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THE CAMPBELL-HAUSDORFF GROUP AND A POLAR DECOMPOSITION OF GRADED ALGEBRA AUTOMORPHISMS

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Let $A = \prod_{k=k_0}^{\infty} \mathrm{gr}_k(A)$ be a complete graded (associative or Lie) algebra over a field of characteristic zero, filtered by the decreasing filtration $F_j(A) = \prod_{k=j}^{\infty} \mathrm{gr}_k(A)$. We let $\mathrm{Aut}(A)$ denote the group of filtration preserving automorphisms of A, and $\mathrm{Aut}_0(A)$ the subgroup consisting of those elements of $\mathrm{Aut}(A)$ which preserve the grading. In this paper we prove that every element of $\mathrm{Aut}(A)$ has a unique polar decomposition of the form $u_0 \exp(d)$, where $u_0 \in \mathrm{Aut}_0(A)$ and $d: A \to A$ is a filtration increasing derivation.

Our central results, presented in §4, generalize and were inspired by theorems on decompositions of diffeomorphisms and symplectic mappings found in the dynamical systems literature; they also touch on the related topic of one-parameter group extensions. Particularly influential were Broer's treatment of normal forms of vector fields [**4**], and Sternberg's work on the formal aspects of dynamical systems [**11**]. The setting adopted here is that of filtered groups and algebras, for the reason that certain functorial properties of these structures are particularly well suited for the treatment of convergence questions arising from the use of the Campbell-Hausdorff formula. Broer (loc. cit.) credits Gérard and Levelt [**6**] with the first use of filtration techniques in this field.

A second aspect of our work is the introduction of a restricted class of "analytic functions" which map the ground field into appropriate filtered objects. Such functions turn out to be rather peculiar, in that they are always "entire" and have (except when identically zero) only finitely many zeros. Our study is limited to those properties which are relevant to this paper.

Applications of Theorem 4.5 are presented in the last two sections; we offer short proofs of two of these decompositions. That implied by the upper exact sequence of Theorem 5.4 is classical: C. L. Bouton was working on related problems as early as 1916 [**2**] (also see Lewis [**8**] and Sternberg [**11**]). The decomposition implied by Theorem 6.2 is also well-known (see van der Meer [**14**, Lemma 2.11, p. 27]). The novelty of the presentation here is that the relationships between the group structures are formulated explicitly.

The splittings of §5 and §6 are only the first step toward "normal form" theory. How such forms can be achieved by transforming coordinates is well-known (for vector fields and diffeomorphisms see e.g. Takens [**12**] and [**13**], and for Hamiltonian systems see e.g. Moser [**9**]), and will not be addressed here, although we briefly touch on the subject in examples ending these sections. For detailed historical surveys of normal form theory see van der Meer [**14**, pp. 42–45], Brjuno [**3**, pp. 134–9 and 142–3], and Dixon and Esterle [**5**, pp. 152–3].

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1. The categories $\mathscr{G}$, $\mathscr{A}$ and $\mathscr{L}$. Here we collect basic definitions and establish notation. For more detail see [**1**] or [**10**].

Unless stated to the contrary K denotes a field of characteristic zero and an *algebra* simply means a vector space over K which admits a K-bilinear mapping $(x, y) \to xy$ into itself.

A *filtered group* (*vector space*, *algebra*) is a group (vector space, algebra) X together with a decreasing sequence $F_{j_0}(X) = X \supset F_{j_0+1}(X) \supset \cdots$ of normal subgroups (subspaces, ideals) such that

$$\left(F_p(X), F_q(X)\right) \subset F_{p+q}(X)$$

for all $p, q \geq j_0$, where $(x, y) = xyx^{-1}y^{-1}$, and, in the case of an algebra,

$$F_p(X)F_q(X) \subset F_{p+q}(X).$$

Throughout we assume $1 \geq j_0 \in Z$, and when confusion cannot result we write $F_j(X)$ as F_j. The filtration $\{F_j\}$ is *separated* if $\bigcap_j F_j = \{e\}$ (0), the identity (origin) of X.

The order function of a filtered group (vector space, algebra) X will be denoted by ν, i.e. $\nu(x) = \sup\{j : x \in F_j\}$, $x \in X$. We recall the following properties:

(a) $\nu(e) = +\infty$ ($\nu(0) = +\infty$ for vector spaces and algebras);
(b) $\nu(xy^{-1})$ ($\nu(x - y)$ for vector spaces and algebras) $\geq \inf\{\nu(x), \nu(y)\}$;
(c) $\nu(x) = \nu(x^{-1})$ ($\nu(-x)$ for vector spaces and algebras);
(d) $\nu((x, y))$ ($\nu(xy)$ for algebras) $\geq \nu(x) + \nu(y)$;
(e) $\nu(xax^{-1}) \geq \nu(x)$ (for groups); and
(f) $\nu(xy) = \nu(yx)$ (for groups).

In terms of the order function X is separated if and only if $\nu(x) < \infty$ for all $x \neq e$ (0). In this case we set $|x| = 2^{-\nu(x)}$, where $2^{-\infty}$ denotes 0, and we have

(i) $|x| \geq 0$;
(ii) $|x| = 0$ if and only if $x = e$ (0);
(iii) $|xy|$ ($|x + y|$ for vector spaces and algebras) $\leq \max\{|x|, |y|\}$;
(iv) $|\lambda x| = |x|$ for $0 \neq \lambda \in K$ (for vector spaces and algebras); and
(v) $|xy| \leq |x||y|$ (for algebras).

When X is separated the metric $(x, y) \to |xy^{-1}|$ ($|x - y|$) defines the topology on X for which the F_j constitute a basis of (open and closed) neighborhoods of the origin. All algebraic operations are continuous, assuming in the vector space and algebra cases that K is given the discrete topology or, equivalently, the trivial filtration $F_0(K) = K$, $F_1(K) = \{0\}$. X is *complete* if the metric is such.

