A CHARACTERIZATION OF PSEUDO-ANOSOV FOLIATIONS

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Let M be a closed oriented smooth surface of genus $g \geq 2$, and let $\mathcal{MF}$ denote the space of equivalence classes of measured foliations on M. The importance of measured foliations began with Thurston's work on diffeomorphisms of surfaces: he defined the space $\mathcal{MF}$ and recognized the natural action of the mapping class group on $\mathcal{MF}$ as an extension of the action of this group on the Teichmüller space of M. In these investigations, there arose the concept of a pseudo-Anosov map which fixes a pair of transverse projective measured foliation classes on M, and the question evolves of recognizing the foliation classes fixed by some pseudo-Anosov map. Our main result provides a solution to this problem: we give a combinatorial characterization of these projective measured foliation classes. The combinatorial formulation of this problem uses the theory of train tracks.

1. Introduction.

1.1. In §§2 and 3 of this paper, we develop a method which associates a semi-infinite combinatorial "RLS word" to a class of measured foliations. The techniques underlying these RLS words first arose in [**K**] (see also [**HP**]) and depend on the machinery of train tracks; we recall the necessary material in §1.2. The RLS word does not uniquely determine the projective measured foliation class, rather it determines exactly the subset of $\mathcal{MF}$ consisting of the foliations topologically equivalent to the given foliation. (For the definitions and basic properties of measured foliations, we refer the reader to [**FLP**].) Section 4 contains our main results, and we completely characterize the classes of measured foliations left invariant by some pseudo-Anosov map in terms of their RLS words. Roughly, a measured foliation class is invariant under a pseudo-Anosov map if and only if it admits a preperiodic RLS word. Part of the theorem is in some sense constructive, and we describe how to find a pseudo-Anosov map fixing an invariant foliation class. This allows the description of an algorithm for producing representatives of all conjugacy classes of pseudo-Anosov map in §4.4, wherein we also discuss some open problems and likely applications.

1.2. *Train tracks.* In this section, we recall the definitions and some basic facts about train tracks that we shall make use of in this paper; for the details, we refer the reader to [**HP**], [**P**] and [**T**].

A *train track* is a branched 1-submanifold of a surface M, i.e., a differentiable graph embedded in M with local models at the vertices given by Figure 1. We require furthermore that no complementary component of the train track is an annulus without cusps on its boundary or a disk wtih zero, one, or two cusps on its boundary.

$$b_2 \qquad\qquad b_1$$

$$b_3$$

FIGURE 1

Given a train track τ with edges $a_1, \ldots, a_I$, there is a well-defined map $\varphi_\tau \colon E(\tau) \to \mathscr{MF}$, where $E(\tau) \subset \mathbf{R}^I$ is the convex cone of nonnegative weights on the edges of τ which satisfy the condition that at each vertex as in Figure 1, the weight of b_1 is the sum of the weights of b_2 and b_3. φ_τ is a homeomorphism onto its image, and we denote this image by V_τ.

We shall make use of the following terminology for the edges of a train track: an edge is *large* (and otherwise *small*) at an endpoint if each smooth arc in the train track through the endpoint intersects the interior of the edge. In Figure 1, b_2 and b_3 are small, whereas b_1 is large at the given endpoint. There are thus three kinds of edges: large at both endpoints, small at both endpoints, and large at one and small at the other endpoint. We shall refer to edges of the first and second kind respectively as *large* and *small* edges.

Associated to τ, there is a *fibred neighborhood* $N(\tau) \subset M$ equipped with a retraction $N(\tau) \searrow \tau$. $N(\tau)$ has singular points on its boundary, called *cusps*, whose local models are given by Figure 2. The set of cusps of $N(\tau)$ is in natural one-to-one correspondence with the set of vertices of τ. The fibres of the retraction $N(\tau) \searrow \tau$ form a foliation $\mathscr{T}$ of $N(\tau)$ by segments called *ties*, and the ties that pass through the cusps are called the *singular ties* of $\mathscr{T}$. $N(\tau)$ with its foliation is well-defined up to isotopy, and τ can be regarded as a quotient space of $N(\tau)$ when every fibre is identified to a point.
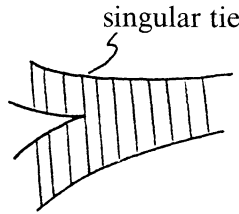
singular tie

FIGURE 2

If $\mathscr{F}$ is an element of $\mathscr{MF}$, we will usually represent $\mathscr{F}$ by a *partial measured foliation* of the surface M. This is a foliation of a subsurface of M having all its singular points on the boundary of the subsurface with local models at these points given by Figure 3. If no complementary region of a partial foliation is a disk with zero or one cusp, then this partial foliation has a well-defined class in $\mathscr{MF}$.
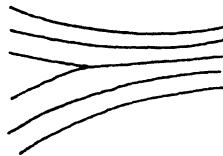
FIGURE 3

We say that $\mathscr{F}$ is *carried by* τ, and we write $\mathscr{F} < \tau$, if $\mathscr{F}$ can be represented by a partial foliation which is contained in a fibred neighborhood $N(\tau)$ of τ and is transverse to the ties. The elements of $V_\tau \subset \mathscr{MF}$ are exactly the elements of $\mathscr{MF}$ that are carried by τ, and a measured foliation class $\mathscr{F}$ corresponds to an element of V_τ which has all of its weights strictly positive if and only if $\mathscr{F}$ can be represented by a partial foliation which is transverse to the ties and whose support is equal to $N(\tau)$.

