$

as in the proof of 3.8. For each $i \ge 1$, let $g_i \in \pi_{4-d(i)} KO \approx Z$ be a generator and note that g_i: $KO_{d(i)} KO/p^\infty \approx KO_4 KO/p^\infty$ since the $\pi_* KO$-module $KO_* KO/p^\infty$ is a direct sum of copies of $\pi_* KO/p^\infty$. For each map b: $KO/p^\infty \to KO/p^\infty$ the diagram

$$\begin{array}{ccc} \pi_{d(i)} KO/p^\infty & \xrightarrow{g_i h} & KO_4 KO/p^\infty \\ \downarrow b_* & & \downarrow b_\# \\ \pi_{d(i)} KO/p^\infty & \underset{\approx}{\xrightarrow{g_i}} & \pi_4 KO/p^\infty \end{array}$$

commutes where h is the Hurewicz monomorphism and $b_\#$ corresponds to b via the universal coefficient isomorphism. Let $G_i \subset KO_4 KO/p^\infty$ denote the image of $g_i h$. It suffices to show that $G_k \cap (G_1 + \cdots + G_{k-1})$ is finite for each k and to let S_k correspond to its counterimage under $g_i h$. Choose $r > 1$ relatively prime to p. Then $\psi^r w = r^{e(i)} w$ for each $w \in G_i$ where $e(i) = (4 - d(i))/2$, and we let $\xi_k \in Z_{(p)}[x]$ be a polynomial with $\xi_k(r^{e(i)}) = 0$ for $1 \le i < k$ and $\xi_k(r^{e(k)}) = p^j$ for some $j \ge 0$. Using the operator $\xi_k(\psi^r)$, one shows that $G_k \cap (G_1 + \cdots + G_{k-1})$ is annihilated by p^j and is therefore finite since $G_k \approx Z/p^\infty$. Next, the existence of f

follows easily since $\pi_4 KO/p^\infty \approx Z/p^\infty$ is divisible. For uniqueness of f it suffices to show $b \simeq 0$ when b: $KO/p^\infty \to KO/p^\infty$ is a map with $b_* = 0$ on $\pi_{d(i)}KO/p^\infty$ for each $i \geq 1$. For $1 \leq j < \infty$, there is a K_*-equivalence A: $\Sigma^q MZ/p^j \to MZ/p^j$ of Adams [1] with $q > 0$, and there is an induced isomorphism

$$A^*\colon [MZ/p^j, KO/p^\infty]_* \approx [MZp^j, KO/p^\infty]_{*+q}.$$

Thus $b_* = 0$ on $\pi_{4t}KO/p^\infty$ for all t, and $b_\#$: $KO_4 KO/p^\infty \to \pi_4 KO/p^\infty$ vanishes on the image of the canonical map $\oplus_t \pi_{4t} KO/p^\infty \to KO_4 KO/p^\infty$. This map is onto since it is a quotient of the isomorphism $\oplus_t \pi_{4t} KO_Q \to KO_4 KO_Q$. Thus $b_\# = 0$ and $b \simeq 0$.

Using the notation of 3.12, we have

LEMMA 3.13. *If $d(j) - d(1) < 2p - 2$ then $S_j = 0$. If f: $L/p^\infty \to L/p^\infty$ induces an automorphism f_*: $\pi_{d(j)}L/p^\infty \approx \pi_{d(j)}L/p^\infty$ for each j with $d(j) - d(1) < 2p - 2$, then f: $L/p^\infty \simeq L/p^\infty$. If $\{\varphi_i\}_{1 \leq i < k}$ is a realizable set of automorphisms, then $\tilde{\varphi}_k$: $S_k \to S_k$ is an automorphism.*

Proof. The first part follows since L/p^∞ splits as in [2] for p odd and since $j = 1$ for $p = 2$. The second part follows using the Adams periodicity of mod-p homotopy groups of L/p^∞ and knowledge of its k-invariants for $p = 2$. The third part follows from the preceding parts.

3.14. *Proof of Theorem* 3.4. For each $p \in P$, $X/p^\infty \simeq L/p^\infty$ since $X_p^\wedge \simeq L_p^\wedge$, and $X_Q \simeq L_Q$ since $\pi_* X \approx \pi_* L_{(J)}$. Thus there are cofibre sequences

$$\begin{array}{ccccc}
X & \xrightarrow{s} & L_Q & \xrightarrow{t} & L/P^\infty \\
\downarrow a & & \downarrow b & & \downarrow c \\
L_{(J)} & \xrightarrow{u} & L_Q & \xrightarrow{v} & L/P^\infty
\end{array}$$

inducing short exact sequences

$$\begin{array}{ccccccccc}
0 & \to & \pi_{d(i)}X & \to & \pi_{d(i)}L_Q & \to & \pi_{d(i)}L/P^\infty & \to & 0 \\
 & & \downarrow \alpha_i & & \downarrow \beta_i & & \downarrow \gamma_i & & \\
0 & \to & \pi_{d(i)}L_{(J)} & \to & \pi_{d(i)}L_Q & \to & \pi_{d(i)}L/P^\infty & \to & 0
\end{array}$$

for each $i \geq 1$. Assuming that P is finite, we construct isomorphisms $(\alpha_i, \beta_i, \gamma_i)$ for $i \geq 1$ such that the above diagram commutes and the p-components of $\{\gamma_i\}_{1 \leq i < k}$ are realizable for each $p \in P$. Let α_1 be an arbitrary isomorphism and let β_1, γ_1 be induced by α_1. Given $(\alpha_i, \beta_i, \gamma_i)$

for $1 \leq i < k$, let $\bar{S}_k \subset \pi_{d(k)}L/P^\infty$ be the sum of the $S_k \subset \pi_{d(k)}L/P^\infty$ for $p \in P$, and let $\tilde{\gamma}_k$ be the isomorphism induced by $\{\gamma_i\}_{1 \leq i < k}$. It induces an isomorphism $\bar{\alpha}_k$: $\pi_{d(k)}X \otimes Z/n \approx \pi_{d(k)}L_{(P)} \otimes Z/n$ where $n = |\bar{S}_k|$. Since P is finite, the quotient map $Z_{(P)} \to Z/n$ restricts to an epimorphism $Z^*_{(P)} \to Z^*_n$ for units. Thus we can choose α_k: $\pi_{d(k)}X \approx \pi_{d(k)}L_{(P)}$ inducing $\bar{\alpha}_k$ and then let β_k, γ_k be induced by α_k. Then γ_k extends $\tilde{\gamma}_k$ and is compatible with $\{\gamma_i\}_{1 \leq i < k}$ on p-components for $p \in P$. After completing this inductive construction, let b: $L_Q \simeq L_Q$ and c: $L/P^\infty \simeq L/P^\infty$ be the equivalences induced by $\{\beta_i\}_{i \geq 1}$ and $\{\gamma_i\}_{i \geq 1}$. Now $ct \simeq vb$ by 2.9 and we obtain a: $X \simeq L_{(P)}$. When P is infinite, similar methods give the required maps $X \to L_{(P)}$ and $L_{(P)} \to X$.

