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TRANSITIVE ISOMETRY GROUPS WITH NON-COMPACT ISOTROPY

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Let G be a connected Lie group acting effectively and transitively by isometries on a riemannian manifold M. Then G is a Lie subgroup of the full isometry group, which is not necessarily closed. In this paper we study the structure of the closure of G in $I(M)$ and illustrate the results with examples, with non-compact isotropy, where the closure is described explicitly.

Introduction. If M is a riemannian manifold the isotropy group in $I(M)$ is compact; hence a homogeneous riemannian M can always be represented as a quotient G'/H' with H' compact.

Assume now that G is a connected Lie group acting effectively and transitively by isometries on M. Then G is a Lie subgroup of $I(M)$ which will be closed in $I(M)$ if and only if the isotropy subgroup H is compact.

In this paper we study in detail the closure of G in $I(M)$. Also if G is any connected Lie group and H a closed subgroup we compare three standard conditions on H which ensure that G/H admits a riemannian invariant structure. The rest of the paper is devoted to illustrate the fact that it is rather common to have transitive, effective, non-closed Lie subgroups of $I(M)$, hence the isotropy subgroup is non-compact. This situation arises quite frequently, even when M is compact (Lemma 1.4). Also, any connected semisimple Lie group with infinite center admits a closed non-compact subgroup H such that G acts effectively on G/H and G/H carries a G-invariant riemannian structure. In this case, that is, when G is semisimple, we give an upper bound for the dimension of $\overline{G}_L$ and provide examples showing that these bounds are sharp (see (2.4), Proposition 2.3 and Remark 2.4).

In **[DMW]** the use of $\overline{G}_L$ proved to be convenient in the study of bounded isometries on a riemannian manifold acted on transitively and effectively by a semisimple Lie group without local compact factors. Also, some examples where G_L is not closed in $I(M)$ (G semisimple) are given in **[DMW]** Example (3.10). The authors would like to thank J. A. Wolf for very useful comments on a first version of this paper and, in particular, for suggesting a simpler proof of Proposition 2.2.

1. Closure of G. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let H be a closed subgroup of G with Lie algebra $\mathfrak{h}$. We will say that H satisfies condition (1) (respectively (2), (3)) if

(1) $\overline{\mathrm{Ad}_G(H)}$ is compact in $\mathrm{GL}(\mathfrak{g})$, the closure with respect to the $\mathrm{GL}(\mathfrak{g})$ topology;

(2) $\mathrm{cl}(\mathrm{Ad}_G(H))$ is compact in $\mathrm{Ad}(G)$, where cl denotes closure with respect to the $\mathrm{Ad}(G)$ topology;

(3) $\mathfrak{h}$ is a compactly embedded subalgebra of $\mathfrak{g}$ (or equivalently see [**H**, p. 130]) $\mathrm{Ad}_G(H_0)$ is compact in $\mathrm{Ad}(G)$), where H_0 denotes the identity component of H.

Along this section we will analyze the relationship among the different conditions (1), (2), (3).

Condition (1). It is well known (cf. [**Ch E**]. Ch. III) that if G acts effectively on G/H then H satisfies (1) if and only if G/H admits a G-invariant riemannian structure. Furthermore for each such structure, G is a Lie subgroup of the full isometry group $I(M)$, $M = G/H$, via $x \to L_x$, L_x the left translation on $M = G/H$. If S is a Lie subgroup of G, denote $S_L = \{L_x \mid x \in S\}$. The next theorem describes the closure of G_L in $I(M)$.

1.1. THEOREM. *Let G be a connected Lie group acting effectively on $M = G/H$, H a subgroup of G satisfying* (1). *Then*

$$\overline{G}_L = \{L_x \circ \varphi \colon x \in G,\ \varphi \in \overline{H}_L\}.$$

In particular, G_L is closed if and only if H is compact. Moreover

(i) *$\overline{H}_L$ is isomorphic to $\overline{\mathrm{Ad}(H)}$ (as Lie groups).*

(ii) *There is an exact sequence*

$$\{1\} \to \{(x, \mathrm{Ad}(x^{-1})) \colon x \in H\} \to G \rtimes \overline{\mathrm{Ad}(H)} \to \overline{G}_L \to \{1\}$$

where $\rtimes$ denotes semidirect product.

Proof. If L_{x_n} converges in $I(M)$ then $x_n H \to x^* H$ in G/H, for some x^* in G. Hence there is a sequence h_n in H such that $x_n h_n \to x^*$ in G. Now $L_{h_n^{-1}} \in H_L$ and since H_L is contained in the isotropy subgroup in $I(M)$, $L_{h_{n_j}^{-1}} \to \phi$, $\phi \in \overline{H}_L$. As a consequence $L_{x_{n_j}} = L_{x_{n_j} h_{n_j}} \circ L_{h_{n_j}^{-1}}$ converges to $L_{x^*} \circ \phi$ and the first assertion follows.