If σ and τ are two train tracks on M, we say that σ is *carried by* τ, and we denote this relation by $\sigma < \tau$, if σ is isotopic to a train track σ' which is contained in a fibred neighborhood $N(\tau)$ of τ and which is transverse to the ties. We shall make use of three elementary moves which produce from a train track τ a train track σ which is carried by τ. These moves are depicted in Figure 4(a), (b) and (c) and are called respectively a *right split*, a *left split* and a *shift*. If a train track σ arises from a train track τ by splitting and shifting, then we say that τ *refines* to σ.
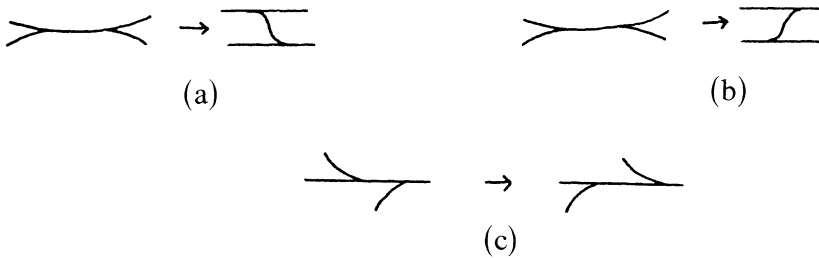
FIGURE 4

If $\sigma < \tau$, there is a natural linear map Ψ: $E(\sigma) \to E(\tau)$ which induces the inclusion map $V_\sigma \subset V_\tau$. The linear map Ψ can be described (in a non-unique way) as the restriction of a linear map Ψ: $R^J \to R^I$, where R^J and R^I are respectively the vector spaces of weights (not necessarily nonnegative) on the edges of σ and τ. We describe precisely such a map Ψ since we will make use of it.

Regard the fibred neighborhood $N(\tau)$ as a union of rectangles arising as the inverse images of edges of τ under the collapse $N(\tau) \searrow \tau$, where each rectangle is foliated by the ties. If $a_1, \ldots, a_I$ are the edges of τ, we make a choice for each edge a_i of a tie above an interior point of a_i; this tie is called the *central tie* associated to the edge a_i. Let $b_1, \ldots, b_J$ be the edges of σ, and $b'_1, \ldots, b'_J$ respectively the corresponding edges of σ', where the correspondence is given by the isotopy which carries σ to σ'.

Isotope σ' slightly so that it is in general position with respect to the central ties of $N(\tau)$ and consider the integral (J by I)-matrix A whose (j,i)th entry is equal to the number of times the edge b'_j of σ' intersects the central tie of $N(\tau)$ which is above the edge a_i of τ. This matrix A defines a linear map Ψ: $R^J \to R^I$ which induces the map Ψ: $E(\sigma) \to E(\tau)$. We again emphasize the fact that the linear map Ψ from R^J to R^I is not canonical, but the induced linear map on the convex cones Ψ: $E(\sigma) \to E(\tau)$ is independent of the choices involved in the definition of Ψ.

2. Combinatorial words associated to foliations and train tracks.

2.1. *The semi-infinite word construction.* Let $\mathscr{F}$ be an element of $\mathscr{MF}$, τ a train track with fibred neighborhood $N(\tau)$ and $\mathscr{T}$ the foliation of $N(\tau)$ by the ties. We say that τ is *suited to* $\mathscr{F}$ if $\mathscr{F}$ can be represented by a partial measured foliation F of support $N(\tau)$, which is transverse to the ties and has no leaves connecting cusps of $N(\tau)$. We recall that this last condition implies that each infinite half-leaf of $\mathscr{F}$ is dense in $N(\tau)$.

Suppose that τ is suited to $\mathscr{F}$, and let $a_1, \ldots, a_I$ denote the (infinite) separatrices of F. The set $\{a_1, \ldots, a_I\}$ is in natural one-to-one correspondence with the set of cusps of $N(\tau)$ as well as with the set of infinite separatrices of any measured foliation representing the class $\mathscr{F}$. Let s be a function from the set $\mathbf{Z}^+$ of positive integers to the set $\{1, \ldots, I\}$. Associated to the triple $(\mathscr{F}, \tau, s)$ we wish to define:

(i) a semi-infinite word $w_\tau^s(\mathscr{F}) = w_1, w_2, \ldots$ in the alphabet

$$\{R_1, \ldots, R_I, L_1, \ldots, L_I, S_1, \ldots, S_I\}.$$

(ii) an infinite sequence $\tau_0, \tau_1, \tau_2, \ldots$ of train tracks so that $\mathscr{F}$ is suited to τ_n for each $n = 0, 1, 2, \ldots$, whence $\mathscr{F}$ admits a representative F_n whose support is a fibred neighborhood N_n of τ_n; in this way, each cusp of τ_n gets a label from $\{1, \ldots, I\}$ depending on which separatrix of F_n issues from that cusp. Furthermore, for each n, τ_{n+1} is obtained from τ_n by a single elementary move, which is respectively a left split, right split or shift on the cusp of τ_n with label $i = s(n)$ if and only if the nth letter w_n of the word $w_\tau^s(\mathscr{F})$ is L_i, R_i or S_i respectively.

The sequence of train tracks $\tau_1 > \tau_1 > \tau_2 > \cdots$ that the construction produces will be naturally associated with a nested sequence $N_0 \supset N_1 \supset N_2 \supset \cdots$ of fibred neighborhoods, so that for all n the foliation of N_{n+1} by the ties will be induced by the foliation of N_n by its ties. The word $w_\tau^s(\mathscr{F})$ is called the RLS *sequence* (or RLS *word*) associated to $(\mathscr{F}, \tau, s)$.

