PACIFIC JOURNAL OF MATHEMATICS
Vol. 131, No. 1, 1988

CURVATURE PROPERTIES OF TYPICAL CONVEX SURFACES

TUDOR ZAMFIRESCU

Here we shall see that on typical convex surfaces the set of points with an infinite sectional curvature in some direction and that of points in which the lower sectional curvature in some direction equals the upper sectional curvature in the opposite direction are dense. Also we shall see that, in a certain sense, most convex surfaces are a.e. "very close" to their tangent hyperplane, closer than vanishing curvature already indicates.

Introduction. This paper completes the description of the curvature behaviour of most convex surfaces, the words "most" and "typical" being used in the sense of "all, except those in a set of first Baire category". It is known that typical convex surfaces are smooth and strictly convex (V. Klee [7]), but not of class C^2 (P. Gruber [4]). The latter result was strengthened by R. Schneider [10] and myself [12], [13]. Schneider proved that for these surfaces there is a dense set of points in which, for every tangent direction, the lower and upper curvatures are 0 and ∞ respectively. In [12] we showed that in each point where a finite curvature exists (and it exists almost everywhere by results of H. Busemann-W. Feller [3] and A. D. Aleksandrov [1]), the curvature is zero. In [13] we proved that the mentioned set in Schneider's result is not only dense, but also residual, i.e. a set of typical points.

For a survey on the use of Baire categories in Convexity, the reader may consult [15].

We consider the space $\mathscr{C}_n$ of all closed convex surfaces in $\mathbf{R}^n$. It is an easy matter to verify that $\mathscr{C}_n$, equipped with the Hausdorff distance, is a Baire space.

Let C be a smooth surface in $\mathscr{C}_n$, $x \in C$, τ be a tangent direction at x. We denote by $\rho_i^\tau(x)$ the lower radius of curvature at x in direction τ of the normal section of C (or of C itself for $n = 2$) along τ (see [2], p. 14 for a definition); analogously, the upper radius of curvature at x in direction τ is denoted by $\rho_s^\tau(x)$. For $n = 2$ there are at every point just two (opposite) tangent directions, which we simply denote by $+$ and $-$. If $\rho_i^\tau(x) = \rho_s^\tau(x)$, we write $\rho^\tau(x)$ for the common value. If $\rho^+(x) = \rho^-(x)$, we denote the common value by $\rho(x)$.

The notations $\operatorname{int} A$, $\bar{A}$, $\operatorname{bd} A$, $\operatorname{conv} A$ will be used for the interior, closure, boundary, and convex hull of the set A, respectively.

We shall repeatedly make use of the following, already mentioned results:

THEOREM A (*Theorem* 2 *in* [**12**]). *For most surfaces in $\mathscr{C}_n$, the curvature of the normal section vanishes a.e. in all tangent directions.*

THEOREM B (*Theorem* 2 *in* [**13**]). *For most $C \in \mathscr{C}_n$, at most points $x \in C$,*

$$\rho_i^\tau(x) = 0 \quad \textit{and} \quad \rho_s^\tau(x) = \infty$$

in each tangent direction τ.

Many thanks are due to the referee, whose comments helped improve this paper.

Lower and upper curvatures in opposite directions.

THEOREM 1. *For most $G \in \mathscr{C}_n$, at each point $x \in C$ and tangent direction τ,*

$$[\rho_i^{-\tau}(x), \rho_s^{-\tau}(x)] \cap [\rho_i^\tau(x), \rho_s^\tau(x)] \neq \varnothing.$$

Proof. Let $\mathscr{C}^*$ be the family of all surfaces $C \in \mathscr{C}_n$ $(n \geq 3)$ on which there is a smooth point x and a tangent direction τ such that

$$[\rho_i^{-\tau}(x), \rho_s^{-\tau}(x)] \cap [\rho_i^\tau(x), \rho_s^\tau(x)] = \varnothing.$$

For $C \in \mathscr{C}^*$ there are two rational numbers p_C, q_C and three circles Γ_1, Γ_2, Γ_3 in the normal plane at x parallel to τ, of radius p_C, q_C, $p_C - q_C$ respectively, such that Γ_1 and Γ_2 are tangent to C at x, Γ_3 is centred in x, $C \cap \Gamma_3$ has exactly two points, and one component of $C \cap \operatorname{conv}\Gamma_3 - \{x\}$ lies outside Γ_1 and the other inside Γ_2 (see Figure 1).

Let $\mathscr{C}_{p,q}$ be the family of all surfaces C in $\mathscr{C}^*$ for which a smooth point x, a tangent direction τ at x, and corresponding numbers $p_C = p$ and $q_C = q$ may be found. Clearly,