4. K_*-Localized Moore spectra. Recall that Adams constructed K_*-equivalences A: $\Sigma^{q_j}MZ/p^j \to MZ/p^j$ in HS for p prime and $j \geq 1$, where $q_j = \operatorname{Max}\{8, 2^{j-1}\}$ for $p = 2$ and $q_j = 2(p-1)p^{j-1}$ for p odd (see [1]). These A's induce equivalences A_K: $\Sigma^{q_j}(MZ/p^u)_K \simeq (MZ/p^j)_K$ demonstrating the periodicity of the spectra $(MZ/p^j)_K$. However, these A's are not canonically determined, and the A_K's need not be compatible for successive j's. In this section we construct a compatible sequence of equivalences α: $\Sigma^{q_j}(MZ/p^j)_K \simeq (MZ/p^j)_K$ together with an associated system of Adams-like maps which will be used to prove Theorem 2.1.

LEMMA 4.1. *For p prime, $j \geq 1$, and $n \equiv 0 \bmod q_j$, there are isomorphisms $\pi_{n-2}(MZ/p^j)_K \approx 0$, $\pi_{n-1}(MZ/p^j)_K \approx Z/p^j$, and $\pi_n(MZ/p^k)_K \approx Z/p^j \oplus G$ where $G = 0$ for p odd and $G = Z/2$ for $p = 2$.*

Proof. By [10, §4] there is a fibre sequence

$$(MZ/p^j)_K \to KO/p^j \xrightarrow{\psi^r - 1} KO/p^j$$

in HS where $r = 3$ for $p = 2$ and where r is a positive integer generating the group of units of Z/p^2 for p odd. The homomorphism $(\psi^r - 1)_*$: $\pi_i KO/p^j \to \pi_i KO/p^j$ is zero for $n - 2 \leq i \leq n + 2$. Thus $\pi_i(MZ/p^j)_K$ has the desired properties for p odd, while $\pi_i(MZ/2^j)_K$ is isomorphic to 0 for $i = n - 2$, to $Z/2^j$ for $i = n - 1$, and to $Z/2^j \oplus Z/2$ or $Z/2^{j+1}$ for $i = n$. Since η^2 acts nontrivially on $\pi_n KO/2^j$, η^2 also acts nontrivially on $\pi_{n-1}(MZ/2^j)_K$. Thus $\pi_n(MZ/2^j)_K$ contains an element of order 2 which is not divisible by 2, and consequently $\pi_n(MZ/2^j)_K \approx Z/2^j \oplus Z/2$.

Choose a sequence of maps

$$MZ/p \xrightarrow{e} MZ/p^2 \to \cdots \to MZ/p^j \xrightarrow{e} MZ/p^{j+1} \to \cdots$$

in HS which is carried by H_0 to the canonical sequence of injections of Z/p^j's.

LEMMA 4.2. *For p prime, $j \geq 1$, and $n \equiv 0 \bmod q_{j+1}$, the image of the homomorphism e_{K*}: $\pi_n(MZ/p^j)_K \to \pi_n(MZ/p^{j+1})_K$ is $p\pi_n(MZ/p^{j+1})_K$.*

Proof. This follows by inspecting the homotopy exact sequence of the cofibering

$$(MZ/p^j)_K \xrightarrow{e_K} (MZ/p^{j+1})_K \to (MZ/p)_K$$

using the groups calculated in 4.1.

PROPOSITION 4.3. *For p prime, there exists a sequence of equivalences α: $\Sigma^{q_j}(MZ/p^j)_K \simeq (MZ/p^j)_K$ for $j \geq 1$ such that the diagrams*

$$\begin{array}{ccc} \Sigma^{q_{j+1}}(MZ/p^j)_K & \xrightarrow{\alpha \circ \cdots \circ \alpha} & (MZ/p^j)_K \\ \downarrow \Sigma^{q_{j+1}} e_K & & \downarrow e_K \\ \Sigma^{q_{j+1}}(MZ/p^{j+1})_K & \xrightarrow{\alpha} & (MZ/p^{j+1})_K \end{array}$$

commute in HS.

Proof. Let α: $\Sigma^{q_1}(MZ/p)_K \to (MZ/p)_K$ be induced by an Adams map, and suppose inductively that the K_*-equivalence α: $\Sigma^{q_j}(MZ/p^j)_K \to (MZ/p^j)_K$ has been constructed. Letting $n = q_{j+1}$, consider the diagram

$$\begin{array}{ccc} \Sigma^n MZ/p^j & \xrightarrow{\alpha \circ \cdots \circ \alpha} & (MZ/p^j)_K \\ \downarrow \Sigma^n e & & \downarrow e_K \\ \Sigma^n MZ/p^{j+1} & \xrightarrow{f} & (MZ/p^{j+1})_K \end{array}$$

and apply 4.1 and 4.2 to construct a map f such that the associated diagram of π_n-groups commutes. Hence $\varepsilon \circ \sigma = 0$ where $\varepsilon = f \circ \Sigma^n e - e_K \circ (\alpha \circ \cdots \circ \alpha)$ is the commutation error and σ is the indicated map in the cofibering

$$\Sigma^n S \xrightarrow{\sigma} \Sigma^n MZ/p^j \xrightarrow{\tau} \Sigma^{n+1} S.$$

Thus there exists δ: $\Sigma^{n+1}S \to (MZ/p^{j+1})_K$ such that $\varepsilon = \delta \circ \tau$, and there clearly exists μ: $\Sigma^n MZ/p^{j+1} \to \Sigma^{n+1}S$ such that $\tau = \mu \circ \Sigma^n e$. Consequently the diagram commutes when f is replaced by $\bar{f} = f - \delta \circ \mu$, and $\bar{f}$

is a K_*-equivalence since $\alpha \circ \cdots \circ \alpha$ is. The inductive step is completed by letting α: $\Sigma^n(MZ/p^{j+1})_K \to (MZ/p^{j+1})_K$ correspond to $\bar{f}$.

Using our results on the surjectivity of the K_*-localization map $[X,Y]_i \to [X_K, Y_K]_i$ for sufficiently large i when X and Y are finite CW-spectra (see **[13]**) we can deduce that each of the equivalences α: $\Sigma^{q_j}(MZ/p^j)_K \to (MZ/p^j)_K$ in 4.3 is induced by some K_*-equivalence A: $\Sigma^{q_j}MZ/p^j \to MZ/p^j$. However, to achieve the required compatibility, we shall instead use the following lemma to construct our system of Adams-like maps.