Clearly, if H is compact, $\overline{H}_L = H_L \subset G_L$. Conversely if G_L is closed in $I(M)$ then H_L is compact in G_L since H_L is the intersection of G_L with the isotropy subgroup in $I(M)$.

Next we prove (i). Let $\mathfrak{m}$ be an $\overline{\mathrm{Ad}(H)}$ invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$ where $\mathfrak{h}$ denotes the Lie algebra of H. It is not hard to show that the canonical map $\overline{H}_L \to \overline{\mathrm{Ad}(H)} \mid \mathfrak{m}$ sending $\varphi \to (d\varphi)_{eH}$ is an isomorphism of Lie groups. Hence (i) will follow once we prove that $\overline{\mathrm{Ad}(H)}$ and $\overline{\mathrm{Ad}(H)} \mid \mathfrak{m}$ are isomorphic.

Let π: $\overline{\mathrm{Ad}(H)} \to \overline{\mathrm{Ad}(H)} \mid \mathfrak{m}$ denote the restriction homomorphism. By compactness of $\overline{\mathrm{Ad}(H)}$ it follows that π is onto. If T is in $\overline{\mathrm{Ad}(H)}$ and $T \mid \mathfrak{m}$ is the identity, then $T \mid \mathfrak{m}_1$ is the identity for any $\mathfrak{m}_1$, an $\overline{\mathrm{Ad}(H)}$ invariant complement of $\mathfrak{h}$. Hence we may take $\mathfrak{m}_1$ to be the orthogonal complement of $\mathfrak{h}$ with respect to an $\overline{\mathrm{Ad}(H)}$ invariant inner product and consequently $\mathfrak{m}_1$ will satisfy $[\mathfrak{h}, \mathfrak{m}_1] \subset \mathfrak{m}_1$. If $X \in \mathfrak{h}$, $Y \in \mathfrak{m}_1$

$$[TX - X, Y] = [TX, Y] - [X, Y] = T[X, Y] - [X, Y] = 0.$$

Hence $\mathrm{Ad}(\exp(TX - X))$ is the identity in $\mathfrak{m}_1$ or equivalently $(dL_{\exp(TX-X)})_{eH}$ is the identity in G/H. Since we are assuming that the action of G on G/H is effective it follows that $\exp(TX - X) = e$ for all X in $\mathfrak{h}$, hence T is the identity on $\mathfrak{g}$.

Now (ii) follows immediately from the description of $\overline{G}_L$, after observing that the kernel of the Lie group homomorphism $G \rtimes \overline{H}_L \to \overline{G}_L$, $(x, \varphi) \to L_x \circ \varphi$ consists of pairs $(x, L_{x^{-1}})$ where x is in H. □

Let G be a connected Lie group and H a subgroup of G satisfying (1). Set

$$\mathrm{Aut}(G, H) = \{\varphi \in \mathrm{Aut}(G) : \varphi(H) = H, (d\varphi)_e \in \overline{\mathrm{Ad}(H)}\}.$$

If $\varphi \in \mathrm{Aut}(G, H)$, $\overline{\varphi}$: $G/H \to G/H$ will denote the induced diffeomorphism, $\overline{\varphi}(xH) = \varphi(x)H$. If G acts effectively on $M = G/H$ then, for any G-invariant riemannian structure on M, $\overline{\varphi}$ is an isometry of M. Furthermore, if $(d\varphi)_e = \lim \mathrm{Ad}(h_n)$, the sequence $L_{h_n} \to \psi$, ψ in $\overline{H}_L$ (by taking a subsequence if necessary), hence $(dL_{h_n})_{eH} \to (d\psi)_{eH}$, that is, $\overline{\varphi} = \psi$ since they both fix eH.

1.2. PROPOSITION. *Assume the hypotheses of Theorem* 1.1 *hold. Then the correspondence* $\varphi \to \overline{\varphi}$, $\mathrm{Aut}(G, H) \to \overline{H}_L$ *is a group isomorphism.*

Proof. It is clearly a homomorphism. To check it is surjective assume $\psi = \lim L_{h_n}$, $h_n \in H$. By taking a subsequence, if necessary, we have that $\mathrm{Ad}(h_n) \to T$, T an automorphism of $\mathfrak{g}$. If ϕ denotes the lifting of T to $\tilde{G}$, $\tilde{G}$ the simply connected covering group of G, then ϕ is the identity restricted to the center of $\tilde{G}$. In fact, let $\tilde{h}_n$ in $\tilde{G}$ be such that $d(I_{\tilde{h}_n})_e = \mathrm{Ad}(h_n)$. Then $I_{\tilde{h}_n}(\exp X) = \exp \mathrm{Ad}(h_n) X \to \exp TX = \phi(\exp X)$ and since

$\tilde{G}$ is connected $I_{\tilde{h}_n}$ converges pointwise to ϕ; the assertion follows. As a consequence ϕ induces an automorphism φ: $G \to G$ lifting T and since I_{h_n} converges pointwise to φ then $\varphi(H) \subset H$. By repeating the above argument with T^{-1} we conclude that $\varphi \in \mathrm{Aut}(G, H)$. Also from $I_{h_n}(x) \to \varphi(x)$, x in G, it follows that $L_{h_n}(xH) \to \bar{\varphi}(xH)$ and consequently $\bar{\varphi} = \psi$.