$$\mathscr{C}^* = \bigcup_{p,q} \mathscr{C}_{p,q}.$$

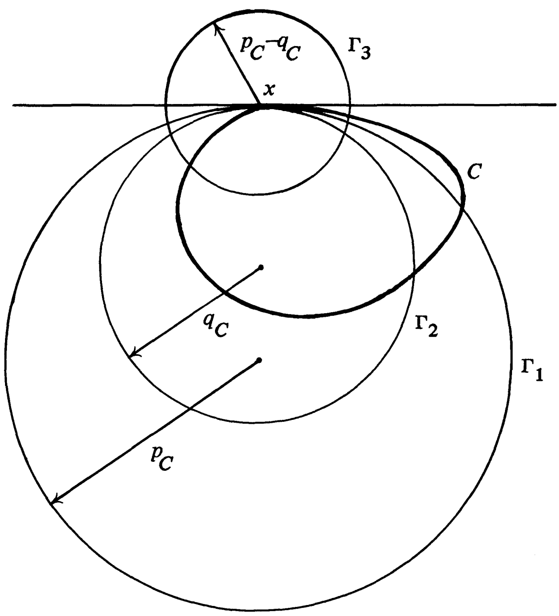

FIGURE 1

We show now that $\overline{\mathscr{C}_{p,q}} \subset \mathscr{C}_{p,q} \cup \mathscr{C}^\dagger$, where $\mathscr{C}^\dagger$ is the set of all non-smooth surfaces in $\mathscr{C}_n$. Let $\{C_i\}_{i=1}^\infty$ be a sequence of surfaces in $\mathscr{C}_{p,q}$ converging to some $C \in \mathscr{C}_n$. For each C_i there exists a smooth point $x_i \in C_i$, a tangent direction τ_i and three circles Γ_{1i}, Γ_{2i}, Γ_{3i} in the normal plane at x_i parallel to τ_i, of radius p, q, $p-q$ respectively, such that Γ_{1i} and Γ_{2i} are tangent to C_i at x_i, Γ_{3i} is centred in x_i, $C_i \cap \Gamma_{3i}$ has exactly two points and, within Γ_{3i}, one component of $C_i \cap \operatorname{conv}\Gamma_{3i} - \{x_i\}$ lies outside Γ_{1i} and the other inside Γ_{2i}. By taking a subsequence $\{C_{i_j}\}_{j=1}^\infty$ of $\{C_i\}_{i=1}^\infty$, we arrange that $\{x_{i_j}\}_{j=1}^\infty$ converges to a point $x \in C$ and that the sequence $\{\tau_{i_j}\}_{j=1}^\infty$ converges, whence $\{\Gamma_{1i_j}\}_{j=1}^\infty$, $\{\Gamma_{2i_j}\}_{j=1}^\infty$, $\{\Gamma_{3i_j}\}_{j=1}^\infty$ converge to the circles Γ_1, Γ_2, Γ_3. Now, if C is smooth at x, Γ_1, Γ_2, Γ_3 play the roles from the definition of $\mathscr{C}_{p,q}$ and $C \in \mathscr{C}_{p,q}$. Hence $\overline{\mathscr{C}_{p,q}} \subset \mathscr{C}_{p,q} \cup \mathscr{C}^\dagger$. Observe the analogy with the proof of Theorem 1 in [**12**].

We show $\overline{\mathscr{C}_{p,q}}$ is nowhere dense. It is indeed well-known that each convex surface may be approximated by algebraic convex surfaces, which obviously neither belong to $\mathscr{C}_{p,q}$ nor to $\mathscr{C}^\dagger$.

Thus $\mathscr{C}_{p,q}$ is nowhere dense and $\mathscr{C}^*$ is of first Baire category, which concludes the proof for $n \geq 3$. The case $n = 2$ is similar, but simpler since $\mathscr{C}_{p,q}$ is then easily seen to be closed.

Infinite curvatures in the planar case. We already know that typical convex curves have many points with vanishing curvature [**12**] and many points with infinite upper curvature [**13**]. Do they also possess points with infinite curvature?

Throughout all the paper λ and μ will denote linear and $(n-1)$-dimensional Lebesgue measure, respectively.

We consider here the space $\mathscr{C}_2$.

THEOREM 2. *For each smooth, strictly convex curve C for which the curvature is zero a.e., there is an uncountable dense set of points on C where the curvature is infinite.*

Proof. Let B be a small arc on C, and

$$g\colon [a,b] \to \mathbf{R}$$

be a convex function with graph B. Since C is smooth, g is differentiable and let $f = g'$. Since the existence of curvature in some point of B is equivalent with the existence of g'' in the corresponding point of $[a,b]$, and the curvature equals $g''(1+g'^2)^{-3/2}$, we have $f' = 0$ a.e.

If M is a measurable set on $[a,b]$ such that

$$\sup_{x \in M} f_s'(x) = \delta < \infty,$$

where f_s' is the upper Dini derivative of f, then we have (see [**6**], p. 269)

$$\lambda(f(M)) \le \delta\lambda(M).$$

Let

$$M_n = \{x \in [a,b] : 0 < f_s'(x) < n\}.$$

Clearly, $\lambda(M_n) = 0$ for each natural number n and

$$f(M_\infty) = f\left(\bigcup_{n=1}^{\infty} M_n\right) = \bigcup_{n=1}^{\infty} f(M_n);$$

thus

$$\lambda(f(M_\infty)) \le \sum_{n=1}^{\infty} \lambda(f(M_n)) = 0.$$

Let

$$L_0 = \{x \in [a,b] : f'(x) = 0\},$$

$$L_\infty^s = \{x \in [a,b] : f_s'(x) = \infty\},$$

$$L_\infty = \{x \in [a,b] : f'(x) = \infty\}.$$

Then $\lambda(L_0) = b - a$, but, since f^{-1} is almost everywhere differentiable,

$$\lambda(f(L_0)) = 0.$$

It follows that

$$\lambda(f(L_\infty^s)) = f(b) - f(a).$$

Finally, since f has a derivative if and only if its inverse has one, and since f^{-1} is a.e. differentiable,

$$\lambda(f(L_\infty)) = f(b) - f(a)$$

too.

Hence g'' is infinite at uncountably many points and the theorem follows.

From Theorems A and 2 we derive the following result, which provides full information about the curvature of a typical convex curve at those points where it exists (see Theorem 1 in [**12**]).

THEOREM 3. *On most convex curves the curvature is zero a.e. and infinite at an uncountable dense set of points.*

Infinite curvatures in higher dimensions. We shall make use here of a topological space associated with a smooth surface, usually called *sphere bundle*, consisting of all pairs (x, τ), where x belongs to the surface and τ is a unit tangent vector at x.

The next result extends Theorem 2 to higher dimensions.