LEMMA 4.4. *For each finite CW-spectrum Y, there exists a sequence of K_*-equivalences of finite CW-spectra $Y = Y_0 \to Y_1 \to Y_2 \to \cdots$ whose homotopy direct limit is the K_*-localization of Y in HS. Thus* $\operatorname{colim}_n[X, Y_n]_* \approx [X, Y_K]_*$ *for each finite CW-spectrum X.*

Proof. Let L_0, L_1, $L_2, \ldots$ be a sequential listing of the homotopy cofibres of the Adams maps $\Sigma^i A$: $\Sigma^{i+q_1}MZ/p \to \Sigma^i MZ/p$ for p prime and $i \in Z$. By **[10]**, a spectrum $E \in HS$ is K_*-local if and only if $[L_i, E] = 0$ for each $i \in Z$. Let $Y_0 = Y$ and suppose inductively that the finite CW-spectrum Y_n is given. Let F_n denote the K_*-acylic finite CW-spectrum $\bigvee_{i,f} L_{i,f}$ for $0 \leq i \leq n$, $f \in [L_i, Y_n]$, and $L_{i,f} = L_i$. Then construct $Y_n \to Y_{n+1}$ as the homotopy cofibre of a map $F_n \to Y_n$ acting by f on each $L_{i,f}$. The homotopy colimit of the resulting sequence $Y = Y_0 \to Y_1 \to Y_2 \to \cdots$ is K_*-local by the above criterion, and the lemma follows easily.

The following proposition will provide our system of Adams-like maps of finite CW-spectra. For notational convenience, we take $j \geq b$ where $b = 1$ for p odd and $b = 4$ for $p = 2$.

PROPOSITION 4.5. *For each prime p, there exists an array of finite CW-spectra W_j^i for $j \geq b$ and $0 \leq i \leq p$, together with K_*-equivalences u: $W_j^i \to W_j^{i+1}$ and a: $\Sigma^{q_j}W_j^i \to W_j^{i+1}$ and maps λ: $W_j^p \to W_{j+1}^1$ in HS such that $W_j^0 = MZ/p^j$ and the diagrams*

$$\begin{array}{ccc} \Sigma^{q_j}W_j^i & \xrightarrow{a} & W_j^{i+1} \\ \downarrow{\scriptstyle \Sigma^{q_j}u} & & \downarrow{\scriptstyle u} \\ \Sigma^{q_j}W_j^{i+1} & \xrightarrow{a} & W_j^{i+2} \end{array} \qquad \begin{array}{ccc} W_j^0 \vee \Sigma^{q_{j+1}}W_j^0 & \xrightarrow{u^p \vee a^p} & W_j^p \\ \downarrow{\scriptstyle e \vee \Sigma^{q_{j+1}}e} & & \downarrow{\scriptstyle \lambda} \\ W_{j+1}^0 \vee \Sigma^{q_{j+1}}W_{j+1}^0 & \xrightarrow{u \vee a} & W_{j+1}^1 \end{array}$$

commute.

Proof. Let α: $\Sigma^{q_j}(MZ/p^j)_K \simeq (MZ/p^j)_K$ for $j \geq b$ be a compatible sequence of equivalences, and for each j let

$$MZ/p^j = V_t^0 \xrightarrow{t} V_j^1 \xrightarrow{t} V_j^2 \xrightarrow{t} \cdots \to (MZ/p^j)_K$$

be a sequence of K_*-equivalences of finite CW-spectra as in 4.4. For $j = b$ construct a finite subsequence

$$MZ/p^b = W_b^0 \xrightarrow{u} W_b^1 \xrightarrow{u} \cdots \xrightarrow{u} W_b^p \to (MZ/p^b)_K$$

together with maps a: $\Sigma^{q_b}W_b^i \to W_b^{i+1}$ for $0 \leq i < p$ such that the diagram

$$\begin{array}{ccccccccc}
\Sigma^{q_b}W_b^0 & \to & \Sigma^{q_b}W_b^1 & \to & \cdots & \to & \Sigma^{q_b}W_b^{p-1} & \to & \Sigma^{q_b}(MZ/p^b)_K \\
\downarrow a & & \downarrow a & & & & \downarrow a & & \downarrow \alpha \\
W_b^1 & \to & W_b^2 & \to & \cdots & \to & /W_b^p & \to & (MZ/p^b)_K
\end{array}$$

commutes. Next for sufficiently large i, choose a map c such that the diagram

$$\begin{array}{ccc}
\Sigma^{q_{b+1}}MZ/p^{b+1} & \xrightarrow{c} & V_{b+1}^i \\
\downarrow & & \downarrow \\
\Sigma^{q_{b+1}}(MZ/p^{b+1})_K & \xrightarrow{\alpha} & (MZ/p^{b+1})_K
\end{array}$$

commutes. Now the solid arrow diagram

$$\begin{array}{ccccc}
MZ/p^b \vee \Sigma^{q_{b+1}}MZ/p^b & \xrightarrow{u^p \vee a^p} & W_b^p & \to & (MZ/p^b)_K \\
\downarrow & & \downarrow \lambda & & \downarrow \\
MZ/p^{b+1} \vee \Sigma^{q_{b+1}}MZ/p^{b+1} & \xrightarrow{t_i \vee c} V_{b+1}^i \to & V_{b+1}^k & \to & (MZ/p^{b+1})_K
\end{array}$$

commutes, and for sufficiently large k there is a map λ such that the two subdiagrams commute. Let u: $W_{b+1}^0 \to W_{b+1}^1$ denote the map t^k: $MZ/p^{b+1} \to V_{b+1}^k$, and define λ: $W_p^b \to W_{b+1}^1$ and a: $\Sigma^{q_{b+1}}W_{b+1}^0 \to W_{b+1}^1$ from the diagram. Continuing in the obvious way, one inductively constructs the required array.