The construction is recursive, and we begin with $\tau_0 = \tau$, $N_0 = N(\tau)$, and $F_0 = F$. For the recursion step, we are given the train track τ_n with the bi-foliated neighborhood N_n and an order on the cusps. We examine the separatrix α of F_n issuing from the $s(n)$th cusp of N_n. There are several cases indicated in Figure 5(a). These cases are distinguished by considering the first intersection of α with the singular ties of N_n: case (i) (respectively (ii)) occurs when the first edge of τ_n that is covered by α (for the natural colapse $N_n \searrow \tau_n$) is large (recall the definition given in §1.2), and α travels to the left (respectively right) in N_n after this first intersection point. Otherwise case (iii) occurs. To define the bi-foliated neighborhood $N_{n+1} \subset N_n$, we separate the bi-foliated neighborhood N_n along an arc $\beta \subset \alpha$ whose initial point is at the cusp and whose endpoint lies between the first and second points of intersection of the separatrix with the singular ties. The opening operation is illustrated in Figure 5(b), and one thinks of points of $\{\mathring{\beta} \cup$ the $s(n)$th separatrix$\}$ as giving rise to *two* points of ∂N_{n+1}. Collapsing ties of N_{n+1} gives the train track τ_{n+1}. w_n is given by $L_{s(n)}$, $R_{s(n)}$ or $S_{s(n)}$ if we have case (i), (ii), or (iii) respectively. The cusps of N_{n+1} inherit a natural order from the order on the cusps of N_n.
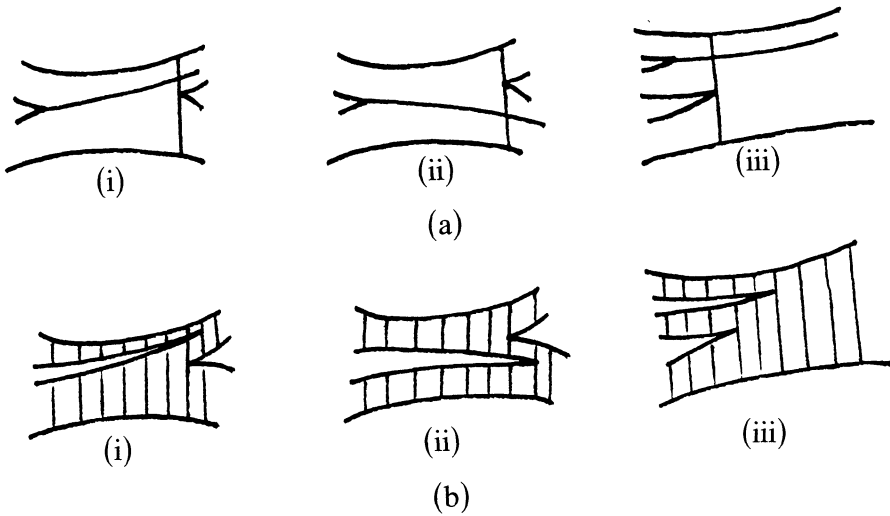
(i)   (ii)   (iii)

(a)

(i)   (ii)   (iii)

(b)

FIGURE 5

Notice that the foliation F_{n+1} induced on N_{n+1} by F_n is isotopic to F_n even though τ_{n+1} is not isotopic to τ. The leaves of the tie-foliation $\mathcal{T}_{n+1}$ on N_{n+1} are subleaves of the tie foliation $\mathcal{T}_n$ on N_n. The train tracks τ_n, $n \geq 1$, are said to arise from τ by *refinement along the separatrices* of $\mathcal{F}$.

2.2. *Refinement of train tracks suited to foliations.* We would like next to associate a *finite* combinatorial word to the following situation: $\mathcal{F} < \sigma < \tau$, and τ is suited to $\mathcal{F}$. This finite word describes a finite number of elementary moves by which τ refines to σ. We require some technical results.

2.1. LEMMA. *Suppose $\mathcal{F} < \sigma < \tau$, where σ is a train track contained in $N(\tau)$ transverse to the ties, and τ is suited to $\mathcal{F}$. There is then a fibred neighborhood $N(\sigma)$ of σ with $N(\sigma) \subset \overset{\circ}{N}(\tau)$ and a family $\{\Gamma_i\}_1^I$ of arcs disjointly embedded in $N(\tau) - \overset{\circ}{N}(\sigma)$ transverse to the ties so that $\{\partial\Gamma_i\}_1^I$ gives a pairing of cusps of $N(\tau)$ and cusps of $N(\sigma)$. For a fixed $N(\sigma) \subset \overset{\circ}{N}(\tau)$ the family $\{\Gamma_i\}_1^I$ is unique up to isotopy fixing the endpoints.*

Proof. Insofar as $\sigma \subset N(\tau)$ is transverse to the ties, we may choose a tie neighborhood $N(\sigma)$ with $N(\sigma) \subset \overset{\circ}{N}(\tau)$, where the ties of $N(\sigma)$ are subarcs of the ties of $N(\tau)$. Fix a cusp of $N(\tau)$ and consider the corresponding singular tie t. There is a point of $t \cap \partial N(\sigma)$ on each side of the cusp, for otherwise, as $\mathcal{F} < \sigma$, there would exist a representative F of $\mathcal{F}$ contained in $N(\tau)$ transverse to the ties which does not pass through every tie of $N(\tau)$. This would contradict the uniqueness of the system of weights on the edges of τ representing $\mathcal{F}$.
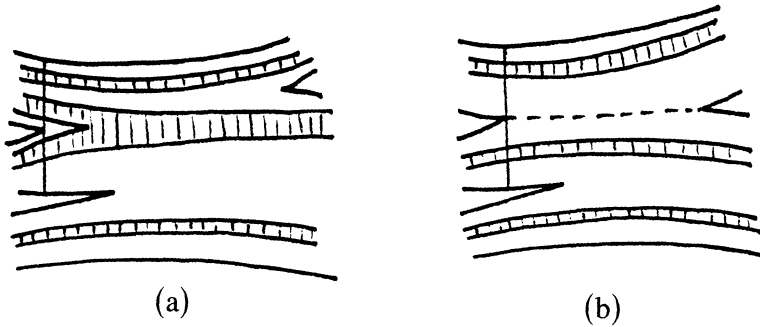
(a)   (b)

FIGURE 6

Consider the edges of $\partial N(\sigma)$ which intersect t nearest the cusp on either side as in Figure 6(a). We claim that these edges of $\partial N(\sigma)$ coalesce at a cusp of $N(\sigma)$ as in Figure 6(a). If not, then these edges must traverse distinct paths and so eventually diverge as in Figure 6(b). In this case, consider a representative F of $\mathcal{F}$ supported in $N(\sigma)$ and transverse to the ties. From this foliation, we can construct another representative supported on $N(\tau)$ transverse to the ties which has a leaf which connects two cusps of $N(\tau)$. This contradicts the unicity of the measure on τ describing $\mathcal{F}$.