A *morphism* φ: $X \to Y$ between filtered groups (vector spaces, algebras) is a homomorphism which preserves filtrations; in particular, all morphisms are continuous. Our interest will be in the following categories:

$\mathscr{G}$: The category of filtered, separated complete groups;
$\mathscr{A}$: The category of filtered, separated complete associative algebras with identity (always denoted 1), with all morphisms preserving 1; and
$\mathscr{L}$: The category of filtered, separated and complete Lie algebras.

If $A \in \mathscr{A}$ we denote by A_L the Lie algebra obtained from A by using the commutator $[x, y] = xy - yx$ as bracket. The filtration on A is compatible with this multiplication; hence $A_L \in \mathscr{L}$.

An infinite product $\prod x_j$ in an object of $\mathscr{G}$ converges if and only if $x_j \to e$. Likewise, an infinite series $\sum x_j$ in an object of $\mathscr{A}$ or $\mathscr{L}$ converges if and only if $x_j \to 0$. Moreover, in either case we have

(1.1) $$|x| \leq \max_j\{|x_j|\},$$

where x is the limit of the product or series in question. An important consequence is that if $A \in \mathscr{A}$, then any power series $\sum a_j x^j$ with coefficients in A converges for all x in the neighborhood F_1 of 0. In fact the series will also converge for all $x \in A$ satisfying $x^n \in F_1$ for some $n \geq 1$. Because of this property F_1 will play a particularly special role in our considerations. This is the reason we assume w.l.o.g. that $j_0 \leq 1$ for the first level F_{j_0} of a filtration; otherwise we could define F_j to be F_{j_0} for $j = 1, \ldots, j_0$.

To any graded group (vector space, algebra) $X = \prod_{j \geq j_0} X_j$ we associate the filtration $F_j(X) = \prod_{n \geq j} X_n$, and to any filtered group (vector space, algebra) we associate the graded group (vector space, algebra) $\mathrm{gr}(X) = \prod_j \mathrm{gr}_j(X)$, where $\mathrm{gr}_j(X) = F_j/F_{j+1}$. Note that, contrary to standard usage, $\mathrm{gr}(X)$ represents the product rather than the direct sum of the $\mathrm{gr}_j(X)$. For a filtered, separated and complete vector space X we have $X \simeq \mathrm{gr}(X)$, although the isomorphism is not natural.

2. The Campbell-Hausdorff group. For an arbitrary set S let $V = V_S$ be the vector space freely generated by S over K and let $\tilde{T}_S = K \oplus V \oplus (V \otimes V) \oplus \cdots$ denote the tensor algebra of S. $\tilde{T}_S$ is graded but not complete under the associated decreasing filtration; the completion T_S is the graded algebra of formal series $a_0 + a_1 + \cdots$, where $a_n \in V^{\otimes n}$.

2.1. PROPOSITION. *The mapping $S \to F_1(T_S)$ is universal in the sense that for any $A \in \mathscr{A}$ and any mapping $\alpha : S \to F_1(A)$ there is a unique morphism $T_S \to A$ in $\mathscr{A}$ extending α.*

The extension will also be denoted by α.

Proof. The universal property of $\tilde{T}_S$ implies that α extends uniquely to a homomorphism $\tilde{T}_S \to A$. Since we assume $\alpha(S) \subset F_1(A)$, induction shows this extension to be filtration preserving, hence uniformly continuous. Since $\tilde{T}_S$ is dense in T_S, the claim follows. □

Let L_S denote the closure in $(T_S)_L$ of the Lie subalgebra generated by S. Since L_S is the completion of the free Lie algebra of S ([7], p. 225) we have the following analogue in $\mathscr{L}$ of Proposition 2.1.

2.2. PROPOSITION. *The mapping $S \to F_1(L_S)$ is universal: for any $L \in \mathscr{L}$ and any mapping $\alpha : S \to F_1(L)$ there is a unique morphism $L_S \to L$ extending α.*

When $S = X = \{X_1, \ldots, X_n\}$ is a finite set, the tensor algebra $T_S = T_X$ is simply the algebra of formal non-commutative power series f in the variables $X_1, \ldots, X_n$. In view of Proposition 2.1 such an f defines a (universal) function $f_A : F_1(A)^n \to A$ through the substitution of variables $X \to x \in F_1(A)^n$. More explicitly: if $\alpha : T_X \to A$ is the morphism extending $\alpha : X_i \to x_i$, then $f_A(x) = \alpha(f)$.

2.3. PROPOSITION. *For $f \in F_p(T_X)$ the mapping f_A is continuous. Moreover, the collection $\{f_A\}_{A \in \mathscr{A}}$ defines a natural transformation between the functors $F_1(A)^n$ and $F_p(A)$ on $\mathscr{A}$, i.e. for any morphism $\varphi : A \to B$ the* *diagram*

$$\begin{array}{ccc} F_1(A)^n & \overset{\varphi\times\cdots\times\varphi}{\rightarrow} & F_1(B)^n \\ f_A \downarrow & & \downarrow f_B \\ F_p(A) & \overset{\varphi}{\rightarrow} & F_p(B) \end{array}$$

commutes. The analogous statements hold when T_X is replaced by L_X.

Proof. Let $x = (x_1, \dots, x_n) \in F_1(A)^n$ and $h = (h_1, \dots, h_n) \in F_j(A)^n$. Then for any finite sequence $i_1, \dots, i_k$ of the integers $1, \dots, n$ and any non-negative integers $q_1, \dots, q_k$ the difference $\prod_{m=1}^k (x_{i_m} + h_{i_m})^{q_m} - \prod_{m=1}^k x_{i_m}^{q_m}$ belongs to the closed subspace $F_j(A)$ of A, and the same holds for the difference $f_A(x + h) - f_A(x)$, which is a convergent series of such monomials. Continuity follows.