THEOREM 4. *For each smooth, strictly convex surface in $\mathscr{C}_n$, for which the sectional curvature in every tangent direction is zero a.e., the associated sphere bundle includes an uncountable dense set of pairs (x, τ) with $\rho^{\pm\tau}(x) = 0$. Thus, most surfaces in $\mathscr{C}_n$ enjoy this property.*

Proof. Let $C \in \mathscr{C}_n$, $x_0 \in C$, τ_0 be a tangent direction at x_0, T the tangent hyperplane at x_0 and A an $(n-1)$-cell in C containing x_0 in its interior such that the orthogonal projection $p\colon A \to T$ is injective and $p(A)$ is convex. Let B be the set of points in A where the sectional curvature in every tangent direction is zero. We have

$$\mu(p(A - B)) = 0.$$

Putting $T = t_0 \times T_0$, t_0 being a line in direction τ_0 and T_0 an orthogonal $(n-2)$-plane, both through x_0, Fubini's theorem says that

$$\lambda(p(A - B) \cap (t_0 \times \{y\})) = 0$$

for almost every $y \in T_0$. Choose such an y in $\operatorname{int} p(A)$. (See Figure 2.)

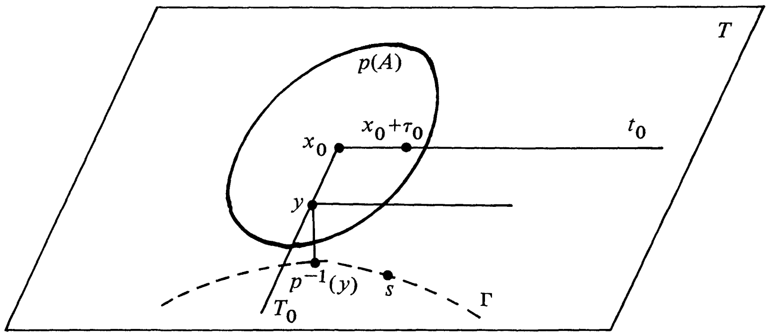

FIGURE 2

By Meusnier's theorem (see [2], p. 15), at each point of $B \cap p^{-1}(t_0 \times \{y\})$ the curvature of $\Gamma = A \cap p^{-1}(t_0 \times \{y\})$ vanishes. By Theorem 2, we get in Γ an uncountable dense set S of points with infinite curvature. For a point $s \in S$ close to $p^{-1}(y)$, the tangent directions $\pm\tau'$ at s to Γ are close to the tangent directions at $p^{-1}(y)$ to Γ (because Γ is of class C^1) and these are close to $\pm\tau_0$ if A is small enough (because C is of class C^1).

Finally, the curvature of Γ in s being ∞, the curvature of the normal section of C at s in both directions $\pm\tau'$ is also ∞, by Meusnier's theorem.

By Theorem A, most convex surfaces satisfy the hypotheses of Theorem 4; it follows that they also enjoy the property of its conclusion.

We do not know whether on typical convex surfaces in $\mathbf{R}^n$ ($n \geq 3$), there must be points where the upper indicatrix reduces to a point. We know what occurs for any differentiable, strictly convex surface: there is a dense set of points where the upper indicatrix is bounded. For typical surfaces this set is both of measure zero and of first category.

It is interesting to mention here a result of V. Klima and I. Netuka [8] concerning the partial derivatives of most convex functions. On one hand their result only implies that the upper Dini derivative of any partial derivative must be ∞ at a dense set of points. But on the other it says that this upper Dini derivative is in a certain sense "very large", in no way "majorizable" on any open set. Observe also the relationship with Theorem 12.

A technical lemma. Let $C \in \mathscr{C}_2$ be smooth and strictly convex, and fix a direct sense on C. The set of diameters of C, i.e. chords with parallel supporting lines at their endpoints, forms a spread in Grünbaum's sense (see [5], [14]).

Let $L(x)$ be the diameter of C with an endpoint at x and let $-x$ be the other endpoint of $L(x)$. Also, let $y \neq \pm x$ be a point on C, $\{z\} = L(x) \cap L(y)$ and

$$\beta(y, x) = \frac{d(x, z)}{d(-x, z)},$$

where d denotes Euclidean distance. Put

$$\gamma_i^-(x) = \liminf \beta(y, x),$$

where y converges to x in the direct sense on C. $\gamma_s^-(x)$, $\gamma_i^+(x)$, and $\gamma_s^+(x)$ are then defined in an obvious way.

The following lemma will be useful.

LEMMA. *Suppose the smooth, strictly convex curve C includes the arc B of a circle and let $x \in C$ be such that $-x \in$ int B. Then*

$$\gamma_i^{\pm}(x) \leq \frac{\rho_i^{\pm}(x)}{\rho(-x)}; \qquad \gamma_s^{\pm}(x) \geq \frac{\rho_s^{\pm}(x)}{\rho(-x)}.$$

Proof. First suppose $\rho_i^+(x) < \rho_s^+(x)$. Let Γ', Γ'' be two circles tangent to C at x, of radius r', r'' respectively, such that

$$\rho_i^+(x) < r' < r'' < \rho_s^+(x).$$

Let $x_1, x_2, x_3, \ldots$ be points lying in this order (the indirect sense) on C, with $x_n \to x$, such that $x_1, x_3, x_5, \ldots$ lie on Γ'' and $x_2, x_4, x_6, \ldots$ lie on Γ'. Let Γ_2' be the smallest circle tangent to C in x and in a point x_2' of the arc x_1x_3 of C. Let r_2' be the radius of Γ_2'. Clearly $r_2' \leq r'$. With similarly defined r_n', we also have $r_n' \leq r'$ for all even n's. By an easy similarity argument,

$$\beta(x_n', x) = \frac{r_n'}{\rho(-x)}$$

for corresponding points x_n' and even n. Thus $\gamma_i^+(x) \leq r'/\rho(-x)$ and, since r' was chosen arbitrarily close to $\rho_i^+(x)$,

$$\gamma_i^+(x) \leq \frac{\rho_i^+(x)}{\rho(-x)}.$$

Let now Γ_3'' be the largest circle tangent to C in x and in a point x_3'' of the arc x_2x_4 of C. Let r_3'' be the radius of Γ_3''. Obviously $r_3'' \geq r''$. Continuing in the same way as before, we get

$$\gamma_s^+(x) \geq \frac{\rho_s^+(x)}{\rho(-x)}.$$

The other two inequalities are analogous.