Assume $\bar{\varphi}$ is the identity on G/H. Then $y^{-1}\varphi(y) \in H$, for all y in G. Hence

$$\frac{d}{dt}\bigg|_{t=0} \exp(-tY)\varphi(\exp tY) = -Y + (d\varphi)_e Y \in \mathfrak{h}$$

for all Y in $\mathfrak{g}$. As a consequence $(d\varphi)_e | \mathfrak{m}$ is the identity, $\mathfrak{m}$ an $\mathrm{Ad}(H)$ invariant complement of $\mathfrak{h}$. Hence $(d\varphi)_e$ is the identity on $\mathfrak{g}$ (see the proof of (i), Theorem 1.1) or equivalently, $\varphi = \mathrm{Id}$. □

Condition (2). Let Ad: $G \to \mathrm{Aut}(\mathfrak{g})$ be the adjoint representation, Z the center of G, ad: $\mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ the derivative of Ad. Since $\mathrm{Ad}(G)$, the connected Lie subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$, is isomorphic to G/Z, condition (2) is equivalent to

(2′) $\qquad \pi(\overline{ZH})$ compact in G/Z,

where π: $G \to G/Z$ is the standard projection. Furthermore, since $\mathrm{Aut}(\mathfrak{g})$ carries the relative topology of $\mathrm{Gl}(\mathfrak{g})$, Condition (2) implies Condition (1) (the converse does not hold in general, see Example 1.4 below). For subgroups H of G satisfying (2), one can give a simple description of $\bar{G}_L$, in terms of G and the closure of ZH in G (see also **[DMW]**, (3.5)).

1.3. Proposition. *Let G be a connected Lie group, acting effectively on* $M = G/H$, *H satisfying* (2). *Then* $\bar{G}_L = \{L_x \circ R_{y^{-1}}\colon\ x \in G,\ y \in \overline{ZH}\}$ *and there is an exact sequence*

$$\{1\} \to S \to G \times \overline{ZH} \to \bar{G}_L \to \{1\}$$

where $S = \{(z, zh) \mid z \in Z, h \in H\}$. *Furthermore, if H is not compact,* $\overline{ZH}/H$ *is an abelian Lie subgroup of positive dimension of* $Z(\bar{G}_L)$, *the center of* $\bar{G}_L$.

Proof. We observe first that for any G-invariant structure on $M = G/H$, if $y \in \overline{ZH}$ the map R_y is in the closure of G in $I(M)$, where $R_y(zH) = zyH$. In fact, since $\overline{ZH}$ is contained in $N(H)$, the normalizer of H, R_y is a diffeomorphism. Furthermore the condition $y = \lim z_n h_n$ says that $xyH = \lim L_{z_n}(xH)$, hence $R_y = \lim L_{z_n}$ in $I(M)$([H], p. 203, Lemma 2.4).

The map $\alpha\colon G \times \overline{ZH} \to \overline{G}_L$, $\alpha(x, y) = L_x \circ R_{y^{-1}}$ is a Lie group homomorphism. To check that α is surjective it is enough (by Theorem 1.1) to consider φ in $\overline{H}_L$, $\varphi = \lim L_{h_n}$. Since H satisfies (2′), there exists a subsequence h_{n_j} such that $\pi(h_{n_j}) \to \pi(u)$, $u \in \overline{ZH}$. Hence $z_{n_j}h_{n_j} \to u$ for some z_{n_j} in Z and $L_{h_n}(xH) = h_n z_n x z_n^{-1} h_n^{-1} H \to L_u \circ R_{u^{-1}}(xH)$, that is, $\varphi = \alpha(u, u)$. Let (x, y) in $\ker\alpha$. Then $z^{-1}xzy^{-1} \in H$ for all z in G. Since $yx^{-1} \in H$ (taking $z = e$) it follows that $L_x \circ R_{x^{-1}} \in \mathrm{Aut}(G, H)$ induces the identity on G/H. By Proposition 1.2, $L_x \circ R_{x^{-1}}$ is the identity on G and the description of $S = \ker\alpha$ follows.

It is not hard to check that $\overline{ZH}/H$ is an abelian Lie group which is a Lie subgroup of $Z(\overline{G}_L)$. Assume $\dim \overline{ZH}/H = 0$. The map

$$\beta\colon Z \times H \to \overline{ZH}, \qquad (z, h) \to zh$$

is a one-to-one Lie group homomorphism. By regularity $\beta(Z \times H) = ZH$ is open in $\overline{ZH}$, hence closed. Now if ZH is closed and H satisfies (2′) it follows that H is compact. This completes the proof. □

We will next show that, even the case M compact riemannian, it is quite common to have a representation with non-compact isotropy subgroup. This question was posed to us by Cristian Sanchez.