To prove naturality simply note that when $x \in F_1(A)^n$ the unique morphism $T_X \to B$ sending x_i to $\varphi(x_i)$ is $\varphi \circ \alpha$. Therefore

$$f_B(\varphi(x_1), \dots, \varphi(x_n)) = (\varphi \circ \alpha)(f) = \varphi(\alpha(f)) = \varphi(f_A(x)). \qquad \square$$

In view of this naturality we drop the subscript A from f_A.

As an example consider the Neumann series $f(X) = 1 + X + X^2 + \cdots \in K[[X]]$. This defines the function $x \to (1 - x)^{-1} = 1 + x + x^2 + \cdots$ on $F_1(A)$ for any $A \in \mathscr{A}$, and shows that all elements of the neighborhood $1 + F_1(A)$ of 1 are invertible.

The exponential and logarithmic series $\exp(X)$ and $\log(1 + X)$ will play an important role in the sequel. The corresponding functions establish homomorphisms between the neighborhoods $F_1(A)$ of the origin and $1 + F_1(A)$ of 1. Moreover, for any $\mathscr{A}$-morphism $\varphi: A \to B$ the diagram

$$\begin{array}{ccc} F_1(A) & \overset{\varphi}{\rightarrow} & F_1(B) \\ \exp\downarrow\uparrow\log & & \exp\downarrow\uparrow\log \\ 1 + F_1(A) & \overset{\varphi}{\rightarrow} & 1 + F_1(B) \end{array} \tag{2.4}$$

commutes.

2.5. REMARK. For $A \in \mathscr{A}$ define the *radical* of $F_1(A)$ as $\operatorname{rad}(F_1(A)) = \{a \in A : a^n \in F_1(A) \text{ for some } n \geq 1\}$; this is an ideal if A is commutative. The power series defining $(1 - x)^{-1}$, $\exp(x)$ and $\log(1 + x)$ converge for $x \in \operatorname{rad}(F_1(A))$, and we may regard these functions as being defined on these (possibly) larger domains.

Let $T_{X,Y}$ be the complete free algebra in two variables X, Y and consider the formal power series $W = W(X, Y) = \log(\exp(X)\exp(Y)) \in T_{X,Y}$. By the Campbell-Hausdorff formula ([7], p. 227) we have $W \in F_1(L_{X,Y})$, where

$$(2.6) \qquad W = X + Y + (1/2)[X, Y] + (1/12)[X, [X, Y]] + (1/12)[Y, [Y, X]] + \cdots .$$

By Proposition 2.3 the power series W defines a natural transformation between the functors $F_1(L) \times F_1(L)$ and $F_1(L)$ on $\mathscr{L}$ which we denote by $*$.

2.7. THEOREM. *$*$ is a group operation, and gives $F_1(L)$ the structure of a filtered, separated complete group with filtration $\{F_j(L)\}_{j \geq 1}$. Moreover, the assignment $L \in \mathscr{L} \to (F_1(L), *) \in \mathscr{G}$ is functorial, and is characterized by the property that the identity*

$$(2.8) \qquad x * y = \log(\exp(x)\exp(y))$$

holds for any $L \in \mathscr{L}$ of the form A_L for some $A \in \mathscr{A}$.

We call $(F_1(L), *)$ the *Campbell-Hausdorff group* of L.

Proof. First assume $L = A_L$ for some associative $A \in A$. If $x, y \in F_1(L) = F_1(A)$, then the $\mathscr{A}$-morphism $\alpha : T_{X,Y} \to A$ sending X to x and Y to y restricts to an $\mathscr{L}$-morphism sending W to $x * y$. Naturality of exp, log and $*$ then gives (2.8) for Lie algebras of this special form, as well as for Lie subalgebras thereof; the group properties follow immediately.

Now suppose $L \in \mathscr{L}$ is arbitrary and $x, y, z \in F_1(L)$. Then we can find an associative $A \in \mathscr{A}$, an $\mathscr{L}$-subalgebra $\tilde{L} \subset A_L$ and an $\mathscr{L}$-morphism $\varphi : \tilde{L} \to L$ with range including x, y and z, e.g. take $A = T_{X,Y,Z}$ and $\tilde{L} = L_{X,Y,Z}$. By the previous paragraph $(F_1(\tilde{L}), *)$ is a group, hence $(x * y) * z = x * (y * z)$ by naturality; the other group properties for $(F_1(L), *)$ are proven in a similar manner.

To verify the required commutator relation for $\{F_j(L)\}_{j \geq 1}$ assume $x \in F_p(L)$, $y \in F_q(L)$ and set $z = x * y * (-x) * (-y)$. From (2.6) we see that for $a \in F_r(L)$ and $b \in F_s(L)$ we have $a * b \equiv a + b + (1/2)[a, b] \pmod{F_{r+s+1}(L)}$, and so

$$\begin{aligned} z &\equiv (x + y + (1/2)[x, y]) * (-x - y + (1/2)[x, y]) \\ &\equiv [x, y] \quad \big(\mathrm{mod}\, F_{p+q+1}(L)\big). \end{aligned}$$

But $[x, y] \in F_{p+q}(L)$, hence $z \in F_{p+q}(L)$ as desired.

As for uniqueness, let $(F_1(L), \#)$ be a second functor with the same properties. By (2.8) $\#$ and $*$ must agree on Lie algebras of the form A_L, and by naturality on Lie subalgebras thereof. For arbitrary $L \in \mathscr{L}$ and $x, y \in L$ simply construct $\tilde{L}$ and $\varphi : \tilde{L} \to L$ as above; then $x\#y = x * y$ by naturality. □

REMARK. In ([10], Prop. 2.3) it is shown that if G is a filtered group then $L = \prod_j F_j/F_{j+1}$ is a graded Lie algebra over Z, where the Lie bracket on L is induced by the commutator on G. The above proof shows that $(\mathrm{gr}(F_1(L)), *) \simeq \mathrm{gr}(L)$ in positive degrees, so that if $L = \prod_j L_j$, $j \geq 1$, then $F_1(L) = L$ and $\mathrm{gr}(L, *) \simeq L$. In this sense the Campbell-Hausdorff group can be viewed as a partial inverse of the functor $\mathrm{gr} : \mathscr{G} \to \mathscr{L}_Z$, where $\mathscr{L}_Z$ is the category of filtered complete Lie algebras over Z. As we will not pursue this matter we leave it to the reader to formulate the precise statement describing the relationship between these functors.