Suppose now $\rho_i^+(x) = \rho_s^+(x) = \rho^+(x)$. Let

$$0 < r' < \rho^+(x) < r'' < \infty$$

(if $\rho^+(x)$ is 0 or ∞, we take just r'' or r') and let Γ', Γ'' be the circles tangent to C at x and with radii r', r''. There is a neighbourhood N of x, such that the points of $C \cap N$ following x (according to the fixed sense) form together with x itself an arc A lying between Γ' and Γ''.

Suppose for each neighbourhood $N' \subset N$ of x there is a circle Γ between Γ' and Γ'', tangent to both at x, such that $A \cap \Gamma - \{x\}$ has at least two points in N'. Then, by an argument rather similar to that used for the case $\rho_i^+(x) < \rho_s^+(x)$,

$$\gamma_i^+(x) \leq \frac{\rho^+(x)}{\rho(-x)}; \qquad \gamma_s^+(x) \geq \frac{\rho^+(x)}{\rho(-x)}.$$

Suppose now there is a neighbourhood $N' \subset N$ of x, such that each circle Γ between Γ' and Γ'', tangent to both at x, cuts $A' = A \cap N' - \{x\}$ in at most one point. Then, clearly, A' traverses all circles it meets. Let $r(y)$ be the radius of the circle $\Gamma(y)$ tangent at x to C and passing through $y \in A'$. Then r is a monotone function of y. By definition,

$$\rho^+(x) = \lim_{y \to x} r(y).$$

Suppose, for example, r is decreasing when y moves in the direct sense (away from x). (The other case is analogous.) Then, for each $y \in A'$, the acute angles $\alpha_C(y)$ and $\alpha_\Gamma(y)$ between the tangents in x and y to C and $\Gamma(y)$ satisfy $\alpha_C(y) \geq \alpha_\Gamma(y)$. There exists a point z in the arc xy of A' such that $\alpha_C(z) = \alpha_\Gamma(y)$. Since the line through y and $-z$ determines on $L(x)$ a ratio equal to $r(y)/\rho(-x)$, we have $\beta(z, x) \leq r(y)/\rho(-x)$, whence

$$\gamma_i^+(x) \leq \frac{\rho^+(x)}{\rho(-x)}.$$

The line Λ_y through the point v of $L(x)$ verifying $d(v, x)/d(v, -x) = r'/\rho(-x)$ and y meets $L(x)$, Γ', A in this order. It follows that there must exist points $y \in A'$ as close to x as we want, such that the acute angle $\alpha_{\Gamma'}(y)$ between the tangents to Γ' in $\Lambda_y \cap \Gamma' \cap N'$ and x satisfies $\alpha_{\Gamma'}(y) \geq \alpha_C(y)$. Then $L(y) \cap L(x)$ lies in the segment with endpoints v and $-x$. Thus $\beta(y, x) \geq r'/\rho(-x)$, whence $\gamma_s^+(x) \geq r'/\rho(-x)$; from the arbitrary choice of r' it follows that

$$\gamma_s^+(x) \geq \frac{\rho^+(x)}{\rho(-x)}.$$

The proof is complete.

This lemma was already stated without proof and used in [**14**].

When lower curvature in one direction equals upper curvature in the opposite direction. Let C be a smooth, strictly convex curve in $\mathscr{C}_2$ and, for each natural number n, let

$$A_n = \{x \in C: \exists r > n \text{ with } \rho_s^-(x) \le r \le \rho_i^+(x)\},$$

$$A'_n = \{x \in C: \exists r > n \text{ with } \rho_s^+(x) \le r \le \rho_i^-(x)\},$$

$$B_n = \{x \in C: \exists r < n^{-1} \text{ with } \rho_s^-(x) \le r \le \rho_i^+(x)\},$$

$$B'_n = \{x \in C: \exists r < n^{-1} \text{ with } \rho_s^+(x) \le r \le \rho_i^-(x)\}(r \in \mathbf{R}).$$

THEOREM 5. *For each smooth, strictly convex curve C, for which the set of points x where $\rho_i^\pm(x) = 0$ and $\rho_s^\pm(x) = \infty$ is dense on C, the sets A_n, A'_n, B_n, B'_n are dense too.*

Proof. Let A be a small arc on C and complete A by an arc B including a half-circle of radius c to a smooth convex curve C'.

Let $x \in \operatorname{int} A$ be such that

$$\rho_i^-(x) = \rho_i^+(x) = 0; \qquad \rho_s^-(x) = \rho_s^+(x) = \infty.$$

By the Lemma in the preceding section, applied to C',

$$\gamma_i^-(x) = \gamma_i^+(x) = 0; \qquad \gamma_s^-(x) = \gamma_s^+(x) = \infty.$$

Then, let $x_1, x_2, x_3, \ldots$ be a sequence of points converging in the indirect sense of C to x, such that, putting

$$\{z_n\} = L(x_n) \cap L(x),$$

$z_{2n} \to x$, $z_{2n+1} \to -x$. Since

$$\beta(x_{2n}, x) \to 0, \qquad \beta(x_{2n+1}, x) \to \infty,$$

we find for every positive integer n_0 an integer m such that

$$\beta(x_{2n}, x) \le n_0^{-1}, \quad \beta(x_{2n+1}, x) \ge n_0, \quad \beta(x_{2n+2}, x) \le n_0^{-1},$$

and $x_n \in \operatorname{int} A$ for all $n \ge m$.