1.4. Lemma. *Let G be a compact, connected semisimple Lie group, H a closed subgroup of G such that G acts effectively on G/H. Assume that $M = G/H$ carries a G-invariant riemannian structure such that* $\mathrm{rank}(U) \geq \mathrm{rank}(H) + 2$, *where $U = \{x \in N(H)\colon R_x \text{ is an isometry}\}$. Then M can be represented as a quotient G'/H', where G' acts effectively by isometries on G'/H' and H' is a closed noncompact subgroup. Furthermore, if $\phi\colon G' \to I(M)$ is the inclusion as a Lie subgroup, then $\phi(G') = G_L \cdot S_R$ where S is a torus in U.*

Proof. Let $\mathfrak{u}$ denote the Lie algebra of U and let T be a torus in U such that $\dim(T) \geq 2$ and $\mathfrak{t} \cap \mathfrak{h} = 0$ (here $\mathfrak{t}$ is the Lie algebra of T). Fix $X \in \mathfrak{t}$ so that $t \to \exp(tX)$ is one-to-one.

Set $G' = G \times \mathbf{R}$; let $\phi\colon G' \to I(M)$ be given by $\phi(g, t) = L_g \circ R_{\exp(-tX)}$. We assert that ϕ is a monomorphism. Indeed, $(g, t) \in \ker(\phi)$ if and only if $x^{-1}gx\ \exp(-tX) \in H$, for any x in G. Thus $g = h\ \exp(tX) \in U$, for some h in H and $\mathrm{Ad}(g)(Z) - Z \in \mathfrak{h}$, for any $Z \in \mathfrak{g}$. Arguing as in the proof of Proposition 1.2 one concludes that $\mathrm{Ad}(g) = \mathrm{Id}$, or $g \in Z(G)$. On the other hand, the one-parameter group generated by X

intersects H in the identity element, since $T \cap H$ is a finite group. This easily implies that $h = \exp(tX) = e$, or $t = 0$. Hence M is isometric to G'/H' where $H' = \{(h\exp(tX), t): h \in H,\ t \in \mathbf{R}\}$.

Finally, by Proposition 1.3 it is easy to check that if S denotes the closure of $\{\exp(tX): t \in \mathbf{R}\}$, then

$$\overline{G'_L} = G_L \cdot (S \times \{0\})_R.$$

From this it easily follows that $\overline{\phi(G')}$ is as asserted in the lemma. This completes the proof. □

1.5. Remark. (i) As a particular case in the lemma we may take $H = \{e\}$. That is, $M = G$ is any compact, semisimple, connected Lie group endowed with a left invariant metric such that $\operatorname{rank}(U) \geq 2$. For instance, if $\operatorname{rank}(G) \geq 2$ and we use a bi-invariant metric, then G is a riemannian symmetric space admitting a representation G'/H', with H' non-compact.

(ii) The proof above only uses that $Z(G)$ is finite and that H is compact. Hence the lemma remains valid under these assumptions on G and H, only.

We recall that in a variety of cases Conditions (1) and (2) are equivalent. For instance when $\operatorname{Ad}(G)$ is closed in $\operatorname{GL}(\mathfrak{g})$ since both topologies coincide. This happens for G semisimple since any derivation of $\mathfrak{g}$ is inner. Also $\operatorname{Ad}(G)$ is closed if G is reductive. It may fail to be so when G is solvable as the next example shows.

1.6. Example. We provide here an example of a subgroup H of a connected Lie group G, satisfying (1) and not satisfying (2). Moreover since G acts effectively on G/H, we apply Theorem 1.1 to compute the closure of G in $I(M)$, $M = G/H$.

If $t_i \in S^1$, $z_i \in \mathbf{C}$ for $1 \leq i \leq n+1$, denote

$$s(t_1, \dots, t_{n+1}) = \begin{vmatrix} t_1 & & & \\ & \ddots & & \\ & & t_{n+1} & \\ & & & 1 \end{vmatrix},$$

$$n(z_1, \dots, z_{n+1}) = \begin{vmatrix} 1 & & & z_1 \\ & \ddots & & \vdots \\ & & 1 & z_{n+1} \\ & & & 1 \end{vmatrix}$$

and set

$$S = \{s(t_1,\dots,t_{n+1}) \colon t_i \in S^1\}, \quad N = \{n(z_1,\dots,z_{n+1}) \colon z_i \in \mathbf{C}^{n+1}\},$$

$G_1 = NS$. Since

$$s(t_1,\dots,t_{n+1})n(z_1,\dots,z_{n+1})s(t_1,\dots,t_{n+1})^{-1} = n(t_1z_1,\dots,t_{n+1}z_{n+1})$$

then it is clear that G_1 is a closed subgroup of $\mathrm{GL}(n+2,\mathbf{C})$ isomorphic to the semidirect product $N \rtimes S$ where S acts on N by

$$s(t_1,\dots,t_{n+1})\cdot(n(z_1,\dots,z_{n+1})) = n(t_1z_1,\dots,t_{n+1}z_{n+1}).$$

We note that if $g = ns$, g commutes with N if and only if s commutes with N, that is, $s = 1$. Hence, if G_2 is a Lie subgroup with $N \subset G_2$ and if, for each j, there is $s(t_1,\dots,t_{n+1})$ in $S \cap G_2$ with $t_j \neq 1$, then $Z(G_2) = 1$.