We have shown that edges of $\partial N(\sigma)$ nearest cusps of $N(\tau)$ along singular ties coalesce at cusps of $N(\sigma)$; conversely, one sees easily that each cusp of $N(\sigma)$ is associated in this way to a cusp of $N(\tau)$. There is thus a pairing of the cusps of $N(\sigma)$ and the cusps of $N(\tau)$, and each pair in this pairing determines a region W_i, $i = 1, \ldots, I$, of $N(\tau) - \overset{\circ}{N}(\sigma)$ bounded by subarcs of edges of $\partial N(\sigma)$ and singular ties of $N(\tau)$ (see Figure 7). Collapsing subties of these regions gives the required collection of arcs.   □
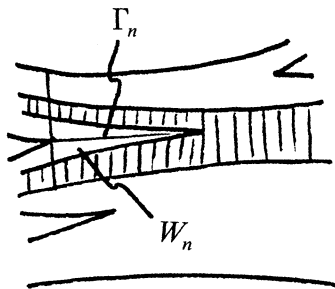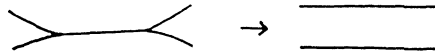

FIGURE 7

FIGURE 8

If σ and τ are two train tracks, recall that τ refines to σ if σ can be obtained from τ by a finite sequence of splits and shifts. There is another move, called a *collision*, which we might perform on a train track and which is illustrated in Figure 8. If a train track σ arises from a train track τ by splitting, shifting and colliding, then we say τ *refines* to σ *with collisions*.

2.2. PROPOSITION. *Suppose $\mathcal{F} < \sigma < \tau$ with $\sigma \subset N(\tau)$ transverse to the ties and with τ suited to $\mathcal{F}$. In this situation, τ refines to σ (without collisions), σ is suited to $\mathcal{F}$, and σ may be isotoped along the ties of $N(\tau)$ to a train track σ' so that σ' arises from τ by refinement along the separatrices of $\mathcal{F}$. In fact, the refinement to σ' may be done by refining first along separatrix one, then separatrix two, ..., and finally along separatrix I.*

Proof. Let W_i, $i = 1, \ldots, I$, be the regions of $N(\tau) - \mathring{N}(\sigma)$ constructed in Lemma 2.1, and consider the quotient N' of $N(\sigma)$ obtained by collapsing the subties on each region W_i. Collapsing ties of N' gives a train track $\tau' < \tau$ which refines (perhaps with collisions) to σ and hence carries $\mathcal{F}$. Furthermore, the switches of τ' coincide exactly with the cusps of $N(\tau)$ by construction.

We claim that the collapse of ties of $N(\tau)$ restricts to a homeomorphism of τ' to τ, and it suffices to show that exactly one branch of τ' intersects each tie of $N(\tau)$. If not, then we can find a pair of adjacent intersections. The corresponding adjacent edges of $N(\tau) - \tau'$ eventually diverge by construction, so we can find an arc in $N(\tau) - \tau'$ transverse to the ties connecting cusps of $N(\tau)$, which is impossible as before. Thus, τ' is homeomorphic to τ, and so τ itself refines to σ. Since the switches of τ and σ are in one-to-one correspondence by Lemma 2.1, this refinement involves no collisions.

Now, consider the compatible tie neighborhood $N(\sigma) \subset \mathring{N}(\tau)$ of σ with its foliation representing $\mathcal{F}$. The collapse of subties induces a foliation on N' which also represents $\mathcal{F}$. Expanding N' along ties to agree with $N(\tau)$ (without changing the topology of N') gives a new foliaton of support $N(\tau)$ representing $\mathcal{F}$, so this new foliation differs from a given representative of $\mathcal{F}$ with support $N(\tau)$ by isotopy along the ties. It follows that σ may be isotoped along the ties of $N(\tau)$ to some σ'

so that σ' arises from τ by refinement along the separatrices of $\mathscr{F}$; it follows easily that $\mathscr{F}$ is suited to σ.

Finally, rather than collapsing the regions W_i, $i = 1, \ldots, I$, all at once, we might collapse them in reverse order: first W_I, then $W_{I-1}, \ldots$, and finally W_1. We require here that the W_i are pairwise disjoint. The last assertion of the proposition then follows. $\square$

2.3. *The zipper construction.* Let $\mathscr{F}$, τ and σ be as in §2.2; we now associate to this triple:

(i) a finite sequence $\tau = \tau_0 > \tau_1 > \cdots > \tau_L = \sigma$ of train tracks with each τ_i suited to $\mathscr{F}$, together with a sequence $N_0 > N_1 > \cdots > N_L$ of bi-foliated neighborhoods with an ordering on the cusps

(ii) a finite RLS-word $w_\tau(\sigma) = w_1 \cdots w_L$, where the ith letter w_i describes the carrying $\tau_{i-1} > \tau_i$.

Let β_i the subarc of the ith separatrix of the foliation on $N(\tau)$ representing $\mathscr{F}$ along which we refine to pass from τ to the train track σ' guaranteed by Proposition 2.2. Set $i = 0$ and $\tau_0 = \tau$. Recursively, let p_{i+1} be the total number of intersections of β_{i+1} with singular ties of $N(\tau_i)$, and let τ_{i+1} be the refinement of τ_i along the subarc β_{i+1} of the $(i + 1)$th separatrix.

We generate in this way a sequence $p_1, \ldots, p_I$ of nonnegative integers and define

$$s: \left\{ 0, 1, \ldots, \sum_1^I p_i \right\} \to \{1, \ldots, I\}$$
$$: j \mapsto k, \quad \text{if } \sum_1^{k-1} p_i < j \le \sum_1^k p_i.$$

Finally, perform the first $\sum_1^I p_i$ stages of the recursion in the construction of §2.1 to generate the associated train tracks, the bi-foliated neighborhoods, and finite RLS-word $w_\tau(\sigma)$.

3. **Convergent RLS sequences.** Let $\mathscr{F}$, τ and s be as in §2.1, $w_\tau^s(\mathscr{F})$ the associated RLS-sequence, and $\tau_0, \tau_1, \ldots$ the associated sequence of train tracks. We say that the semi-infinite word $w_\tau^s(\mathscr{F})$ is *convergent* if it contains infinitely many R or L letters.