Since β is a continuous function on $C \times C$ minus the diagonal,

$$\sup_{y \in x_{2n}x_{2n+2}} \beta(y, x)$$

is attained in some point x'_{2n+1} of the arc $x_{2n}x_{2n+2}$ of C. Of course, $\beta(x'_{2n+1}, x) \geq n_0$. Thus $\beta(x'_{2n+1}, x) \to \infty$. It follows that $\beta(x, x'_{2n+1}) \to \infty$ too, whence, given n_0,

$$\beta(x, x'_{2n+1}) > n_0 c$$

for a suitable n.

The definition of x'_{2n+1} shows that, for y between x_{2n} and x_{2n+2}, $L(y)$ does not cut $L(x)$ between $L(x) \cap L(x'_{2n+1})$ and $-x$. It follows that, for y between x_{2n} and x'_{2n+1}, $L(y)$ does not cut $L(x'_{2n+1})$ between $L(x'_{2n+1}) \cap L(x)$ and $-x'_{2n+1}$, and, for y between x'_{2n+1} and x_{2n+2}, $L(y)$ does not cut $L(x'_{2n+1})$ between x'_{2n+1} and $L(x'_{2n+1}) \cap L(x)$. Consequently,

$$\gamma_s^+(x'_{2n+1}) \leq \beta(x, x'_{2n+1}), \qquad \gamma_i^-(x'_{2n+1}) \geq \beta(x, x'_{2n+1}).$$

In view of the Lemma, we then must have

$$\rho_s^+(x'_{2n+1}) \leq \frac{\beta(x, x'_{2n+1})}{c} \leq \rho_i^-(x'_{2n+1}).$$

Thus, we see that A'_n is dense on C. If $\{x_n\}_{n=1}^{\infty}$ converged in the direct sense on C to x, it would result that A_n is dense on C. A slight modification of the above proof yields that B_n and B'_n are dense on C too.

THEOREM 6. *For most $C \in \mathscr{C}_2$ and every number $n \in \mathbf{N}$, the sets*

$$\{x \in C\colon \rho_i^-(x) = 0,\ n < \rho_s^-(x) = \rho_i^+(x) < \rho_s^+(x) = \infty\}$$

and

$$\{x \in C\colon \rho_i^-(x) = 0 < \rho_s^-(x) = \rho_i^+(x) < n^{-1}, \rho_s^+(x) = \infty\}$$

are dense on C. The same is true if "+" is replaced by "−" and vice versa.

Proof. By results of Klee [7] and Gruber [4], most convex curves C are smooth and strictly convex. By Schneider's theorem in [10] (or by Theorem B), most curves C have also the property that the set of points x where $\rho_i^{\pm}(x) = 0$ and $\rho_s^{\pm}(x) = \infty$ is dense on C. By Theorem 5, for these C's the sets A_n and B_n are dense too.

Let D_n and E_n be the sets mentioned in the statement. Now, on one hand, by Theorem 1 in [12], in each point x of most curves C, $\rho_i^{\pm}(x) = 0$ or $\rho_s^{\pm}(x) = \infty$, on the other hand, by Theorem 1,

$$\rho_s^-(x) \geq \rho_i^+(x).$$

It follows that, for most curves C,

$$A_n = D_n, \qquad B_n = E_n,$$

which concludes the proof.

The higher dimensional version.

THEOREM 7. *For each smooth, strictly convex surface $C \in \mathscr{C}_n$, for which the set of points y where $\rho_i^\sigma(y) = 0$ and $\rho_s^\sigma(y) = \infty$ in every tangent direction σ at y is dense in C and for each positive integer m, the sphere bundle associated to C includes a dense set of pairs (x, τ) satisfying*

$$\rho_s^{-\tau}(x) \leq r \leq \rho_i^\tau(x),$$

with $r > m$. (Analogously, changing $r > m$ into $r < m^{-1}$.)

Proof. This is similar to the proof of Theorem 4. Let $C \in \mathscr{C}_n$, $x_0 \in C$, τ_0 be a tangent direction at x_0, T the tangent hyperplane at x_0 and A an $(n-1)$-cell in C containing x_0 such that the orthogonal projection $p\colon C \to T$ is injective and $p(A)$ is convex. Let B be the set of points y in the statement. The proof (in [**13**]) of Theorem B shows that B is residual in A. Since p is a homeomorphism, $p(B)$ is residual in $p(A)$. Putting $T = t_0 \times T_0$, t_0 being a line in direction τ_0 and T_0 an orthogonal $(n-2)$-plane, the Kuratowski-Ulam theorem ([**9**], p. 67) shows that $p(B) \cap (t_0 \times \{y\})$ is residual in $p(A) \cap (t_0 \times \{y\})$ for most $y \in T_0$. Choose such a point y in $p(A)$, near x_0. By Meusnier's theorem, at each point of $B \cap p^{-1}(t_0 \times \{y\})$, the lower curvature of

$$\Gamma = A \cap p^{-1}(t_0 \times \{y\})$$

is zero and the upper curvature of Γ is ∞.