5. Cospectra associated with Moore spaces. Continuing toward a proof of Theorem 2.1, we now obtain a system of cospectra associated with Z/p^j-Moore spaces where p is a fixed prime. We work simplicially and assume familiarity with the elementary theory of simplicial sets (see **[20]**, **[17]**). Let $s.sets_*$ denote the category of pointed simplicial sets and recall that each $L \in s.sets_*$ has a geometric realization $|L|$ which is a pointed CW-complex. Let

$$M(Z/p,1) \xrightarrow{e} M(Z/p^2,1) \xrightarrow{e} M(Z/p^3,1) \xrightarrow{e} \cdots$$

be a sequence of cofibrations in $s.sets_*$ such that each $|M(Z/p^j,1)|$ is a Moore space of type $(Z/p^j,1)$ and such that the sequence is carried by H_1 to the canonical sequence of injections $Z/p \to Z/p^2 \to Z/p^3 \to \cdots$. Let $S^1 \in s.sets_*$ be the standard 1-sphere whose only non-degenerate simplicies are a vertex and 1-simplex, and for $n \geq 2$ let $S^n \in s.sets_*$ denote the n-fold smash product $S^1 \wedge \cdots \wedge S^1$. For $m > 1$ and $j \geq 1$, let $M(Z/p^j,m) = M(Z/p^j,1) \wedge S^{m-1}$ and let e: $M(Z/p^j,m) \to M(Z/p^{j+1},m)$ denote $e \wedge 1$. An *augmented* cospectrum X consists of sequence of objects $X^n \in s.sets_*$ and maps σ: $S^1 \wedge X^{n+1} \to X^n$ for $n \geq 0$, together with an object $X^a \in s.sets_*$ and map ε: $X^a \to X^0$. A *map* f: $X \to Y$ of augmented cospectra consists of a sequence of maps f^n: $X^n \to Y^n$ such that $f^n\sigma = \sigma(1 \wedge f^{n+1})$ for each $n \geq 0$, together with a map f^a: $X^a \to Y^a$ such that $\varepsilon f^a = f^0\varepsilon$. For an augmented cospectrum X and object $L \in s.sets_*$, let $X \wedge L$ denote the augmented cospectrum with $(X \wedge L)^i = X^i \wedge L$ and $(X \wedge L)^a = X^a \wedge L$. As in 4.5 let $b = 1$ for p odd and let $b = 4$ for $p = 2$. Also let $q_j = 2(p-1)p^{j-1}$ for p odd and let $q_j = \operatorname{Max}\{8, 2^{j-1}\}$ for $p = 2$. Our goal in this section is to prove

PROPOSITION 5.1. *For each prime p, there exists an increasing sequence of positive integers m_j for $j \geq b$ together with a sequence of augmented cospectra $P(j)$ and maps ρ: $P(j) \wedge S^{d_j} \to P(j+1)$ with $d_j = m_{j+1} - m_j$ such that the following conditions hold: $P(j)^a = M(Z/p^j, m_j)$; the maps ε: $P(j)^a \to P(j)^0$ and σ: $S^1 \wedge P(j)^{n+1} \to P(j)^n$ are K_*-equivalences; the maps ρ^a: $M(Z/p^j, m_j) \wedge S^{d_j} \to M(Z/p^{j+1}, m_{j+1})$ equal e; the maps ρ^n: $P(j)^n \wedge S^{d_j} \to P(j+1)^n$ are cofibrations; and for each $j \geq b$ and $n \geq 0$ there exists a K_*-equivalence $S^i \wedge M(Z/p^j, m_j) \to P(j)^n$ where i is the integer with $0 \leq i < q_j$ and $n \equiv -i \bmod q_j$.*

The proof is completed in 5.5. We begin by obtaining a rigid simplicial version of Proposition 4.5. A diagram

$$\begin{array}{ccc} A & \to & C \\ \downarrow & & \downarrow \\ B & \to & D \end{array}$$

is called a *pre-cofibration* if it commutes and the induced map $B \amalg_A C \to D$ is a cofibration.

LEMMA 5.2. *For each prime p, there exists an increasing sequence of positive integers m_j for $j \geq b$ together with an array of objects $B_j^i \in s.sets_*$, for $j \geq b$ and $0 \leq i \leq p$, and an array of maps $u\colon B_j^i \to B_j^{i+1}$, $a\colon S^{q_j} \wedge B_j^i \to B_j^{i+1}$, and $\lambda\colon B_j^p \wedge S^{d_j} \to B_{j+1}^1$ with $d_j = m_{j+1} - m_j$ such that the following conditions hold: $B_j^0 = M(Z/p^j, m_j)$; the maps $u\colon B_j^i \to B_j^{i+1}$ are K_*-equivalences and cofibrations; the maps $a\colon S^{q_j} \wedge B_j^i \to B_j^{i+1}$ are K_*-equivalences; the maps $\lambda\colon B_j^p \wedge S^{d_j} \to B_{j+1}^1$ are cofibrations; and the diagrams*

$$\begin{array}{ccc} S^{q_j} \wedge B_j^i & \xrightarrow{a} & B_j^{i+1} \\ \downarrow{\scriptstyle 1 \wedge u} & & \downarrow{\scriptstyle u} \\ S^{q_j} \wedge B_j^{i+1} & \xrightarrow{a} & B_j^{i+2} \end{array}$$

$$\begin{array}{ccc} \left(B_j^0 \wedge S^{d_j}\right) \vee \left(S^{q_{j+1}} \wedge B_j^0 \wedge S^{d_j}\right) & \xrightarrow{(u^p \wedge 1) \vee (a^p \wedge 1)} & B_j^p \wedge S^{d_j} \\ \downarrow{\scriptstyle e \vee (1 \wedge e)} & & \downarrow{\scriptstyle \lambda} \\ B_{j+1}^0 \vee \left(S^{q_{j+1}} \wedge B_{j+1}^0\right) & \xrightarrow{u \vee a} & B_{j+1}^1 \end{array}$$

are pre-cofibrations.

Proof. First construct a system of finite CW-spectra W_j^i and associated maps in HS satisfying the conditions of Proposition 4.5. Then desuspend to give a system of pointed CW-complexes X_j^i and associated maps in the pointed homotopy category Ho satisfying conditions like those in the present lemma, but without cofibration properties. Then inductively rigidify, using 5.3 below, to give the required system in $s.sets_*$.

LEMMA 5.3. *Let $i\colon A \to B$ and $f\colon A \to C$ be maps in $s.sets_*$ and let $\beta\colon |B| \to X$ and $\gamma\colon |C| \to X$ be homotopy classes to some $X \in Ho$ such that $\gamma|f| = \beta|i|$ in Ho. If i is a cofibration, then there exists an equivalence $\delta\colon |D| \simeq X$ in Ho for some $D \in s.sets_*$ together with maps $g\colon B \to D$ and $j\colon C \to D$ in $s.sets_*$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ B & \xrightarrow{g} & D \end{array}$$

is a pre-cofibration with $\delta|g| = \beta$ and $\delta|j| = \gamma$ in Ho.

The proof is straightforward.