Now let

$$\phi \colon \mathbf{R}^{n+1} \to G_1, \ \phi(x_1,\dots,x_{n+1}) = s(e^{2\pi i x_1},\dots,e^{2\pi i x_{n+1}}).$$

Let $\mathbf{V}$ be an n-dimensional subspace of $\mathbf{R}^{n+1}$ and let $v \in \mathbf{V}$ be such that ϕ is injective when restricted to $\mathbf{V}$ and $\mathbf{R}v + \sum_1^{n+1}\mathbf{Z}e_j$ is dense in $\mathbf{R}^{n+1}$, $\{e_1,\dots,e_{n+1}\}$ the standard basis of $\mathbf{R}^{n+1}$; (see appendix, Lemma A.4).

Set $G = N\phi(\mathbf{V})$, $H = \phi(\mathbf{R}v)$. Then G is a normal dense Lie subgroup of G_1 and H is dense in S. On the other hand, H is a closed Lie subgroup of G. Now, by the choice of H, it is not hard to see that $\overline{\mathrm{Ad}(H)} = \mathrm{Ad}_G(S)$ and this is, in turn, isomorphic to T^{n+1}. Hence (1) holds. However, given that $Z = \{e\}$, and H is not compact, (2) is not satisfied.

One checks that $\overline{\mathrm{Ad}(H)} = \{I_s \colon s \in S\}$. From this and the fact that G acts effectively on G/H, it follows (see 1.1 and 1.2) that

$$\overline{G}_L = \{L_x \circ \bar{I}_s \colon x \in G, s \in S\}.$$

By a calculation one checks that $Z(\overline{G}_L) = \{e\}$ in contrast to the case when (2) is satisfied (see Proposition 1.3).

1.7. Remark. Clearly a connected subgroup H of G satisfying condition (3) will satisfy (2) (then (1)). Now if G acts effectively on G/H, the conditions $\mathrm{Ad}_G(H)$ compact and H compact are equivalent. Hence if H satisfies (3), G_L will be closed in $I(M)$. As 1.4, 1.5, and 1.6 show, it is not true in general that if G/H is riemannian homogeneous then H satisfies (3). On the other hand, we see later (see Remark 2.3) that if G is any simply connected semisimple Lie group with infinite center there exists H a non-compact, Lie subgroup satisfying (2) with G acting effectively on G/H. Such an H will not satisfy (3).

As a final observation we describe a way to construct new examples of pairs (G, H), H satisfying (1), H non-compact.

1.8. LEMMA. *Let (G, H) be a pair of a connected Lie group G and a closed subgroup H so that* (1) *holds. Let $(\sigma, \mathbf{V})$ be a finite dimensional representation of G such that H acts unitarily. Set $G_\sigma = \mathbf{V} \rtimes_\sigma G$. Assume that* $\ker(\sigma) \cap \ker($*action of G on G/H*$) = \{e\}$. *Then G_σ acts effectively on G_σ/H and H satisfies* (1). *Furthermore, if* $\ker(\sigma) \cap Z(G) = \{e\}$, *$G$ acts without non-zero fixed vectors and H is not compact, then* (2) *will not be satisfied.*

Proof. Clearly H is closed in G_σ and $\mathrm{Ad}_{G_\sigma}(H) \subset O(\mathbf{V}) \times \mathrm{Ad}_G(H)$, hence (1) holds. On the other hand

$$(1.9) \qquad \begin{aligned} &(0, g_1) \cdot (v, g) \cdot (0, g_1^{-1}) = (\sigma(g_1)v, g_1 g g_1^{-1}), \\ &(v_1, 1) \cdot (v, g) \cdot (-v_1, 1) = (v + v_1 - \sigma(g)v_1, g). \end{aligned}$$

Thus

$$\bigcap_{x \in G_\sigma} xHx^{-1} = \Big(\bigcap_{x \in G} xHx^{-1}\Big) \cap \ker\sigma = \{e\},$$

by the assumption. Hence the action is effective. On the other hand, by (2.5)

$$Z(G_\sigma) = \{(g_0, v_0);\ g_0 \in Z(G) \cap \ker(\sigma),\ \sigma(g)v_0 = v_0 \text{ for any } g \in G\}$$

and under the assumptions $Z(G_\sigma) = \{e\}$, hence the second assertion is clear. □

1.9. REMARKS. (a) We note that Example 1.6 is a particular instance of the construction in the lemma with $(G, H) = (\mathbf{R}^n, \mathbf{R})$. In this case, the representation of $\mathbf{R}^n$ in $\mathbf{C}^{n+1}$ is faithful and unitary and the action of G on G/H is far from being effective but the action of G_σ on G_σ/H is so. This procedure could naturally be iterated by taking a finite dimensional representation $(\tau, \mathbf{W})$ of G_σ, unitary on H, thus getting a new pair $((G_\sigma)_\tau, H)$.