By Theorem 5, the set A_m is dense on Γ. For a point $x \in A_m$ near $p^{-1}(y)$, x is near x_0 and the tangent τ in x to Γ is near τ_0. By Meusnier's theorem again, from

$$\rho_s^-(x) \leq r' \leq \rho_i^+(x) \qquad (r' > m)$$

on Γ, it follows

$$\rho_s^{-\tau}(x) \leq \frac{r'}{\cos\alpha} \leq \rho_i^\tau(x),$$

α being the acute angle between the normal at x to C and any 2-plane orthogonal to T_0. Clearly $r'/\cos\alpha > m$ and the theorem is proved.

THEOREM 8. *For most surfaces in $\mathscr{C}_n$ and every positive integer m, the associated sphere bundle includes two dense sets of pairs (x, τ) and (y, σ) respectively, such that*

$$\rho_i^{-\tau}(x) = 0, \quad m < \rho_s^{-\tau}(x) = \rho_i^\tau(x) < \infty, \quad \rho_s^\tau(x) = \infty,$$

$$\rho_i^{-\sigma}(y) = 0, \quad 0 < \rho_s^{-\sigma}(y) = \rho_i^\sigma(y) < m^{-1}, \quad \rho_s^\sigma(y) = \infty.$$

We omit the proof, because it is an obvious adaptation of the proof of Theorem 6.

Nearest and furthest points. This small section is included, since its theorem follows directly from results on the curvature of most convex surfaces obtained until now.

On a closed surface S, not necessarily convex, embedded in the Euclidean space $\mathbf{R}^n$, we always have nonempty sets N, F of points representing the nearest, respectively furthest points of S from certain points in $\mathbf{R}^n - S$. Sometimes these sets may equal S. If S is of class C^2 then $N = S$. If, moreover, the Gauss curvature is strictly positive everywhere on S, then $F = S$ too.

Now consider $C \in \mathscr{C}_n$, chosen arbitrarily. Let N_i (N_e) be the set of points of C nearest to points in the interior (exterior) of C. It is easily seen that N_e equals C and N_i is uncountable and dense in C. In the plane, if C is strictly convex, then F is also uncountable and dense in C. Let us explain the last assertion. Let O be open in C. If the lower curvature at some point $x \in O$ is positive in both directions, then $x \in F$. If for all $x \in O$ the lower curvature in some direction is zero, then at almost all points of O the curvature exists in both directions and is zero [**3**]. A suitable version of Theorem 2 guarantees then the existence of points in O with infinite curvature: they belong to F. The same assertion is true in higher dimensions and can be proved by providing locally supporting spheres touching the surface near a given point of it.

The following theorem describes the situation for a typical convex surface.

THEOREM 9. *For most convex surfaces, N_i and F are disjoint and of first category.*

Proof. No point x with $\rho_i^\tau(x) = 0$ for some τ belongs to N_i and no point x with $\rho_s^\tau(x) = \infty$ for some τ belongs to F; because, by Theorem B, at most points x, $\rho_i^\tau(x) = 0$ and $\rho_s^\tau(x) = \infty$, N_i and F are of first category on most convex surfaces. Since, on these surfaces, by Theorem 1 in [**12**], at each x, $\rho_i^\tau(x) = 0$ or $\rho_s^\tau(x) = \infty$ for any τ, $N_i \cap F = \varnothing$.

Notice that R. Schneider and J. A. Wieacker [**11**] proved that for most convex surfaces the Hausdorff dimension of F vanishes.

Contact properties of most convex surfaces. In this final chapter we see by using the techniques of [**12**] that, for most convex surfaces, not only the curvature vanishes, but, moreover, the order of contact with the supporting hyperplane is, in a sense to be rendered precise, ∞ a.e.

Let $f, g\colon [-\varepsilon, \varepsilon] \to [0, \infty)$ ($\varepsilon > 0$) be two convex functions with $f(0) = g(0) = \{0\}$, g being also symmetric in the sense that $g(x) = g(-x)$ for all $x \in (0, \varepsilon)$. We say that the graph A of f has a *g-contact* at $(0, 0)$ if there is no neighbourhood of 0 on the x-axis where

$$f(x) \geq g(x).$$

If, for some g, the derivatives of all orders of g exist and vanish at 0, then we say that A has *infinite contact order* at $(0, 0)$. If C is a convex surface in $\mathbf{R}^n$, we say that C has a *g-contact* (*infinite contact order*) at $x \in C$ if each normal section at x has a g-contact (infinite contact order) there.

Let $T_1(S)$ be the sphere bundle associated to the $(n-1)$-dimensional sphere S. Consider the space $\mathscr{D}$ of all closed convex curves in $\mathbf{R} \times [0, \infty)$ passing through $(0, 0)$. Let $w, W\colon T_1(S) \to \mathscr{D}$ be two continuous functions such that

(i) $w(y, \tau) \subset [0, \infty) \times [0, \infty)$, $(0, 1) \in w(y, \tau)$ and $w(y, \tau)$ has a right angle at $(0, 0)$,

(ii) $(-1, 0) \in W(y, \tau)$, $(0, 0)$ is an extreme point of conv $W(y, \tau)$ and $W(y, \tau)$ is smooth at $(0, 0)$. (See Figure 3.)