3. Analytic functions. The initial results in this section do not require our standing hypothesis that the characteristic of K be zero. V will be a filtered, separated and complete K-vector space.

We will consider functions $f(t)$ on K defined by power series $\Sigma a_j t^j$, where $a_j \in V$, and we will call such functions *analytic*. Note, however, that if such a power series converges at a point $0 \neq t_0 \in K$ then it converges for all $t \in K$, Indeed, we have $|a_j t^j| = |a_j| = |a_j t_0^j| \to 0$. Thus "analytic" is equivalent to "entire" in our context. More general definitions can obviously be formulated, e.g. assuming domains in a filtered algebra A, but this will be sufficient for our purposes.

3.1. LEMMA. *Let $t_0, \ldots, t_n \in K$ be distinct and let $b_0, \ldots, b_n \in V$. Then there is a unique polynomial $p(t) = \Sigma_{j=0}^n a_j t^j$, $a_j \in V$, such that $p(t_i) = b_i$ for $i = 0, \ldots, n$. Moreover,*

$$|a_j| \leq \max\{|b_i|\}, \qquad j = 0, \ldots, n.$$

Proof. Simply view $p(t_i) = b_i$ as a set of n linear equations for $a_0, \ldots, a_n$; the system has a unique solution since the coefficient matrix is the invertible Vandermonde matrix (t_i^j). Since each a_j can be expressed as a linear combination of the b_i, the estimates follow from (iii) and (iv) of §1. □

3.2. *Elementary properties of analytic functions.* Assume $f(t) = \Sigma a_j t^j$ is analytic and let $p_n(t) = \Sigma_{j=0}^{n} a_j t^j$, $r_n(t) = \Sigma_{j=n+1}^{\infty} a_j t^j$.

(a) *$p_n \to f$ uniformly.* By (1.1) and (iv) of §1 we have $|r_n(t)| \leq \max_{j \geq n+1}\{|a_j|\}$, and $|a_j| \to 0$.

(b) *All a_j vanish if f has infinitely many distinct zeros. In particular, a non-trivial analytic function has only finitely many distinct zeros.* Suppose $t_0, \ldots, t_n$ are distinct zeros of f, and set $b_j = -r_n(t_j)$, $j = 0, \ldots, n$. Then $f(t_j) = 0$ may be written as $p_n(t_j) = b_j$, hence $|a_j| \leq \sup_t\{|r_n(t)|\}$ by Lemma 3.1, and the result then follows from (a).

(c) *When K is infinite the coefficients of f are uniquely determined.* This is immediate from (b).

In view of (c) we henceforth assume K is infinite.

The space of analytic functions is filtered by $F_j = \{f: K \to F_j(V)\}$. It is clear that the associated metric is defined by the supremum norm $|f| = \sup_t\{|f(t)|\}$.

(d) *We have* $|f| = \max_j\{|a_j|\} = \max_t\{|f(t)|\}$. This is immediate from (c) when $|f| = 0$, so assume $|f| > 0$ and note from (1.1) and (iv) of §1 that $|f| \leq \sup_j\{|a_j|\}$. To obtain the reverse inequality first observe using (a) that $|r_n| \to 0$, and so $|r_n| \leq |f|$ for $n \geq n_0$. But then (iii) of §1 implies $|p_n(t)| \leq \max\{|f_n(t)|, |r_n(t)|\} \leq |f|$, whereas Lemma 3.1 gives $|a_j| \leq \max_k\{|p_n(t_k)|\} \leq |f|$ provided $j \leq n$ and the t_k, $k = 0, \ldots, n$, are distinct points of K. Therefore $|f| \geq \sup_j\{|a_j|\}$. Sup can be replaced by max since 0 is the only accumulation point of the values of the metric on V and $|f| > 0$.

(e) *The space of analytic functions is complete.* If $\{f_n\}$ is Cauchy then for each j the coefficient sequence of t^j also has this property, hence converges to some element $a_j \in V$. One now checks that $f_n \to \Sigma a_j t^j$.

(f) *If $A \in \mathscr{A}$ the space of analytic $f: K \to A$ is an algebra.* If $f(t) = \Sigma a_j t^j$ and $g(t) = \Sigma b_n t^n$ then it is a straightforward verification that $f(t)g(t) = \Sigma c_n t^n$, where $c_n = \Sigma a_j b_{n-j}$.

Henceforth we assume K has characteristic zero. Since K is infinite the formal definition $f'(t) = \Sigma n a_n t^{n-1}$ of the derivative of an analytic function $f(t) = \Sigma a_n t^n$ is unambiguous by 3.2(c). One has $f^{(n)}(0) = n! a_n$, and as a result the Taylor formula

$$f(t) = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) f^{(n)}(0) t^n \tag{3.3}$$

holds.

If one wished to define f' analytically, rather than formally, a topology on K would be needed. However, since the $F_j(V)$ are linear subspaces, (iv) of §1 shows that *the only topology which makes scalar multiplication continuous is the discrete topology*, so that $h \to 0$ in K if and only if h is eventually identically 0. With this understanding it is easy to see that

$$(3.4) \qquad f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} := \left.\frac{f(t+h)-f(t)}{h}\right|_{h=0}.$$

This last expression is symbolic for the following recipe: factor h from $f(t+h) - f(t)$; divide by h; then set $h = 0$.