(b) As another illustration of the lemma, let G be semisimple and H a closed non-compact subgroup such that (2) holds. Let $(\sigma, \mathbf{V})$ be a non-trivial, finite dimensional, irreducible representation of G. We have that $Z(G_\sigma) = \{(0, z) \mid z \in Z \cap \ker\sigma\}$. Hence

$$p_\sigma\big(\overline{Z(G_\sigma)H}\big) = \{0\} \times p\big(\overline{(Z \cap \ker\sigma)H}\big)$$

is compact, where p_σ: $G_\sigma \to G_\sigma/Z(G_\sigma)$ is the canonical projection. Hence (2) holds and if, for instance, G acts effectively or σ is faithful the action of G_σ is effective.

2. Non compact isotropy. In this section we consider the case when G is semisimple. As observed in §1, for H a closed subgroup of G Conditions (1) and (2) are equivalent. The first result we prove is a characterization of connected non-compact subgroups H satisfying (2). We also give a way to compute $\overline{G}_L$ explicitly, in the light of Proposition 1.2 and a classical result (see Theorem A.1 in the appendix). We also construct families of examples where $\dim \overline{G}_L - \dim G_L$ is the maximum possible.

Let then G be a connected semisimple Lie group. If $G = K \cdot \exp(\mathfrak{p})$ is a Cartan decomposition, K is compact if and only if Z, the center of G, is finite. Denote by $\mathfrak{k}$ the Lie algebra of K and let $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$. Then $\mathfrak{k} = z_\mathfrak{k} \oplus \mathfrak{k}'$, $z_\mathfrak{k}$ the center of $\mathfrak{k}$. Denote by Z_K and K', respectively, the connected Lie subgroups of K corresponding to $z_\mathfrak{k}$ and $\mathfrak{k}'$. Then K' is compact and if furthermore, G is simply connected, K is isomorphic to the direct product of Z_K and K'. In this case Z_K is a vector group.

2.1. Proposition. *Let G be a simply connected semisimple Lie group and H a closed connected subgroup such that* (2) *is satisfied. Then there is a Cartan decomposition $G = K$* $\exp(\mathfrak{p})$ *such that*

$$H = \exp(\mathbf{V}) \cdot (H \cap K') \tag{2.1}$$

where V is a vector subspace of the center of $\mathfrak{h}$, exp: $\mathbf{V} \to \exp \mathbf{V}$ is an isomorphism onto a closed subgroup and $\exp(\mathbf{V}) \cap (H \cap K') = 1$. *Furthermore $\overline{ZH}$ can be explicitly computed.*

Proof. Let p: $G \to G/Z$ be the canonical projection. Since $p(\overline{ZH}) = \overline{p(H)}$ is compact, by a theorem of Iwasawa, there is a maximal compact subgroup K_0 of G/Z such that $p(\overline{ZH}) \subset K_0$. Thus $\overline{ZH} \subset K = p^{-1}(K_0)$. Let $K = Z_K \cdot K'$ as above. If $Z_k = \{e\}$, then K is compact, hence H, and (2.1) is clear. We thus assume from now on that $z_\mathfrak{k} \neq 0$. Now, $\mathfrak{h}$ admits an $\mathrm{Ad}(H)$-invariant inner product, hence $\mathfrak{h} = z_0 \oplus \mathfrak{h}'$ where z_0 is the center of $\mathfrak{h}$ and $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$. Let H' (resp. Z_H) be the connected Lie subgroup of H corresponding to $\mathfrak{h}'$ (resp. z_0). Then H' is compact and $H' \subset K'$. Write $z_0 = \mathbf{V} \oplus z'$ with $z' = z_0 \cap \mathfrak{k}'$ and $\mathbf{V}$ a vector subspace of z_0. Fix $\{(U_i, Y_i)\colon U_i \in z_\mathfrak{k},\ Y_i \in \mathfrak{k}',\ 1 \le i \le m\}$ a basis of $\mathbf{V}$. Then the set $\{U_i\colon 1 \le i \le m\}$ is linearly independent since $\mathbf{V} \cap \mathfrak{k}' = \{0\}$. Also $[Y_i, Y_j] = 0$, for all i, j. Assume now that

$$\exp\left(\sum_1^m t_i(U_i, Y_i)\right) = \exp\left(\sum_1^m t_i(U_i, 0)\right)\exp\left(\sum t_i(0, Y_i)\right) \in K'.$$

Since $\exp(\Sigma_1^m t_i U_i) = 1$, then, $t_i = 0$ for $1 \leq i \leq m$. Hence, $\exp|_{\mathbf{V}}$ is one-to-one and $\exp(\mathbf{V}) \cap K' = \{1\}$. This implies that $Z_H \cap K' = \exp(z')$, that is, $Z_H = \exp \mathbf{V} \cdot (Z_H \cap K')$.