Choose some representative (F, μ) of $\mathscr{F}$ (μ being the transverse measure on F), and let $\Delta_{\mathscr{F}}$ denote the subset of $\mathscr{MF}$ consisting of the classes of measured foliations which admit representatives of the form (F, μ'), where μ' is some invariant transverse measure on F. The set $\Delta_{\mathscr{F}}$ does not depend on the choice of representative (F, μ) of $\mathscr{F}$. This section

is devoted to proving

3.1. THEOREM. *If the word $w_\tau^s(\mathscr{F})$ is convergent, then*

$$\bigcap_{\iota=0}^{\infty} V_{\tau_\iota} = \Delta_{\mathscr{F}}.$$

REMARK. If the word is not convergent, then it is clear that $\bigcap_{i=0}^{\infty} V_{\tau_\iota}$ strictly contains $\Delta_{\mathscr{F}}$, since a shift on the train track τ_i does not change the set $V_{\tau_\iota} \subset \mathscr{MF}$.

For the proof of the theorem, we first remark that the construction of the τ_ι does not depend on the transverse measure but only on the trajectory structure of the separatrices of $\mathscr{F}$ relative to τ. It follows that $\Delta_{\mathscr{F}} \subset \bigcap_{\iota=0}^{\infty} V_{\tau_\iota}$.

To prove the reverse inclusion, we require some lemmas.

3.2. LEMMA. *Let $\mathscr{F}$ and $\mathscr{G}$ be two elements of $\mathscr{MF}$ carried by a train track τ with corresponding systems of weights (x_k) and (y_k), $k = 1, \dots, K$, respectively. Then we have the inequality*

$$i(\mathscr{F}, \mathscr{G}) \leq \sum_{k=1}^{K} x_k y_k.$$

$(i(\cdot, \cdot)$ denotes the intersection function on $\mathscr{MF} \times \mathscr{MF}$; see [**FLP**].)

Proof. We may suppose that all the x_k are strictly positive; the general case then follows by continuity of the intersection function. Represent $\mathscr{F}$ by a foliation F of a fibred neighborhood $N(\tau)$ of τ so that F is transverse to the ties. Recall that the vertices of τ are in natural one-to-one correspondence with the cusps of $N(\tau)$.

We embed τ in $N(\tau)$ in such a way that:

each vertex of τ is at the corresponding cusp of $N(\tau)$;

each edge of τ is either in a leaf of F or transverse to F and is embedded in $N(\tau)$ in one of the five ways indicated in Figure 9. Consider now the system of weights (y_k) on the train track τ. Edge number k of τ has transverse measure (with respect to F) at most x_k. The inequality

$$i(\mathscr{F}, \mathscr{G}) \leq \sum_{k=1}^{K} x_k y_k$$

then follows easily from the definition of the intersection function, first by supposing that $\mathscr{G}$ is the class of a simple closed curve, and then by continuity.   □

FIGURE 9

3.3. PROPOSITION. *With the same hypotheses as in Theorem* 3.1, *if* $\mathscr{G}$ *is an element of* $\bigcap_{n=0}^{\infty} V_{\tau_n}$, *then* $i(\mathscr{F}, \mathscr{G}) = 0$.

Proof. For each $n = 0, 1, 2, \ldots$, let (x_k^n) and (y_k^n), $k = 1, \ldots, K$, be the systems of weights on the train track τ_n for $\mathscr{F}$ and $\mathscr{G}$ respectively. We want to show that as n tends to infinity, $\sum_{k=1}^{K} x_k^n y_k^n$ tends to zero. Since each train track τ_{n+1} is obtained from its predecessor τ_n by splitting or shifting, we have

$$\max_k \left\{ x_k^n \right\} \leq \max_k \left\{ x_k^0 \right\}$$

and

$$\max_k \left\{ y_k^n \right\} \leq \max_k \left\{ y_k^0 \right\}$$

for all $n = 1, 2, \ldots$. To prove the proposition, it is therefore sufficient to prove:

3.4. LEMMA. *As* n *tends to infinity, we have* $\max_k \{ x_k^n \}$ *tends to zero.*

Proof. Recall that in the construction of the sequence $\tau_0, \tau_1, \ldots$ given in §2.1, at each stage there is a foliation F_n in the class $\mathscr{F}$ which is supported in a fibred neighborhood N_n of the train track τ_n. F_n is obtained from F_{n-1} by opening the fibred neighborhood N_{n-1} along a separatrix segment; there is also a foliation $\mathscr{T}_n$ by the ties which is transverse to F_n. We want to show that the largest transverse F_n-measure of the ties of $\mathscr{T}_n$ tends to zero as n tends to infinity.

Put a Lebesgue-equivalent transverse measure ν of full support on the foliation $\mathscr{T}$ so that the total transverse ν-measure of a branch of τ is equal to one. In this way, the total transverse ν-measure between two successive singular ties in a positive integer. For each n, there is an induced transverse measure on $\mathscr{T}_n$.

If we can show that there exists a sequence of successive separatrix segments of F which are involved in the refining process so that the total transverse ν-measure of this sequence is infinite, then the lemma will be

proved. Indeed, each semi-infinite separatrix of F is dense in the support of F as well as in the support of each foliation F_n. If we now take a segment of arbitrarily small F_n-measure ε at the center of each tie of each central tie of N_n, then these segments will be separated at some later stage n' in the refining process; it follows that

$$\max_k \left\{ x_k^{n'} \right\} \leq \tfrac{1}{2} \max_k \left\{ x_k^n \right\} + \varepsilon.$$

Thus, to prove Proposition 3.3 it remains to prove the

3.5. LEMMA. *There exists a chain of separatrix segments of F whose total transverse ν-measure is infinite which are involved (in the sense above) in the construction of the word $w_\tau^s(\mathscr{F})$.*