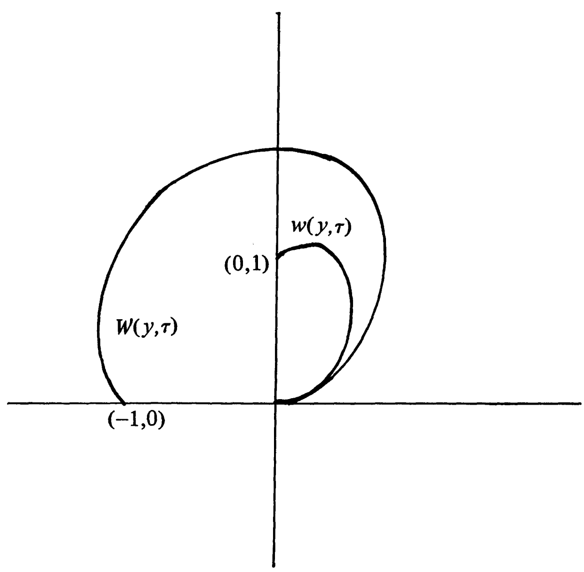

FIGURE 3

For $C \in \mathscr{C}_n$ let

$$\text{out}\, C = \overline{\mathbf{R}^n - \operatorname{conv} C}.$$

Now, let x be a smooth point of C and τ be a tangent direction at x. Consider the congruence c from $\mathbf{R}^2$ to the plane $\Pi_\tau(x)$ containing the outer normal $n_e(x)$ at x and parallel to τ, satisfying

$$c(0,0) = x, \quad c(\{0\} \times (-\infty, 0]) = n_e(x), \quad c(1,0) = x + \tau.$$

Denote

$$v_k(x,\tau) = c(k \cdot w(\nu(x),\tau)),$$

$$V_k(x,\tau) = c(k \cdot W(\nu(x),\tau)),$$

$\nu \colon C \to S$ being the spherical image and $k \in \mathbf{R}$.

Let $\mathscr{C}^*$ be the set of all surfaces $C \in \mathscr{C}_n$ such that there exist $k \in \mathbf{N}$, $x \in C$ and a tangent direction τ at x satisfying

$$v_{k^{-1}}(x,\tau) \subset \operatorname{conv} C$$

and

$$V_k(x,\tau) \subset \operatorname{out} C.$$

In [**12**] we already used the notions of a *semidisk* and a *corner-disk* in case $w(y,\tau)$ is a half-circle plus a diameter and $W(y,\tau)$ is a $3\pi/2$-arc of a circle plus two line-segments, tangent at its endpoints. The same arguments as those used in the proof of Theorem 1 from [**12**] lead to the following.

PROPOSITION. *$\mathscr{C}^*$ is of first category in $\mathscr{C}_n$.*

Obviously, Theorem 1 of [**12**] is a consequence of this Proposition. Moreover, it permits us to derive the following strengthening of Theorem A.

Consider an arbitrary symmetric convex function g as defined at the beginning of this chapter and the functions g_k defined by $g_k(t) = kg(t/k)$ ($k \in \mathbf{N}$).

THEOREM 10. *Most convex surfaces have a.e. a vanishing curvature in all tangent directions and, for any $k \in \mathbf{N}$, a g_k-contact.*

Proof. Let A be the set of points where $C \in \mathscr{C}_n$ has a g_k-contact for any $k \in \mathbf{N}$. If we take as $w(y,\tau)$ a semidisk and as $W(y,\tau)$ a curve defined near $(0,0)$ in $[0,\infty) \times \mathbf{R}$ by the function g, then $\mathscr{C}_n - \mathscr{C}^*$ consists of surfaces of $\mathscr{C}_n$ such that at each point x and tangent direction τ, $\rho_i^\tau(x) = 0$ or the normal section of C at x in direction τ has a g_k-contact at x for every $k \in \mathbf{N}$. Since a.e. on $C \in \mathscr{C}_n - \mathscr{C}^*$, in each tangent direction the curvature exists and is not ∞, it follows that $C - A$ has measure zero, which together with Theorem A ends the proof.

Taking as g a function with $g^{(n)}(0) = 0$ for all $n \in \mathbf{N}$, we get the following.

COROLLARY. *Most convex surfaces have an infinite contact order a.e.*

Let us say that a convex body K *contacts* the surface $C \in \mathscr{C}_n$ at $x \in C$ if $x \in K$ and $K \subset \operatorname{conv} C$.

For every convex body K there are surfaces $C \in \mathscr{C}_n$ such that, for every $x \in C$, there is a translate of K contacting C at x: take C to be homothetic to and larger than bd K. Given K and any surface $C \in \mathscr{C}_n$ surrounding K, there are points on C at which translates of K contact C. Also, given K and any $C \in \mathscr{C}_n$, there is a dense set of points on C at which sets homothetic to K contact C. What do such sets of contact points look like, for typical $C \in \mathscr{C}_n$?

THEOREM 11. *Given the smooth convex body K, for most surfaces $C \in \mathscr{C}_n$ the set of points at which sets congruent with K contact C is nowhere dense in C.*

Proof. This follows from the easy observation that the set of contact points mentioned in the statement is closed, and from the next theorem.

THEOREM 12. *Given the smooth convex body K, for most surfaces $C \in \mathscr{C}_n$ there is no (non-empty) open set of points on C at which sets similar to K contact C.*

Proof. For $z \in$ bd K and a tangent direction σ, let $s(z,\sigma)$ be the normal section of bd K at z in direction σ, completed by a segment along the normal to give rise to a convex curve. The length of this segment depends continuously on z and does not vanish when z describes bd K. So its infimum ζ also does not vanish. The nature of our statement allows us to apply an appropriate homothety on K to assure $\zeta \geq 1$.

Let t be the congruence from the plane of $s(z,\sigma)$ to $\mathbf{R}^2$ such that

$$t(z) = (0,0), \quad t(n_e(z)) = \{0\} \times (-\infty, 0], \quad t(z+\sigma) = (1,0).$$

Put

$$w(y,\tau) = \operatorname{bd} \bigcap_{(z,\sigma)} \operatorname{conv} t(s(z,\sigma))$$

for all (y,τ); hence w is constant.