We also define the *primitive* of $f(t) = \Sigma a_n t^n$ formally, i.e. $\int_0^t f(s)\,ds = \Sigma(a_n/(n+1))t^{n+1}$; (iii) and (iv) of §1 show this to be an analytic function. If $Q_n(t) = \Sigma_{j=0}^{n-1}(a_j/(j+1))t^{j+1}$ is the nth-partial sum, then $|Q_n(t)| \le \max_{0\le j\le n-1}\{|a_j/j+1|\} = \max_{0\le j\le n-1}\{|a_j|\} \le |f|$, hence

$$(3.5) \qquad \left|\int_0^t f(s)\,ds\right| \le |f|.$$

3.6. THEOREM. *Suppose $G\colon V \to V$ is Lipschitz with Lipschitz constant contained in $[0,1)$, and suppose $G\circ\xi\colon K \to V$ is analytic whenever $\xi\colon K\to V$ is analytic. Then for each $v \in V$ the initial-value problem*

$$\dot\eta = G(\eta), \quad \eta(0) = v$$

has a unique solution.

Proof. The space H of analytic $\xi\colon K \to V$ is complete, and $T\colon H\to H$, defined by $T\xi(t) = v + \int_0^t G(\xi(s))\,ds$, is a uniform contraction by (3.5) and the Lipschitz assumption on G. The result thus follows from the contraction mapping principle. □

4. The polar decomposition of Aut(A). In this section we let A denote a fixed element of $\mathscr{A}$ or $\mathscr{L}$, and we let end(A) denote the space of endomorphisms of A *as a filtered vector space*. (We write "end(A)" rather than "End(A)" as a reminder that elements of end(A) are unrelated to the multiplicative structure of A.) If for $j \ge 0$ we let $F_j(\text{end}(A)) = \{u \in \text{end}(A) : u(F_n) \subset F_{n+j}\}$, then $\{F_j(\text{end}(A))\}$ is a separated filtration of end(A), and the resulting metric space is complete. Thus end(A) $\in \mathscr{A}$. If $A = \Pi_j A_j$ is graded so is end(A): set $\text{end}_j(A) = \{u \in \text{end}(A) : u(A_n) \subset A_{n+j}\}$. Notice that $u = \Sigma_{j=0}^\infty u_j$, where for $x = \Sigma x_n \in \Pi_n A_n$ we have

$$(4.1) \qquad u_j(x) = \sum_n (u(x_n))_{n+j}.$$

Any $u \in \text{end}(A)$ induces an endomorphism on F_j/F_{j+1}, and as a result an endomorphism $\Pi u \in \text{end}_0(\text{gr}(A))$. When A is graded and $u = u_0 + u_1 + \cdots$ we simply have $\Pi u = u_0$. Since $A \simeq \text{gr}(A)$ as vector spaces the homomorphism $\Pi : \text{end}(A) \to \text{end}_0(\text{gr}(A))$ must be surjective, and we obviously have $\text{Ker}(\Pi) = F_1(\text{end}(A))$. In other words, the sequence

$$0 \to F_1(\text{end}(A)) \to \text{end}(A) \overset{\Pi}{\to} \text{end}_0(\text{gr}(A)) \to 0 \tag{4.2}$$

is exact.

4.3. Lemma ([**1**], p. 178) *$u \in \text{end}(A)$ is invertible if and only if $\Pi u \in \text{end}_0(\text{gr}(A))$ has this property.*

Now let $\text{Aut}(A) \subset \text{end}(A)$ denote the group of filtered algebra automorphisms of A, and as a filtration on $\text{Aut}(A)$ let $F_0(\text{Aut}(A)) = \text{Aut}(A)$ and $F_j(\text{Aut}(A)) = (1 + F_j(\text{end}(A))) \cap \text{Aut}(A)$. To see that $\text{Aut}(A) \in \mathcal{G}$ first notice that for $1 + u \in F_p(\text{Aut}(A))$ we have $(1+u)^{-1}$ given by the Neumann series, and as a consequence whenever $1 + v \in F_q(\text{Aut}(A))$ we have $(1+u, 1+v) \equiv 1 + [u, v] \pmod{F_{p+q+1}(\text{Aut}(A))}$. This gives the required commutator property, and since the topology on $\text{Aut}(A)$ is that it inherits as a closed subspace of $\text{end}(A)$, $\text{Aut}(A) \in \mathcal{G}$ follows.

Let $\text{Der}(A)$ denote the collection of filtration preserving derivations of A. This is a closed Lie subalgebra of $(\text{end}(A))_L$, and therefore belongs to $\mathcal{L}$.

4.4. Proposition. *For any $a \in \text{Der}(A) \cap \text{rad}(F_1(\text{end}(A)))$ we have $\exp(a) \in \text{Aut}(A)$. In particular,* $\exp: F_1(\text{Der}(A)) \to \text{Aut}(A)$.

Proof. The argument is standard and uses the Leibniz rule $a^n(xy) = \sum_{j=0}^n \binom{n}{j} a^j(x) a^{n-j}(y)$. The condition that $a \in \text{rad}(F_1(\text{end}(A)))$ insures convergence in the following computation:

$$\begin{aligned}\exp(a)(xy) &= \sum_n \left(\frac{1}{n!}\right) \sum_{j=0}^n \binom{n}{j} a^j(x) a^{n-j}(y) \\ &= \sum_{j=0}^\infty \sum_{n=j}^\infty \left(\frac{1}{j!}\right) a^j(x) \left(\frac{1}{(n-j)!}\right) a^{n-j}(y) \\ &= \exp(a)(x)\exp(a)(y). \qquad \square\end{aligned}$$

Let $A \in \mathscr{A}$ be graded and let $\mathrm{Aut}_0(A)$ denote those automorphisms of A which preserve degree. We endow $\mathrm{Aut}_0(A)$ with the trivial filtration and thereby regard it as an object of $\mathscr{G}$.