Let now $v_n \in \mathbf{V}$ be so that $\exp(v_n) \to u \in G$. Then $u \in Z_H$, hence $u = \exp(v)u'$, $u' \in Z_H \cap K'$. Write

$$v_n v^{-1} = \sum_1^m t_{i,n}(U_i, Y_i).$$

Since $\exp(v_n v^{-1}) \to u' \in K'$ it follows that $\exp(\Sigma_1^m t_{i,n} U_i) \to 1$ in Z_K. Hence $t_{i,n} \to 0$, for $1 \leq i \leq m$ (again we use that $U_1, \ldots, U_m$ are linearly independent). Therefore $u' = 1$ and $\exp(\mathbf{V})$ is closed in G. This completes the proof of the first assertion. We now describe how to compute $\overline{ZH}$.

Fix $Z' \subset Z \cap Z_K$, a subgroup such that $|Z/Z'| < \infty$ and let Z'' be a system of representatives of Z/Z'. Then

$$\overline{ZH} = \overline{Z'H} \cdot Z'' = \overline{Z' \cdot \exp \mathbf{V} \cdot H \cap K'} \cdot Z''. \tag{2.2}$$

Let $\{(U_i, Y_i) \,|\, 1 \leq i \leq m\}$ be as above. Also let $\mathfrak{s} = \Sigma_1^m \mathbf{R} Y_i$, $T' = \overline{\exp(\mathfrak{s})}$ and let $\mathfrak{t}'$ be the Lie algebra of T'. Choose $\{X_j\colon 1 \leq j \leq d\}$ a basis of $\mathfrak{t}'$ so that $\Sigma \mathbf{Z} X_j = \ker(\exp|_{\mathfrak{t}'})$ and $\{V_i\colon 1 \leq i \leq c\}$ a basis of $z_{\mathfrak{k}}$ such that $Z' = \exp(\Sigma \mathbf{Z} V_i)$.

Thus

$$\overline{Z' \exp(\mathbf{V})} = \exp\overline{\left(\sum_1^c \mathbf{Z} V_i + \sum_1^d \mathbf{Z} X_j + \mathbf{V}\right)} \tag{2.3}$$

and it then follows from (2.2) and (2.3) that $\overline{ZH}$ can be computed (at least in principle) by means of Theorem A.1 in the appendix. □

2.2. Remark. Assume that G is semisimple and the hypotheses of Proposition 1.3 hold. Then, given any G-invariant riemannian metric on G/H.

$$\dim \overline{G}_L \leq \dim G + \operatorname{rank}(K) - \operatorname{rank}(H) \tag{2.4}$$

where $G = K \cdot \exp(\mathfrak{p})$ is a Cartan decomposition of G. In fact, since H satisfies (2), $\overline{G}_L$ is given by Proposition 1.3, hence

$$\dim \overline{G}_L = \dim G + \dim \overline{ZH}/H.$$

Now $\overline{ZH} \subset K = p^{-1}(K_0)$, K_0 a maximal compact subgroup of G/Z, $p(\overline{ZH}) \subset K_0$. Let $\bar{\mathfrak{h}}$ denote the Lie algebra of $\overline{ZH}$. It is clear that $\mathfrak{h}$ is an ideal of $\bar{\mathfrak{h}}$ and there is an abelian ideal $\mathfrak{a}$ of $\bar{\mathfrak{h}}$ with $\bar{\mathfrak{h}} = \mathfrak{a} \oplus \mathfrak{h} \subset \mathfrak{k}$. Hence $\dim \overline{G}_L = \dim G + \dim \mathfrak{a} \leq \dim G + \operatorname{rank}(K) - \operatorname{rank}(H)$ as asserted.

We next provide a family of examples where $\dim \overline{G}_L - \dim G_L$ is the maximum possible. This will show that the bound in (2.4) is sharp.

2.2. PROPOSITION. *Let G be a simply connected semisimple Lie group such that the center of each simple factor is infinite. Then there exists a discrete subgroup $H \simeq \mathbf{Z}$ such that $\overline{ZH} = T$, a compact Cartan subgroup, and G acts effectively on G/H. Furthermore if G is not locally isomorphic to* $\mathrm{SL}(2, R)$, *there exists a closed subgroup $H' \simeq \mathbf{R}$ with $\overline{ZH'} = T$ and G acting effectively on G/H'. Then for any G-invariant riemannian metric on G/H (resp. G/H')*

$$\dim \overline{G}_L = \dim G + \operatorname{rank}(K) \qquad \big(\text{resp. } \dim \overline{G}_L = \dim G + \operatorname{rank}(K) - 1\big).$$

Furthermore $\overline{G}_L = I(G/H)_0$ (resp. $I(G/H')_0$).