5.4. *Periodic systems of simplicial sets.* Suppose henceforth that we are given a system of objects B_j^i and associated maps in $s.sets_*$ satisfying the conditions of 5.2. For $j \geq b$ consider the infinite diagram

$$\begin{tikzcd}
 & & \cdots \arrow[d, dashed, "a"] \\
 & B_j^0 \arrow[r, "u"] \arrow[d, dashed, "a"] & B_j^1 \\
B_j^0 \arrow[r, "u"] & B_j^1 &
\end{tikzcd}$$

where a vertical map $X \dashrightarrow Y$ denotes a map $S^{q_j} \wedge X \to Y$. Now extend the diagram infinitely to the right by inserting successive push-out squares, and let $C_j \in s.sets_*$ denote the colimit of each row. Let ε: $M(Z/p^j, m_j) \to C_j$ and a: $S^{q_j} \wedge C_j \to C_j$ denote the induced maps, and note that they are K_*-equivalences. Next, for $j \geq b$ consider the infinite diagram

$$\begin{tikzcd}
 & & & & & \cdots \arrow[d, dashed] & & \cdots \arrow[d, dashed] & \cdots \arrow[d, dashed] \\
 & & & & B_j^0 \arrow[r] & B_j^1 \arrow[r] \arrow[dl, dashed] & \cdots \arrow[r] & B_j^{p-1} \arrow[r] & B_j^p \\
 & & & B_j^0 \arrow[r] \arrow[d, dashed] & B_j^1 \arrow[d, dashed] & & & & \\
 & & & \cdots \arrow[d, dashed] & \cdots \arrow[d, dashed] & & & & \\
 & B_j^0 \arrow[r] \arrow[d, dashed] & \cdots \arrow[r] & B_j^{p-2} \arrow[r] \arrow[d, dashed] & B_j^{p-1} \arrow[d, dashed] & & & & \\
B_j^0 \arrow[r] & B_j^1 \arrow[r] & \cdots \arrow[r] & B_j^{p-1} \arrow[r] & B_j^p & & & &
\end{tikzcd}$$

and extend the diagram infinitely to the right by inserting successive pushout squares. Then for $0 \leq k < p$ let $D_j^k \in s.sets_*$ denote the direct limit of the $(k+1)$st row from the bottom. Let ε: $M(Z/p^j, m_j) \to D_j^0$, a: $S^{q_j} \wedge D_j^0 \to D_j^{p-1}$, and a: $S^{q_j} \wedge D_j^k \to D_j^{k-1}$ for $1 \leq k \leq p-1$ be the induced maps, and note that they are all K_*-equivalences. Next observe that our first extended diagram maps injectively to the present extended diagram, and for $0 \leq k < p$ let d: $C_j \to D_j^k$ be the induced map. Note that d is a cofibration and K_*-equivalence. Moreover, $a(1 \wedge d) = da$ and $d\varepsilon = \varepsilon$. Finally, observe that the objects D_j^0 and the maps a^p: $S^{q_{j+1}} \wedge D_j^0 \to D_j^0$ and ε: $M(Z/p^j, m_j) \to D_j^0$ can also be constructed by starting with the diagram

$$\begin{tikzcd}
 & & \cdots \arrow[d, dashed, "a^p"] \\
 & B_j^0 \arrow[r, "u^p"] \arrow[d, dashed, "a^p"] & B_j^p \\
B_j^0 \arrow[r, "u^p"] & B_j^p &
\end{tikzcd}$$

where a vertical map $X \dashrightarrow Y$ now represents a map $S^{q_{j+1}} \wedge X \to Y$. Thus there is an induced map $c\colon D_j^0 \wedge S^{d_j} \to C_{j+1}$. Moreover, c is a cofibration such that $a(1 \wedge c) = c(a^p \wedge 1)$ and $c(\varepsilon \wedge 1) = \varepsilon$.

5.5. *Proof of* 5.1. For $j \geq b$, let $C(j)$ be the augmented cospectrum such that $C(j)^a = M(Z/p^j, m_j)$ and $C(j)^n = S^i \wedge C_j$ for $n \equiv -i \bmod q_j$ with $0 \leq i < q_j$; let $D(j)$ be the augmented cospectrum such that $D(j)^a = M(Z/p^j, m_j)$ and $D(j)^n = S^i \wedge D_j^k$ for $n \equiv kq_j - i \bmod q_{j+1}$ with $0 \leq k < p$ and $0 \leq i < q_j$; and let $\tilde{D}(j)$ be the augmented cospectrum such that $\tilde{D}(j)^a = M(Z/p^j, m_j)$ and $\tilde{D}(j)^n = S^i \wedge D_j^0$ for $n \equiv -i \bmod q_{j+1}$ with $0 \leq i < q_{j+1}$. Let $d\colon C(j) \to D(j)$, $c\colon \tilde{D}(j) \wedge S^{d_j} \to C(j+1)$, and $\varphi\colon \tilde{D}(j) \to D(j)$ be the obvious maps of augmented cospectra, and note that d and c are termwise cofibrations while d and φ are termwise K_*-equivalences. Consider the infinite diagram

$$\begin{array}{ccccc} & & \tilde{D}(b+1) & \overset{dc}{\dashrightarrow} & \cdots \\ & & \downarrow \varphi & & \\ \tilde{D}(b) & \overset{dc}{\dashrightarrow} & D(b+1) & & \\ \downarrow \varphi & & & & \\ D(b) & & & & \end{array}$$

where $dc\colon \tilde{D}(j) \dashrightarrow D(j+1)$ denotes the termwise cofibration $dc\colon \tilde{D}(j) \wedge S^{d_j} \to D(j+1)$. Now extend the diagram infinitely to the right by inserting successive pushout squares, and let

$$P(b) \overset{\rho}{\dashrightarrow} P(b+1) \overset{\rho}{\dashrightarrow} P(b+2) \dashrightarrow \cdots$$

denote the bottom row of the extended diagram. One easily checks that this sequence has the required properties. In particular, for each $j \geq b$ there are termwise K_*-equivalences $C(j) \xrightarrow{d} D(j) \to P(j)$, and thus there are K_*-equivalences

$$S^i \wedge M\big(Z/p^j, m_j\big) \xrightarrow{1 \wedge \varepsilon} S^i \wedge C_j = C(j)^n \to P(j)^n$$

for $0 \leq i < q_j$ and $n \equiv -i \bmod q_j$.

6. Proof of Theorem 2.1. Our proof of Theorem 2.1 will depend on certain natural constructions involving *s.s.* spectra and topological spectra. As noted in [**16**, §2], these two types of spectra have equivalent homotopy theories in the sense of Quillen [**23**], and their homotopy categories are equivalent to HS.