4.5. THEOREM. *For graded A we have a split exact sequence*

$$0 \to F_1(\mathrm{Der}(A)) \overset{\exp}{\to} \mathrm{Aut}(A) \overset{\Pi}{\to} \mathrm{Aut}_0(A) \to e$$

of $\mathscr{G}$-morphisms, where $F_1(\mathrm{Der}(A))$ is the Campbell-Hausdorff group of $\mathrm{Der}(A)$. *In particular,* $\mathrm{Aut}(A)$ *is the semidirect product of $F_1(\mathrm{Der}(A))$ with* $\mathrm{Aut}_0(A)$.

Stated less formally, the conclusion is that any $u \in \mathrm{Aut}(A)$ has a unique "polar decomposition" $u = u_0 \exp(a)$ with $u_0 \in \mathrm{Aut}_0(A)$ and $a \in F_1(\mathrm{Der}(A))$. Obviously we can also write u as $\exp(b)u_0$, where $b = u_0 a u_0^{-1}$.

Proof. (2.4) and Proposition 4.4 imply $\exp : F_1(\mathrm{Der}(A)) \to \mathrm{Aut}(A)$ is an injection, and since $\mathrm{Aut}_0(A) \subset \mathrm{Aut}(A)$, $\Pi : \mathrm{Aut}(A) \to \mathrm{Aut}_0(A)$ must be a surjection. Moreover, the inclusion $\exp(F_1(\mathrm{Der}(A))) \subset \mathrm{Ker}(\Pi)$ is obvious, and the inclusion $\mathrm{Aut}_0(A) \subset \mathrm{Aut}(A)$ provides the section defining the splitting. It remains only to prove that $\mathrm{Ker}(\Pi) \subset \exp(F_1(\mathrm{Der}(A)))$.

To this end consider $g = 1 + u \in \mathrm{Ker}(\Pi) \subset \mathrm{Aut}(A)$, and notice that $g(xy) = g(x)g(y)$ implies $u(xy) = u(x)y + xu(y) + u(x)u(y)$. Now write $u = u_1 + \cdots$, apply this last equality to homogeneous elements $x \in A_p$, $y \in A_q$, and consider the $(1 + p + q)$th-component; it follows that u_1 must be a derivation. On the other hand, (2.4) implies that $g = \exp(a)$ for some $a = a_1 + a_2 + \cdots \in F_1(\mathrm{end}(A))$, and on comparing degrees we see that $a_1 = u_1$, hence $a_1 \in F_1(\mathrm{Der}(A))$. Now assume, by induction, that $a_i \in F_i(\mathrm{Der}(A))$ for $1 \le i \le j - 1$ and set $b = a_1 + \cdots + a_{j-1}$, so that $\exp(-b)$, and therefore $\exp((-b) * a) = (\exp(-b))g$ are in $\mathrm{Aut}(A)$. We observe that $(-b) * a \equiv (-b) * (b + a_j + \cdots) \equiv -b + b + a_j + (1/2)[-b, b + a_j + \cdots] \pmod{F_{j+1}}$, so that $(-b) * a = a_j + \cdots$; therefore $\exp((-b) * a) = 1 + v$, where $v = a_j + \cdots$. But the argument for $1 + u$ can now be applied to $1 + v$ to show that $a_j \in F_j(\mathrm{Der}(A))$, and since $\mathrm{Der}(A)$ is complete it follows that $a = \Sigma a_j \in \mathrm{Der}(A)$. □

We now turn to the closely related question of interpolation. An *(analytic) one-parameter group* in an algebra $A \in \mathscr{A}$ is an analytic homomorphism $\varphi : K \to A$; an element of A *extends to a one-parameter* *group* if it is the value at 1 of such a mapping. Any derivation $a \in \mathrm{rad}(F_1(A))$ defines a one-parameter group by means of the formula $\varphi(t) = \exp(ta)$.

"*Tournants Dangereux*:"

(1) The condition $a \in \mathrm{rad}(F_1(A))$ is necessary to insure convergence of the exponential series.

(2) If A is an algebra over C or R the classical one-parameter groups $t \to e^{kt}$ are *not* analytic in our sense.

4.6. THEOREM. *Let A be an algebra in $\mathscr{A}$ or $\mathscr{L}$, and let $u \in \mathrm{Aut}(A)$. Then u extends to an analytic one-parameter group if and only if $u = \exp(a)$ for some $a \in \mathrm{Der}(A) \cap \mathrm{rad}(F_1(\mathrm{end}(A)))$.*

Proof. Suppose $u = u(1)$, where $u(t)$ is an analytic one-parameter group. Then (3.4) gives

$$u'(t) = (u(t+s) - u(t))/s\,|_{s=0} = s^{-1}(u(s) - 1)u(t)\,|_{s=0} = u'(0)u(t),$$

with $u'(0) = a \in \mathrm{end}(A)$. Moreover, repeated differentiation now gives $u^{(n)}(0) = a^n$. Taylor's formula thus implies $u = \Sigma a^n/n!$, hence $a^n \to 0$, and therefore $a \in \mathrm{rad}(F_1(\mathrm{end}(A)))$. To see that $a \in \mathrm{Der}(A)$ simply note that

$$\begin{aligned} a(xy) &= u'(0)(xy) = (d/dt)(u(t)(xy))\,|_{t=0} \\ &= (d/dt)(u(t)xu(t)y)\,|_{t=0} = a(x)y + xa(y). \end{aligned}$$

The converse is obvious: if $u = \exp(a)$, let $u(t) = \exp(ta)$. □

REMARK. The condition that $a \in \mathrm{rad}(F_1(\mathrm{end}(A)))$ is equivalent to Πa being nilpotent.