Proof. Let T be a Cartan subgroup, $T \subset K$ and let $\mathfrak{t}$ denote the Lie algebra of T. Let $p\colon G \to G/Z$ be the canonical projection. Then T/Z is a torus. Fix X in $\mathfrak{t}$ so that, if $t = \exp(X)$, tZ is a generator of T/Z (i.e. the powers of tZ form a dense subgroup of T/Z). Let H be the group generated by t and let H' be the one-parameter group generated by X. Now $X = X_1 + X_2$, $X_1 \in z_{\mathfrak{k}}$, $X_2 \in \mathfrak{k}'$ and $X_1 \neq 0$, since tZ is a generator. This implies that both H and H' are closed subgroups.

On the other hand G acts effectively on G/H since $\bigcap_{x \in G} x^{-1}Tx = Z$ and by the choice of t it is clear that $H \cap Z = \{e\}$. Similarly, if $H' \cap Z = \{e\}$, the action on G/H' is effective. Otherwise $p(H')$ is a circle and then $p(H') = T$. Therefore G is isomorphic to the universal covering group of $\mathrm{SL}(2, \mathbf{R})$, against our assumption. Thus, by (2.4) $\dim \overline{G}_L$ is as asserted, since, if $S = H$ or H', by Proposition 1.3 $\overline{G}_L = G_L \cdot (\overline{ZS})_R = G_L \cdot T_R$. Finally, by a result of C. Gordon ([**G**], Theorem 4.1)

$$I(M)_0 = G_L \cdot (U_0)_R$$

where, as usual, $U = \{x \in N(S)\colon R_x \text{ is an isometry}\}$. Now

$$T \subseteq U_0 \subset N(S)_0 \subset N(T)_0 = T.$$

Thus the last assertion is clear. This concludes the proof. □

2.3. REMARK. It is easy to modify the example in (3.10) [**DMW**] to show that any connected semisimple Lie group with infinite center G admits a closed, non-compact, discrete subgroup H satisfying (2) and G acting effectively on G/H.

APPENDIX. In this appendix we list some classical facts on the subgroups of $\mathbf{R}^n$. We refer to [B], Ch. VII for a detailed treatment. If x, $y \in \mathbf{R}^n$ denote by $x \cdot y$ the canonical inner product. Given $G \subset \mathbf{R}^n$, a subgroup, set $G^* = \{ g' \in \mathbf{R}^n \mid g' \cdot g \in \mathbf{Z}, \text{ for all } g \in G\}$.

A.1. THEOREM (*see* [B], *p.* 74). *Let* $G \subset \mathbf{R}^n$ *be a subgroup. Then* $\overline{G} = G^{**}$. *In particular* G *is dense if and only if* $G^* = \{0\}$.

A.2. COROLLARY. *Fix* $v = (a_1, \dots, a_n) \in \mathbf{R}^n$ *and let* $\{e_j\}_1^n$ *be the canonical basis. Let* $G_1 = \mathbf{Z}v + \sum_1^n \mathbf{Z}e_j$ (*resp.* $G_2 = \mathbf{R}v + \sum_1^n \mathbf{Z}e_j$). *Then* G_1 (*resp.* G_2) *is dense in* $\mathbf{R}^n$ *if and only if* $1, a_1, \dots, a_n$ *are* $\mathbf{Q}$*-linearly independent* (*resp.* $a_1, \dots, a_n$ *are* $\mathbf{Q}$*-linearly independent*).

A.3. LEMMA. *If* $1 \le k \le n$, *let* ϕ: $\mathbf{R}^n \to \mathbf{R}^k \times T^{n-k}$ *be the canonical map,* $\phi(t_1, \dots, t_k, t_{k+1}, \dots, t_n) = (t_1, \dots, t_k, e^{2\pi i t_{k+1}}, \dots, e^{2\pi i t_n})$. *If* $G \subset \mathbf{R}^n$ *is a subgroup, then* $\overline{\phi(G)} = \overline{\phi(G + \sum_{k+1}^n \mathbf{Z}e_j)}$.

A.4. LEMMA. *There exists an n-dimensional subspace* $\mathbf{V}$ *of* R^{n+1} *and* $v \in \mathbf{V}$ *such that* $\mathbf{V} \cap \sum_1^{n+1} \mathbf{Z}e_j = \{0\}$ *and* $\mathbf{Z}v + \sum_1^{n+1} \mathbf{Z}e_j$ *is dense in* $\mathbf{R}^{n+1}$.

Proof. The lemma follows in a standard fashion from Theorem A.1. The proof will thus be omitted.

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