Proof. Recall the operation of opening the neighborhood described in §2.1. For the purpose of the proof, choose a small number $\varepsilon > 0$ and a sequence of positive numbers ε_i, $i = 1, 2, \ldots$, satisfying $\varepsilon = \sum_{i=0}^{\infty} \varepsilon_i$. We specify that at each stage i of the opening process, the endpoint of the separatrix segment β involved is at some ν-distance at most ε_i from the first intersection point with a singular tie (see Figure 10). With this convention, each separatrix segment involved in the refinement has ν-transverse measure either greater than $1 - \varepsilon$ or less than ε. One sees from the construction that whenever the segment has transverse ν-measure less than ε, the corresponding move on the train track is necessarily a shift. By the convergence of the word $w_\tau^s(\mathscr{F})$, there are infinitely many splits involved and also therefore infinitely many separatrix segments of ν-measure greater than $1 - \varepsilon$. This proves Lemma 3.5 and hence Proposition 3.3.   □

To finally prove Theorem 3.1, it remains to show

3.6. LEMMA. *If τ is suited to $\mathscr{F}$ and if $\mathscr{G}$ satisfies $i(\mathscr{F}, \mathscr{G}) = 0$ with $\mathscr{G} \in V_\tau$, then $\mathscr{G} \in \Delta_{\mathscr{F}}$.*

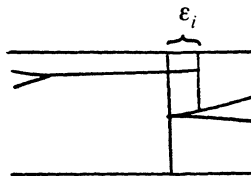

FIGURE 10

Proof. Suppose that $\mathscr{G} \in \Delta_{\mathscr{F}}$. Since $i(\mathscr{F}, \mathscr{G}) = 0$, each representative of $\mathscr{G}$ has a subfoliation which is disjoint from a representative of $\mathscr{F}$. As $\mathscr{F}$ and $\mathscr{G}$ are carried by τ, there exists a measured foliation H in the fibred neighborhood $N(\tau)$ of τ which is transverse to the ties and which is the union of two subfoliations F and F' which are disjoint except on the frontier of their support and where F is in the equivalence class $\mathscr{F}$.

Since τ is suited to $\mathscr{F}$, the separatrices of F are in one-to-one correspondence with the cusps of $N(\tau)$. Thus, F' has no separatrices and is therefore a foliation by closed curves. On the other hand, the fact that τ is suited to $\mathscr{F}$ implies that the complement in $N(\tau)$ of F is a union of annuli, each having one boundary component in $\partial N(\tau)$ and the other one in the frontier of the support of F.

Now, F cannot have closed leaves, so the foliation induced on such a complementary annulus by the ties is a standard foliation as in Figure 11. Therefore there cannot be a smooth closed curve embedded transverse to the ties in such an annulus. Thus follows Lemma 3.6.   □
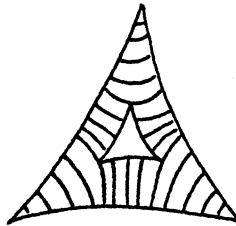

FIGURE 11

The proof of Theorem 3.1 is now complete.   □

4. RLS sequences and pseudo-Anosov homeomorphisms.

4.1. *Train tracks and pseudo-Anosov homeomorphisms.* Recall that a homeomorphism Ψ of M is called *pseudo-Anosov* if there exist a pair of transverse measured foliations (F^u, μ^u) and (F^s, μ^s), called respectively the *unstable* and *stable foliations* of Ψ, and a real number $\lambda > 1$ called the *dilatation* of Ψ, so that

$$\Psi(F^u, \mu^u) = (F^u, \lambda\mu^u)$$

and

$$\Psi(F^s, \mu^s) = (F^s, \lambda^{-1}\mu^s).$$

This section is intended to make explicit some well-known relations between pseudo-Anosov homeomorphisms and train tracks. We prove the

following

   4.1. THEOREM. *If* Ψ *is a pseudo-Anosov homeomorphism and* $\mathscr{F}^u \in \mathscr{MF}$
is the class of its unstable foliation, then there exists a train track τ *suited to*
$\mathscr{F}^u$ *so that* $\Psi\tau < \tau$. *Furthermore, if* $N(\tau)$ *is a fibred neighborhood of* τ,
then $\Psi\tau$ *is isotopic to a train track* τ' *in* $N(\tau)$ *which is transverse to the ties*
so that the matrix describing the linear map from the space of weights on the
edges of τ' *to the space of weights on the edges of* τ *is primitive irreducible*
(*i.e., has some iterate all of whose entries are strictly positive*).

   Proof. Let F^u and F^s denote the unstable and stable foliations of Ψ
respectively. For each singular point P of F^u, consider a small disk D_P
around P having as many cusps on its boundary as the number of
separatrices of F^u at P. D_P is to be embedded in the surface as in Figure
12, where the cusps of ∂D_P are on the F^u-separatrices and the boundary of
D_P is transverse to F^s. Consider the subsurface $M' = M - \bigcup_P \mathring{D}_P$ of M
with the foliation $\mathscr{T}$ induced by F^s. By the density of the leaves of F^s,
each leaf of $\mathscr{T}$ is homeomorphic to a segment and has its endpoints on
the boundary of M'. By general position, we can assume that there are no
leaves of $\mathscr{T}$ connecting two singular points on $\partial M'$. We can regard M' as
the fibred neighborhood $N(\tau)$ of a train track τ with $\mathscr{T}$ the foliation by
the ties; the singular ties correspond to the leaves of $\mathscr{T}$ which pass
through the singular points of $\partial M'$. By "blowing up" F^u at each singular
point and pushing F^u in $N(\tau)$ along F^s-leaves (see Figure 13), we obtain
a partial foliation F in the class $\mathscr{F}^u$ of F_u whose support is $N(\tau)$ and
which is transverse to $\mathscr{T}$. F^u has no saddle connections, and therefore F
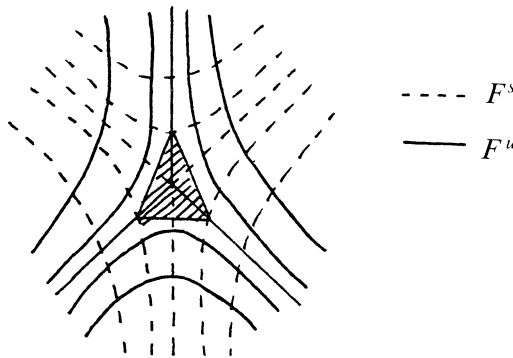has no leaves connecting singular points of $N(\tau)$. We conclude that τ is
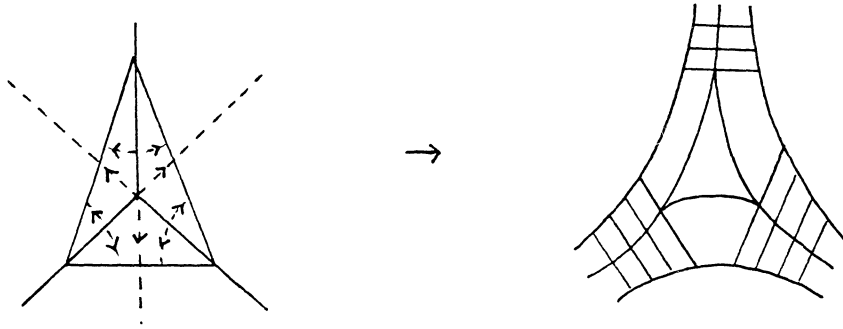