5. Application: Diff(n). Let $A = A(n) = K[[x]] \in \mathscr{A}$ denote the graded K-algebra of formal power series in $x = (x^1, \ldots, x^n)$. Theorem 4.5 guarantees the existence of a split exact sequence

$$0 \to F_1(\mathrm{Der}(A)) \overset{\exp}{\to} \mathrm{Aut}(A) \overset{\Pi}{\to} \mathrm{Aut}_0(A) \to e \tag{5.1}$$

of $\mathscr{G}$-morphisms; the aim of this section is to identify this with a second sequence

$$0 \to F_1(\mathrm{Vect}(n)) \overset{\mathrm{Exp}}{\to} \mathrm{Diff}(n) \overset{\mathrm{Jac}}{\to} \mathrm{Gl}(K^n) \to e \tag{5.2}$$

which we now explain.

First, Vect(n) denotes the space of formal vector fields $X = \Sigma X^i \partial/\partial x^i$ in K^n with the usual Lie bracket and having the origin as an equilibrium point. $\mathrm{Vect}(n) \in \mathscr{L}$ when graded by $\mathrm{Vect}_j(n) = \{X : X^i$ is homogeneous of degree $j+1\}$, $j \geq 0$. In (5.2) $F_1(\mathrm{Vect}(n))$ denotes the Campbell-Hausdorff group.

Secondly, $\mathrm{Diff}(n)$ is the group of non-singular formal power series self-mappings of K^n fixing the origin; the group operation is formal composition (which requires the absence of constant terms to be defined) and the identity is the n-tuple of coordinate functions $x = (x^1, \ldots, x^n)$. We note that $\mathrm{Diff}(n) \subset M_0(n) := F_1(A)^n$, the semigroup (again under composition) of formal power series $\varphi(x) = ax + \cdots$. One has $\varphi \in \mathrm{Diff}(n)$ if and only if the Jacobian $a = \varphi'(0) = \mathrm{Jac}(\varphi)$ is non-singular.

To define $\mathrm{Exp}: F_1(\mathrm{Vect}(n)) \to \mathrm{Diff}(n)$ associate with $X = \sum X^i \partial/\partial x^i$ the element $G = G_X = (X^1, \ldots, X^n) \in M_0(n)$ and consider the initial-value problem

$$\dot{\eta} = G \circ \eta, \quad \eta(0) = x. \tag{5.3}$$

$\mathrm{Exp}(X)$ is the time-one map of the formal flow of X, i.e. $\mathrm{Exp}(X) = \eta(1)$, where $\eta(t)$ is the unique solution of (5.3).

The existence and uniqueness of $\eta(t)$ is a consequence of Theorem 3.6. Indeed, first observe that the self-map $\xi \to G \circ \xi$ of $M_0(n)$ is Lipschitz, with Lipschitz constant at most 1/2, provided $G(x+h) - G(x) \in F_{j+1}$ whenever $h \in F_j$. In analogy with the proof of Proposition 2.3 it suffices to establish this when G is a monomial of the form $x^\alpha v$, $v \in K^n$, but since $X \in F_1(\mathrm{Vect}(n))$ we have $|\alpha| \geq 2$, and so in this case the assertion is obvious. As for $G \circ \xi$ being analytic whenever $\xi : K \to M_0(n)$ is such, simply write $X^i = \sum a^i_\alpha x^\alpha$, $x = (x^1, \ldots, x^n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then $(X^i \circ \xi(t)) = \sum a^i_\alpha (\xi(t))^\alpha$, which is a convergent series in the complete algebra of analytic functions from K into $A(n)$ (cf. (3.2e,f)). Now observe that $\eta(t)$ is an analytic one-parameter group for the usual reason: for fixed $s \in K$ the functions $\eta(t+s)$ and $\eta(t) \circ \eta(s)$ satisfy the same initial-value problem. Thus $\eta(1) = \mathrm{Exp}(X) \in \mathrm{Diff}(n)$.

We remark that $\mathrm{Vect}(n)$ can be identified with $\mathrm{Der}(A)$ as elements of $\mathscr{L}$. Indeed, an element $X \in \mathrm{Vect}(n)$ acts as a derivation $D = D_X$ on $A = A(n)$ in the usual way, and since $X^i = D(x^i)$ the vector field is uniquely determined by this action. On the other hand, any $D \in \mathrm{Der}(A)$ preserves the filtration, so that $X^i = D(x^i) \in F_1(A)$ and $D = \sum X^i \partial/\partial x^i$.

5.4. Theorem. *There is an "anti-isomorphism" of split exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & F_1(\mathrm{Vect}(n)) & \overset{\mathrm{Exp}}{\to} & \mathrm{Diff}(n) & \overset{\mathrm{Jac}}{\to} & \mathrm{Gl}(K^n) & \to & e \\ & & \simeq\downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F_1(\mathrm{Der}(A)) & \overset{\exp}{\to} & \mathrm{Aut}(A) & \overset{\Pi}{\to} & \mathrm{Aut}_0(A) & \to & e, \end{array}$$

where the groups on the left are the Campbell-Hausdorff groups and the center and right vertical arrows denote the anti-isomorphism given by the formal pull-back mapping $\varphi \to \varphi^*$. *In particular*, $\mathrm{Diff}(n)$ *is the semidirect product of* $\mathrm{Gl}(K^n)$ *with* $F_1(\mathrm{Vect}(n))$.

REMARK. The definitions of the mappings in this diagram follow standard usage. However, this results in Exp being an anti-homomorphism.

The theorem is an immediate consequence of the following two propositions. Let $\mathrm{end}(A)$ act on A^n diagonally: if $u \in \mathrm{end}(A)$ and $f = (f^1, \dots, f^n) \in A^n$, write $u \cdot f$ for $(u(f^1), \dots, u(f^n))$.