6.1. *Preliminaries on spectra.* An *s.s. spectrum* M consists of a sequence of objects $M_n \in s.sets_*$ together with maps $\sigma\colon M_n \wedge S^1 \to M_{n+1}$ for $n \geq 0$, and a *map* of *s.s.* spectra $f\colon M \to N$ consists of a sequence of maps $f_n\colon M_n \to N_n$ in $s.sets_*$ such that $\sigma(f_n \wedge 1) = f_{n+1}\sigma$ for $n \geq 0$. Similarly, a *topological spectrum* X consists of a sequence of pointed spaces X_n together with pointed continuous maps $\sigma\colon X_n \wedge S^1 \to X_{n+1}$ for $n \geq 0$, and a *map* of topological spectra $f\colon X \to Y$ consists of a sequence of pointed continuous maps $f_n\colon X_n \to Y_n$ such that $\sigma(f_n \wedge 1) = f_{n+1}\sigma$ for $n \geq 0$. The *singular functor* $\Delta(-)$ and the *geometric realization functor* $|-|$ apply to spectra in the obvious way and provide adjoint functors between the categories of *s.s.* spectra and topological spectra. If M is an *s.s.* spectrum, then $|M|$ is a topological spectrum but need not be a CW-spectrum since $\sigma\colon |M_n| \wedge S^1 \to |M_{n+1}|$ need not be an isomorphism from $|M_n| \wedge S^1$ to a subcomplex of $|M_{n+1}|$. However, $\mathrm{tel}|M|$ is a CW-spectrum where *tel* is the telescope functor of Adams [3, p. 171]. A topological spectrum X is called an *Ω-spectrum*, or *fibrant*, if the structural maps σ': $X_n \to \Omega X_{n+1}$ (adjoint to $\sigma\colon X_n \wedge S^1 \to X_{n+1}$) are weak equivalences for $n \geq 0$. An *s.s.* spectrum M is called *fibrant* if each M_n is a Kan complex and the structural maps $\sigma'\colon M_n \to \Omega M_{n+1}$ are weak equivalences for $n \geq 0$, where ΩM_{n+1} denotes the pointed simplicial function complex $\mathrm{map}_*(S^1, M_{n+1})$. Homotopy groups are defined by $\pi_* X = \mathrm{colim}\, \pi_{*+n} X_n$ for a topological spectrum X, and by $\pi_* M = \pi_*|M|$ for an *s.s.* spectrum M. A map of topological or *s.s.* spectra is a *weak equivalence* if it induces an isomorphism of homotopy groups. If M is an *s.s.* spectrum such that each M_n is a Kan complex, then there is a natural weak equivalence $M \to \overline{\Omega} M$ where $\overline{\Omega} M$ is fibrant with $(\overline{\Omega} M)_n = \mathrm{colim}_i\, \Omega^i M_{n+i}$. Thus if M is any *s.s.* spectrum, there is a natural weak equivalence $M \to \overline{\Omega}\Delta|M|$ where $\overline{\Omega}\Delta|M|$ is fibrant. For an *s.s.* spectrum N and for $J \in s.sets_*$, let $\mathrm{map}_*(J, N)$ be the obvious *s.s.* spectrum with $\mathrm{map}_*(J, N)_n = \mathrm{map}_*(J, N_n)$ for each $n \geq 0$. For a tower

$$M^b \overset{\pi}{\leftarrow} M^{b+1} \overset{\pi}{\leftarrow} \cdots \overset{\pi}{\leftarrow} M^j \overset{\pi}{\leftarrow} M^{j+1} \leftarrow \cdots$$

of fibrant *s.s.* spectra, we obtain a homotopy inverse limit $\mathrm{telim}_{j\to\infty} M^j$ by dualizing the mapping telescope construction. Specifically, we construct $\mathrm{telim}_{j\to\infty} M^j$ by forming the pull-back diagram

$$\begin{array}{ccc} \underset{j\to\infty}{\mathrm{telim}}\, M^j & \rightarrow & \mathrm{map}_*\left(\Delta_1^+, \prod_{j\geq b} M^j\right) \\ \downarrow & & \downarrow (d_0, d_1) \\ \prod_{j\geq b} M^j & \overset{(1,\pi)}{\rightarrow} & \left(\prod_{j\geq b} M^j\right) \times \left(\prod_{j\geq b} M^j\right) \end{array}$$

where Δ_1 is the standard simplicial 1-simplex, $\Delta_1^+ = \Delta_1 \cup *$, d_i is the ith face operator, and $\pi(x_b, x_{b+1}, \ldots) = (\pi x_{b+1}, \pi x_{b+2}, \ldots)$. Note that $\operatorname{telim}_{j\to\infty} M^j$ is a fibrant *s.s.* spectrum. For a tower $X^b \leftarrow X^{b+1} \leftarrow \cdots \leftarrow X^j \leftarrow X^{j+1} \leftarrow \cdots$ of topological spectra, we obtain a homotopy inverse limit

$$L_{j\to\infty} X^j = \operatorname{tel}\left| \operatorname*{telim}_{j\to\infty} \overline{\Omega}\Delta X^j \right|$$

which is an Ω-CW-spectrum.

6.2. *Construction of the functor* Φ_p: $Ho_K \to HS$. For a K_*-local pointed CW-complex X and for a prime p, we shall construct a CW-spectrum $\varphi_p(X)$ which will represent $\Phi_p(X)$ when we pass to homotopy categories. First observe that the singular complex $\Delta X \in s.sets_*$ is a K_*-Kan complex in the sense of [**9**, §12], so each K_*-equivalence $A \to B$ in $s.sets_*$ induces a weak equivalence $\operatorname{map}_*(B, \Delta X) \to \operatorname{map}_*(A, \Delta X)$ of pointed simplicial function complexes. Let $\{P(j) \mid j \geq b\}$ be a system of augmented cospectra and associated maps satisfying the conditions of Proposition 5.1. For each $j \geq b$, let $T^j(X)$ denote the obvious s.s. spectrum with $T^j(X)_n = \operatorname{map}_*(P(j)^n, \Delta X)$ for each $n \geq 0$, and note that $T^j(X)$ is fibrant. Moreover, the given maps ρ: $P(j) \wedge S^{d_j} \to P(j+1)$ induce maps ρ: $S^{d_j} \wedge T^{j+1}(X) \to T^j(X)$ of *s.s.* spectra for $j \geq b$. Consider the tower $\{S^{m_j+1} \wedge T^j(X)\}_{j\geq b}$ of *s.s.* spectra with tower maps $1 \wedge \rho$. Let $\varphi_p(X)$ be the Ω-CW-spectrum

$$\varphi_p(X) = L_{j\to\infty}|S^{m_j+1} \wedge T^j(X)|.$$

If f: $W \to X$ is a weak equivalence of K_*-local pointed CW-complexes, then one easily checks that $\varphi_p(f)$: $\varphi_p(W) \simeq \varphi_p(X)$ in HS. In particular, $\varphi_p(s)$: $\varphi_p(I^+ \wedge X) \simeq \varphi_p(X)$ in HS where $I^+ = I \cup *$ with $I = [0,1]$ and where s: $I^+ \wedge X \to X$ is the projection map. Thus $\varphi_p(d^0)$, $\varphi_p(d^1)$: $\varphi_p(X) \to \varphi_p(I^+ \wedge X)$ represent the same map in HS since $sd^0 = sd^1$ where d^0 and d^1 are the bottom and top maps. Consequently, φ_p respects the homotopy relation and induces a functor Φ_p: $Ho_K \to HS$. To show that Φ_p: $Ho_K \to HS$ takes values in $HS_{K/p}$ we need