FIGURE 12

FIGURE 13

suited to $\mathscr{F}^u$. We represent τ as a train track embedded in the interior of $N(\tau)$ transverse to the ties so that each edge of τ contains a segment which is in a leaf of F^u.

Consider now the image train track $\Psi(\tau)$. Insofar as Ψ preserves the foliation F^s, $\Psi(\tau)$ is transverse to this foliation and does not pass through any singular point of F^s. In each disk D_P, we can therefore push $\Psi(\tau)$ along the F^s-leaves to get a train track τ', which is isotopic to $\Psi(\tau)$ and contained in $N(\tau)$ transverse to the ties. Thus, $\Psi(\tau) < \tau$.

On the other hand, by density of the leaves of F^u in M, the fact that each edge of τ contains a segment in a leaf of F^u, and by the expansion property of F^u-leaves under Ψ, there exists a positive integer l so that if τ_l is the train track obtained from $\Psi^l(\tau)$ by pushing it inside $N(\tau)$ as before, then each edge of τ_l intersects each tie of $\mathscr{T}$. This implies that the matrix describing the linear map from the vector space of weights on the branches of τ_l to the vector space of weights on the branches of τ has all of its entries strictly positive. We conclude that $\Psi(\tau)$ satisfies the requirements of the theorem.   □

4.2. *Foliations invariant by pseudo-Anosov homeomorphisms.* Recall that a foliation F is said to be *arational* if there is no closed curve in the surface M which is the union of segments joining singular points of F. (This is equivalent to the definition given in [**FLP**], Exposé 9, §III.) If this condition is satisfied by F, then it is also satisfied by every other foliation which is Whitehead equivalent to F. We can therefore use this terminology for classees in $\mathscr{MF}$.

4.2. THEOREM. *Let $\mathscr{F}$ be an arational element of $\mathscr{MF}$ and τ a train track suited to $\mathscr{F}$ which admits a preperiodic and convergent RLS-sequence $w_\tau^s(\mathscr{F})$ (for some choice of function s). There then exists a pseudo-Anosov homeomorphism Ψ of M so that $\mathscr{F}$ is the class of the unstable foliation of Ψ.*

Proof. The assumption that the word $w_\tau^s(\mathscr{F})$ is preperiodic means that it admits an expression of the form $w_\tau^s(\mathscr{F}) = uvvv \cdots$, where u is a string of k letters and v a string of p letters from the alphabet. Let $(\tau_l)_{l \geq 0}$ be the sequence of train tracks associated with $w_\tau^s(\mathscr{F})$. Consider the subsequence $(\tau_{k+lp})_{l \geq 0}$ of train tracks, for $p = 1, 2, \ldots$. Up to conjugacy, there exist only finitely many train tracks on M. Therefore, there are positive integers I and J and a diffeomorphism Ψ of M so that $\Psi(\tau_{k+Ip}) = \tau_{k+(I+J)p}$. It follows that $\Psi(\tau_{k+Ip}) < \tau_{k+Ip}$. The corresponding linear map

$$A: E(\tau_{k+Ip}) \rightarrow E(\tau_{k+Ip})$$

induces a map on the set of rays in $E(\tau_{k+Ip})$ which has a fixed point by the Brouwer theorem. There thus exists an element $\mathscr{F}' \in V_{\tau_{k+Ip}}$ so that $\Psi\mathscr{F}' = \lambda'\mathscr{F}'$, where λ' is some positive real number. The measured foliation class $\mathscr{F}'$ is represented by a system of nonnegative weights on the branches of τ_{k+Ip}, and this system of weights is identified with an eigenvector of the matrix A with eigenvalue λ'.

We wish to prove that $\mathscr{F}' \in \bigcap_{i=0}^{\infty} V_{\tau_i}$. The sequence of train tracks $(\tau_{k+I+lJp})_{l \geq 0}$ are all topologically conjugate, and as the sequence of elementary moves which allow one to pass from a train track in this sequence to its successor is independent of l by preperiodicity, the matrices which describe the sequence of carryings $\tau_{k+I+(l+1)Jp} < \tau_{k+I+lJp}$ are identical; these matrices thus have the common eigenvector which describes the measured foliation class $\mathscr{F}'$. Therefore $\mathscr{F}' \in \bigcap_{l=0}^{\infty} V_{\tau_{k+I+lJp}}$, whence $\mathscr{F}' \in \bigcap_{l=0}^{\infty} V_{\tau_l}$.

By Theorem 3.1, $\mathscr{F}'$ is topologically equivalent to $\mathscr{F}$, and therefore $\mathscr{F}'$ is also arational. By [**FLP**], Exposé 9, §IV and V, Ψ is isotopic to either a pseudo-Anosov map or to a periodic map. The latter case cannot occur since the action of Ψ on $V_{\tau_{k+Ip}}$ sends this cone strictly into itself by the convergence of the word $w_\tau^s(\mathscr{F})$. Thus, Ψ is isotopic to a pseudo-Anosov map, and the theorem is proved. $\square$

Since the unstable foliation of a pseudo-Anosov map is uniquely ergodic ([**FLP**], Exposé 12), we have the following

4.3. COROLLARY. *If an arational measured foliation class admits a preperiodic and convergent RLS word, then it is uniquely ergodic.* $\square$

REMARK. It is an interesting and unsolved problem to exactly characterize the uniquely ergodic measured foliation classes in terms of RLS sequences.