Since $\zeta \geq 1$, $(0,1) \in w(y,\tau)$. Also, it is guaranteed that $w(y,\tau)$ has a right angle at $(0,0)$. Indeed, let $H \subset [0,\infty) \times [0,\infty)$ be a halfline starting at $(0,0)$ and making the angle $\alpha < \pi/2$ with $\{0\} \times \mathbf{R}$ (see Figure 4). Obviously, $t(s(z,\sigma))$ depends continuously on (z,σ) and so does the length $l(z,\sigma)$ of $H \cap \operatorname{conv} t(s(z,\sigma))$ too. Since the sphere bundle of bd K is compact, the function l attains its minimum at some (z_0,σ_0). Because $l(z_0,\sigma_0) \neq 0$ and

$$H \cap \operatorname{conv} t(s(z_0,\sigma_0)) \subset \operatorname{conv} w(y,\tau),$$

the angle of $w(y,\tau)$ at $(0,0)$ is at least α. Now α being arbitrary in $[0,\pi/2)$, that angle equals $\pi/2$.

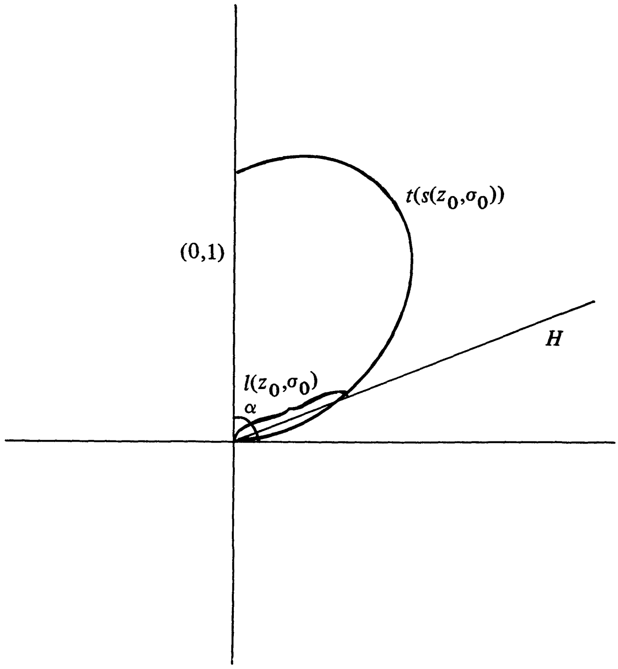

FIGURE 4

Let $W(y,\tau)$ be a corner-disk.

By Theorems A and 4, on a typical surface $C \in \mathscr{C}_n$, the set Q of those points x with $\rho_s^\tau(x) = \infty$ for each tangent direction τ has empty interior. For any $x \in C - Q$, there is a tangent direction τ and a natural number k_0 such that

$$V_k(x,\tau) \subset \operatorname{out} C$$

for all $k \geq k_0$. By the Proposition, for any $x \in C - Q$ and for the direction τ and numbers $k \geq k_0$ we just found, the curves $v_{k^{-1}}(x,\tau)$ are not included in conv C. Hence there is no similarity

$$u \colon \mathbf{R}^2 \to \Pi_\tau(x)$$

($\Pi_\tau(x)$ being again the plane including $n_e(x)$ and parallel to τ), for which

$$u(0,0) = x, \qquad u(\{0\} \times (-\infty, 0]) = n_e(x),$$

x, $x + \tau$ and $u(1,0)$ are collinear, and

$$u(t(s(z,\sigma))) \subset \operatorname{conv} C.$$

Since this holds for all $(z,\sigma) \in T_1(\mathrm{bd}\, K)$, no set similar to K contacts C at x. The proof is complete.

REFERENCES

[1] A. D. Aleksandrov, *Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it* (Russian), Uchenye Zapiski Leningrad. Gos. Univ. Math. Ser. 6 (1939), 3–35.
[2] H. Busemann, *Convex Surfaces*, New York: Interscience Publishers 1958.
[3] H. Busemann and W. Feller, *Krümmungseigenschaften konvexer Flächen*, Acta Math., **66** (1935), 1–47.
[4] P. M. Gruber, *Die meisten konvexen Körper sind glatt, aber nicht zu glatt*, Math. Ann., **229** (1977), 259–266.
[5] B. Grünbaum, *Continuous families of curves*, Canad. J. Math., **18** (1966), 529–537.
[6] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Berlin, Heidelberg, New York: Springer-Verlag 1969.
[7] V. Klee, *Some new results on smoothness and rotundity in normed linear spaces*, Math. Ann., **139** (1959), 51–63.
[8] V. Klima and I. Netuka, *Smoothness of a typical convex function*, Czech. Math. J., **31** (106) (1981), 569–572.
[9] J. C. Oxtoby, *Maß und Kategorie*, Berlin: Springer-Verlag 1971.
[10] R. Schneider, *On the curvatures of convex bodies*, Math. Ann., **240** (1979), 177–181.
[11] R. Schneider and J. A. Wieacker, *Approximation of convex bodies by polytopes*, Bull. London Math. Soc., **13** (1981), 149–156.
[12] T. Zamfirescu, *The curvature of most convex surfaces vanishes almost everywhere*, Math. Z., **174** (1980), 135–139.
[13] _____, *Nonexistence of curvature in most points of most convex surfaces*, Math. Ann., **252** (1980), 217–219.
[14] _____, *Intersecting diameters in convex bodies*, Ann. Discrete Math., **20** (1984), 311–316.
[15] _____, *Using Baire categories in Geometry*, Rend. Sem. Mat. Univ. Politecn. Torino, **43** (1985), 67–88.

Received July 3, 1986 and in revised form November 3, 1986.

UNIVERSITÄT DORTMUND
46 DORTMUND
BUNDESREPUBLIK DEUTSCHLAND

AND

CALIFORNIA STATE UNIVERSITY
LOS ANGELES, CA 90032, U.S.A.