LEMMA 6.3. *Let $Y \in HS$ be an Ω-CW-spectrum. Then Y is K_*-local $\Leftrightarrow$ each $Y_n \in Ho$ is K_*-local. Moreover, Y is both K_*-local and p-adically complete $\Leftrightarrow$ Y is K/p_*-local $\Leftrightarrow$ each Y_n is K/p_*-local.*

Proof. If Y is K_*-loval and p-adically complete, then Y is K/p_*-local since $Y_{K/p} \simeq (Y_K)^\wedge_p$ by [**10**, Proposition 2.11]. The converse is immediate since $K/p = K \wedge MZ/p$ and our p-adic completion is the MZ/p_*-localization. Now suppose for a given $E \in HS$ that there exists a collection

$\{W_\alpha \in Ho\}$ of E_*-acyclic spaces such that a spectrum $X \in HS$ is E_*-local $\Leftrightarrow [\Sigma^\infty W_\alpha, X]_* = 0$ for each W_α. Then an easy formal argument shows that an Ω-CW-spectrum Y is E_*-local $\Leftrightarrow$ each Y_n is E_*-local. This applies when $E = K$ by [**10**, Theorem 4.8] and applies when $E = K/p$ since a spectrum X is p-adically complete $\Leftrightarrow [MZ[1/p], X]_* = 0$.

LEMMA 6.4. *For each $X \in Ho_K$, the spectrum $\Phi_p(X) \in HS$ is K/p_*-local.*

Proof. Since X is K_*-local, $\Delta X \in s.sets_*$ is a K_*-Kan complex. Thus for each $j \geq b$ and $n \geq 0$, $\mathrm{map}_*(P(j)^n, \Delta X)$ is a K_*-Kan complex, and there is a weak equivalence

$$T^j(X)_n = \mathrm{map}_*\big(P(j)^n, \Delta X\big) \to \mathrm{map}_*\big(S^i \wedge M\big(Z/p^j, m_j\big), \Delta X\big)$$

for some i by 5.1. Hence the Ω-CW-spectrum $\mathrm{tel}|T^j(X)|$ is K/p_*-local by 6.3 since it has p-cotorsion homotopy groups and has K_*-local terms $\mathrm{tel}|T^j(X)|_n$. The lemma now follows easily.

LEMMA 6.5. *If f: $V \to W$ is a map in Ho_K such that f_*: $[M(Z/p, m), V] \approx [M(Z/p, m), W]$ for all sufficiently large m, then $\Phi_p(f)$: $\Phi_p(V) \simeq \Phi_p(W)$.*

Proof. Let u: $V \to W$ be a pointed continuous map with $[u] = f$. Using the above natural weak equivalences

$$T^j(X)_n \to \mathrm{map}_*\big(S^i \wedge M\big(Z/p^j, m_j\big), \Delta X\big),$$

we deduce that each $T^j(u)$: $T^j(V) \to T^j(W)$ is a weak equivalence, and thus $\varphi_p(u)$: $\varphi_p(V) \to \varphi_p(W)$ is a weak equivalence.

LEMMA 6.6. *The functor Φ_p: $Ho_K \to HS$ carries homotopy fibre squares to homotopy fibre squares.*

Proof. This follows because φ_p is a composition of functors which preserve homotopy fibre squares.

We must show that $\Phi_p \Omega^\infty X \simeq \hat{X}_p$ for $X \in HS_K$, and for this we shall use double spectra. An *s.s. double spectrum* M consists of objects $M_{m,n} \in s.sets_*$ for m, $n \geq 0$ together with maps σ_1: $M_{m,n} \wedge S^1 \to M_{m+1,n}$ and σ_2: $M_{m,n} \wedge S^1 \to M_{m,n+1}$ in $s.sets_*$ such that the diagram

$$\begin{array}{ccccc} M_{m,n} \wedge S^1 \wedge S^1 & \overset{1 \wedge \tau}{\to} & M_{m,n} \wedge S^1 \wedge S^1 & \overset{\sigma_2}{\to} & M_{m,n+1} \wedge S^1 \\ \downarrow \sigma_1 & & & & \downarrow \sigma_1 \\ M_{m+1,n} \wedge S^1 & & \overset{\sigma_2}{\to} & & M_{m+1,n+1} \end{array}$$

commutes where $\tau\colon S^1 \wedge S^1 \to S^1 \wedge S^1$ is the twisting map. Similarly, a *topological double spectrum* X consists of pointed spaces $X_{m,n}$ for m, $n \geq 0$ together with pointed continuous maps $\sigma_1\colon X_{m,n} \wedge S^1 \to X_{m+1,n}$ and $\sigma_2\colon X_{m,n} \wedge S^1 \to X_{m,n+1}$ such that the corresponding diagram commutes. The double telescope construction of Adams [3, pp. 173–176] produces a CW-spectrum $\mathrm{tel}^{(2)}|M_{*,*}|$ for each *s.s.* double spectrum M. Moreover, there is a natural isomorphism

$$\pi_* \mathrm{tel}^{(2)}|M_{*,*}| \approx \operatorname*{colim}_{m,n} \pi_{*+m+n}|M_{m,n}|$$

where the colimit is for the infinite commutative diagram $\{\pi_{*+m+n}|M_{m,n}|\}$ with homomorphisms

$$(-1)^n \sigma_{1*}\colon \pi_{*+m+n}|M_{m,n}| \to \pi_{*+m+n+1}|M_{m+1,n}|,$$

$$\sigma_{2*}\colon \pi_{*+m+n}|M_{m,n}| \to \pi_{*+m+n+1}|M_{m,n+1}|.$$

Thus if $|M_{m,*}|$ and $|M_{*,n}|$ are Ω-spectra for all m, $n \geq 0$, then the canonical edge maps of CW-spectra

$$\mathrm{tel}|M_{0,*}| \to \mathrm{tel}^{(2)}|M_{*,*}| \leftarrow \mathrm{tel}|M_{*,0}|$$

are weak equivalences.