We have the following converse to Theorem 4.2.

4.4. Theorem. *Let $\mathscr{F} \in \mathscr{M}\mathscr{F}$ be the class of the unstable foliation of a pseudo-Anosov map Ψ of M. There exists a train track τ suited to $\mathscr{F}$ and a function s from $\mathbf{Z}^+$ to the ordered set of infinite separatrices of $\mathscr{F}$ so that the word $w_\tau^s(\mathscr{F})$ is convergent and periodic.*

Proof. By Theorem 4.1, there exists a train track τ suited to $\mathscr{F}$ so that $\Psi(\tau) < \tau$. Let $N(\tau)$ be a fibred neighborhood of τ and τ' a train track isotopic to $\Psi(\tau)$ contained in $N(\tau)$ transverse to the ties. By Lemma 2.1, there is a finite set of arcs $\{\Gamma_i\}_1^I$, transverse to the ties which match the vertices of τ and τ'. By replacing Ψ by one of its iterates, we can suppose that this pairing of vertices is the same as the pairing given by the map followed by isotopy.

Consider now the finite word $w_\tau(\tau')$ obtained from the zipper construction (of §2.3) applied to the present situation; we have the corresponding finite sequence $\tau_0 = \tau_1, \tau_1, \ldots, \tau_L = \tau'$ of train tracks. Construct the semi-infinite word by repeating $w_\tau(\tau')$, and consider the corresponding infinite sequence $\tau_0, \tau_1, \ldots, \tau_L, \tau_{L+1}, \ldots$ of train tracks. For each positive integer k, the train track $\tau_{(k+1)L}$ is topologically conjugate to τ_{kL} by the pseudo-Anosov map Ψ. By the dynamics of pseudo-Anosov maps on $\mathscr{M}\mathscr{F}$, the intersection of the nested sequence V_{τ_i} consists exactly of the element $\mathscr{F} \in \mathscr{M}\mathscr{F}$.

Finally, by construction each element in this intersection admits the semi-infinite word $w_\tau(\tau')w_\tau(\tau') \cdots$, which is periodic. That this word is convergent follows from the fact that the nested intersection of the V_{τ_i} consists of the singleton $\{\mathscr{F}\}$.   □

4.5. *A characterization of pseudo-Anosov foliations carried by a given track.* We can in fact prove the following generalization of Theorem 4.4.

4.5. Theorem. *Let $\mathscr{F} \in \mathscr{M}\mathscr{F}$ be the equivalence class of the unstable foliation of a pseudo-Anosov map of M. Then for every train track σ suited to $\mathscr{F}$, there exists a function s from $\mathbf{Z}^+$ to the ordered set of infinite separatrices of $\mathscr{F}$ so that the word $w_\sigma^s(\mathscr{F})$ is convergent and preperiodic.*

Proof. Fix a neighborhood $N(\sigma)$ of σ with a foliation $\mathscr{T}'$ by the ties, and let F' be a representative of the class $\mathscr{F}$ whose support is equal to $N(\sigma)$ and which is transverse to the ties. Consider the train track τ given by Theorem 4.3 with $N(\tau)$ its fibred neighborhood, F the foliation of

$N(\tau)$ representing $\mathscr{F}$ and $\tau_0, \tau_1, \ldots$ the sequence of train tracks corresponding to the periodic word $w_\tau^s(\mathscr{F})$. As before, the fact that the two representatives F and F' have no leaves connecting singular points implies that these two partial foliations are isotopic. Let Φ be a diffeomorphism of M isotopic to the identity so that $\Phi F = F'$.

From the construction of the infinite sequence of train tracks associated with a RLS word, it is easy to see that as i goes to infinity, the train track τ_i can be embedded in $N(\tau)$ in such a way that the angles made by the edges of τ_i with the leaves of F tend uniformly to zero. This is essentially a consequence of the fact that the refinement operation "follows the leaves" of F.

Now, by compactness of M, Φ has bounded distortion. Therefore, for i sufficiently large, $\Phi(\tau_i)$ makes small angles with F' and is transverse to $\mathscr{F}'$. It follows that $\Phi\tau_i < \sigma$. Take such a large i and consider finally a finite combinatorial RLS word w given by the zipper construction (§2.4), which describes the elementary moves required to pass from σ to τ_i. This finite word w followed by the semi-infinite word induced by $w_\tau^s(\mathscr{F})$ on τ_i then defines the preperiodic semi-infinite word $w_\sigma^s(\mathscr{F})$.   □

4.6. *Concluding remarks.*

In Theorems 4.2 and 4.4 we have given a complete characterization of which classes of measured foliations on M are left invariant by some pseudo-Anosov map of M. Theorem 4.5 further characterizes the invariant foliation classes which are carried by a given train track. Furthermore, the argument of Theorem 4.4 gives a method for producing a pseudo-Anosov map fixing a given invariant foliation class. Thus, our endeavors give a procedure for constructing all conjugacy classes of pseudo-Anosov maps: enumerate representatives $\{\tau_i\}^K$, of all the conjugacy classes of all the train tracks *filling* M (i.e., train tracks all of whose complementary regions are disks); compute the symmetry groups $\{G_i\}^K$, of these tracks; enumerate all the periodic RLS sequences on each τ_i so that the associated linear maps on measures are primitive irreducible, and employ the method of Theorem 4.3 to generate a sequence of pseudo-Anosov maps $\{\Psi_i\}_{i \geq 1}$; finally, consider $\{g \circ \Psi: \Psi$ arises from a RLS sequence on τ_i and $g \in G_i\}$.

We further remark that there is an estimate which says roughly that the dilatation of a pseudo-Anosov map is large if the period of the RLS word is large. This estimate is derived in our paper [PP] and allows one to bound the Teichmüller geometry of moduli space viz. the number of Teichmüller geodesics of moduli space of a fixed length. In this latter regard, it would be useful (and of independent interest) to characterize the

periods which occur for pseudo-Anosov maps (i.e., the periods which determine primitive irreducible linear maps on measures).

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