LEMMA 6.7. *There is a natural equivalence $\Phi_p \Omega^\infty X \simeq X_p^\wedge$ for $X \in HS_K$.*

Proof. Let X be a K_*-local Ω-CW-spectrum. For $j \geq b$, form the obvious *s.s.* double spectrum $D^j(X)$ with $D^j(X)_{m,n} = \mathrm{map}_*(P(j)^m, \Delta X_n)$. Since each X_n is K_*-local by 6.3, $|D^j(X)_{m,*}|$ and $|D^j(X)_{*,n}|$ are Ω-spectra for all m, $n \geq 0$. Hence the canonical edge maps

$$\mathrm{tel}\big|D^j(X)_{*,0}\big| \to \mathrm{tel}^{(2)}\big|D^j(X)_{*,*}\big| \leftarrow \mathrm{tel}\big|D^j(X)_{0,*}\big|$$

are weak equivalences of CW-spectra, and thus their suspensions

$$\begin{aligned} \mathrm{tel}\big|S^{m_j+1} \wedge D^j(X)_{*,0}\big| \quad &\to \quad \mathrm{tel}^{(2)}\big|S^{m_j+1} \wedge D^j(X)_{*,*}\big| \\ &\leftarrow \quad \mathrm{tel}\big|S^{m_j+1} \wedge D^j(X)_{0,*}\big| \end{aligned}$$

are also weak equivalences. The map $\rho\colon P(j) \wedge S^{d_j} \to P(j+1)$ induces a map $\rho\colon S^{d_j} \wedge D^{j+1}(X) \to D^j(X)$ of *s.s.* double spectra for $j \geq b$ and there is an associated tower $\{S^{m_j+1} \wedge D^j(X)\}_{j \geq b}$ of *s.s.* double spectra. It is now straightforward to construct a chain of weak equivalences of

CW-spectra

$$\begin{aligned}
\varphi_p(X_0) &= L_{j\to\infty}\left|S^{m_j+1} \wedge D^j(X)_{*,0}\right| \leftarrow L_{j\to\infty}\mathrm{tel}\left|S^{m_j+1} \wedge D^j(X)_{*,0}\right| \\
&\to L_{j\to\infty}\mathrm{tel}^{(2)}\left|S^{m_j+1} \wedge D^j(X)_{*,*}\right| \leftarrow L_{j\to\infty}\mathrm{tel}\left|S^{m_j+1} \wedge D^j(X)_{0,*}\right| \\
&\to L_{j\to\infty}\left|S^{m_j+1} \wedge D^j(X)_{0,*}\right| \\
&\to L_{j\to\infty}\left|S^{m_j+1} \wedge \mathrm{map}_*\big(M\big(Z/p^j, m_j\big), \Delta X\big)\right| \\
&\to L_{j\to\infty}\left|S^2 \wedge \mathrm{map}_*\big(M(Z/p^j, 1), \Delta X\big)\right| \\
&\leftarrow |S^2| \wedge L_{j\to\infty}\left|\mathrm{map}_*\big(M(Z/p^j, 1), \Delta X\big)\right| \\
&\leftarrow |S^2| \wedge \mathrm{tel}\left|\operatorname*{telim}_{j\to\infty} \mathrm{map}_*\big(M(Z/p^j, 1), \Delta X\big)\right| \\
&\leftarrow |S^2| \wedge \mathrm{tel}\left|\mathrm{map}_*\big(M(Z/p^\infty, 1), \Delta X\big)\right|
\end{aligned}$$

where $M(Z/p^\infty, 1) \in s.sets_*$ is the mapping telescope of the sequence of $M(Z/p^j, 1)$'s from §5. The lemma now follows by passing to homotopy categories. For this, suppose that X and Y are K_*-local Ω-CW-spectra. Since Y is an Ω-spectrum, the homotopy classes from X to Y in HS are the ordinary homotopy classes of topological maps from X to Y, taken without reference to the cofinal subspectra of X. Thus by the argument of 6.2, the preceding constructions on X determine successive functors $HS_K \to HS$ and successive natural transformations between them. We thereby obtain a composite natural equivalence $\Phi_p\Omega^\infty X \simeq X_p^\wedge$ where $\Phi_p\Omega^\infty X$ is represented by $\varphi_p(X_0)$ and $X_p^\wedge$ is represented by $|S^2| \wedge \mathrm{tel}|\mathrm{map}_*(M(Z/p^\infty, 1), \Delta X)|$.

The proof of Theorem 2.1 is completed by

LEMMA 6.8. *If M is a pointed finite CW-complex with p-torsion $\tilde{H}_*(M;\ Z)$ and with a K_*-equivalence α: $\Sigma^t M \to M$ for some $t > 0$, then there is a natural equivalence $\Omega^\infty(\Phi_p W)^M \simeq W^M$ for $W \in Ho_K$.*

Proof. Using the periodicity derived from α, it suffices to construct $\Omega^\infty(\Phi_p W)^M \simeq W^M$ when the given M is replaced by a suspension $\Sigma^s M$. Thus, letting p^k be the stable annihilator of M, we may obtain, for $m \geq 1$ and $j \geq k$, maps

$$r\colon S^{m+1} \wedge M \to M\big(Z/p^j, m\big) \wedge M$$

compatible with suspension and with the maps $e\colon M(Z/p^j, m) \to M(Z/p^{j+1}, m)$, such that the diagram

$$\begin{array}{ccc} (S^m \wedge M) \vee (S^{m+1} \wedge M) & \xrightarrow{0 \vee 1} & (S^m \wedge M) \vee (S^{m+1} \wedge M) \\ \simeq \downarrow i \vee r & & \simeq \downarrow i \vee r \\ M(Z/p^j, m) \wedge M & \xrightarrow{e^k \wedge 1} & M(Z/p^{j+k}, m) \wedge M \end{array}$$

homotopy commutes where i is the canonical injection, and such that each $i \vee r$ is a weak-equivalence. Using the equivalence

$$\Omega^\infty \left| T^j(W)^M \right| \simeq W^{M(Z/p^j, m_j) \wedge M}$$

derived from the augmentation map of $P(j)$, and using the periodicity derived from α, we see that $\pi_*(\Phi_p W)^M$ is mapped isomorphically to the k-fold images in the tower

$$\left\{ \pi_*\left(S^{m_j+1} \wedge \left| T^j W \right|^M \right) \right\}_{j \geq k}.$$

Consequently for $n = m - m_k - 1$ with m sufficiently large, the composite of the maps

$$\Omega^\infty (\Phi_p W)^{S^m \wedge M} \to \Omega^\infty \left| T^k W \right|^{S^n \wedge M} \simeq W^{M(Z/p^k, m-1) \wedge M}$$

$$r^\#\colon W^{M(Z/p^k, m-1) \wedge M} \to W^{S^m \wedge M}$$

is a weak equivalence. This implies the lemma.

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