5.5. PROPOSITION. *The pull-back mapping* $\varphi \to \varphi^*$ *establishes an anti-isomorphism between* $M_0(n)$ *and the semi-group of algebra endomorphisms of* A; *the mapping* $u \to u \cdot x$ *is a left inverse.*

Proof. Simply note that if $\varphi \in M_0(n)$ then $\varphi^* \cdot x = \varphi$, and that when $u \in \mathrm{end}(A)$ then $(u \cdot x)^*(x^i) = u(x^i)$, so that $(u \cdot x)^* = u$ whenever u is an algebra endomorphism. □

5.6. PROPOSITION. *The unique solution* $\eta(t)$ *of* (5.3) *is given by* $t \to \exp(tD) \cdot x$, *where* $D = D_X$.

Proof. We claim that $d^k\eta/dt^k = G_k \circ \eta$, where $G_k = D^k \cdot x$. Indeed, the chain-rule is valid in this context and gives

$$\begin{aligned} d^{k+1}\eta/dt^{k+1} &= d/dt(G_k \circ \eta) = \sum \eta^{i\prime}(t)\big((\partial G_k/\partial x^i) \circ \eta\big) \\ &= \sum (X^i \circ \eta)\big((\partial G_k/\partial x^i) \circ \eta\big) \\ &= \sum \big(X^i(\partial G_k/\partial x^i)\big) \circ \eta = D \cdot G_k \circ \eta = G_{k+1} \circ \eta. \end{aligned}$$

Now apply the Taylor formula, and observe that $G_k \circ \eta(0) = D^k \cdot x$. □

5.7. EXAMPLE If $\varphi, \rho \in \mathrm{Diff}(n)$ have respective polar decompositions $A \circ \mathrm{Exp}(X)$ and $B \circ \mathrm{Exp}(Y)$, then the group structure inherent in Theorem 5.4 immediately gives the polar decomposition of $\varphi \circ \rho$. Indeed, we have

$$\begin{aligned} \varphi \circ \rho &= A \circ \mathrm{Exp}(X) \circ B \circ \mathrm{Exp}(Y) = A \circ B \circ \mathrm{Exp}(B_* X) \circ \mathrm{Exp}(Y) \\ &= (A \circ B) \circ \mathrm{Exp}(Y * (B_* X)). \end{aligned}$$

In particular, if $\varphi = A \circ \mathrm{Exp}(X)$ is in normal form, i.e. if $A_* X = X$, then induction gives $\varphi^n = A^n \circ \mathrm{Exp}(nX)$.

6. Formal symplectic diffeomorphisms. Here we let $x = (q, p) \in K^n \times K^n$ represent a pair of canonical variables in $2n$-dimensional space. In this case the algebra $A = K[[x]]$ of §5 is also a complete graded Lie algebra under the formal Poisson bracket

$$\{f, g\} = \sum((\partial f/\partial q_k)(\partial g/\partial p_k) - (\partial f/\partial p_k)(\partial g/\partial q_k)),$$

and when so viewed it will be convenient to relabel A as $L = \prod_j L_j$, where $L_j = A_{j+2}$ is the space of homogeneous polynomials of degree $j + 2$. We will exploit this double algebra structure.

To begin notice that $\text{end}(L) = \text{end}(A)$, since "end" ignores the multiplicative structure. As a consequence $\text{Der}(A)$, $\text{Der}(L)$, $\text{Aut}(A)$ and $\text{Aut}(L)$ are all included in $\text{end}(A)$. Secondly, if $l \in L$ then $\text{ad}(l) \in \text{Der}(L)$ is also a *ring* derivation of A. But then $\text{ad}|_{F_j(L)}$ has values in $F_j(\text{Der}(A)) \cap F_j(\text{Der}(L))$, and as a result must be a filtered group homomorphism of the Campbell-Hausdorff group of L into $\text{Aut}(A) \cap \text{Aut}(L)$.

6.1. PROPOSITION. $\text{ad}: F_1(L) \to F_1(\text{Der}(A)) \cap F_1(\text{Der}(L))$ *is an isomorphism.*

Proof. For $\text{ad}(l) = 0$ we have $\{l, q_j\} = \{l, p_j\} = 0$, hence all partial derivatives of l must vanish, and therefore l must be constant. But $l \in F_1(L)$ implies $l(0) = 0$, so in fact $l = 0$. Now let $D \in F_1(\text{Der}(A)) \cap F_1(\text{Der}(L))$, and set $Q_i = D(q_i)$, $P_i = D(p_i)$; note that $Q_i, P_i \in F_2(A)$. Applying D to the canonical commutation relations $\{q_i, p_j\} = \delta_{ij}$, $\{q_i, q_j\} = \{p_i, p_j\} = 0$ and using $D \in \text{Der}(L)$ we then obtain

$$\{Q_i, p_j\} + \{q_i, p_j\} = \{Q_i, q_j\} + \{q_i, Q_j\} = \{P_i, p_j\} + \{p_i, P_j\} = 0,$$

hence $\partial P_j/\partial p_i = -\partial Q_i/\partial p_j$, $\partial Q_i/\partial p_j = \partial Q_j/\partial p_i$ and $\partial P_i/\partial q_j = \partial P_j/\partial q_i$. These equations hold for the homogeneous components of the Q_i, P_i, and as a result the one-form $\Sigma(P_i dq_i - Q_i dp_i)$ must be closed in the sense that all of the homogeneous components have this property. But then for each homogeneous component of degree $k \geq 2$ we can find a homogeneous polynomial of degree $k + 1$, denoted by l_{k-1} in accordance with our previous convention, such that for $l = l_1 + \cdots$ we have

$$dl = \sum (P_i dq_i - Q_i dp_i).$$

But then $D = \text{ad}(l)$ and $l \in F_1(L)$. □

In Theorem 5.4 we found that $\text{Diff}(2n) \simeq \text{Aut}(A)$. We now let $\text{Can}(n)$ be the group of formal canonical transformations of K^{2n} fixing the origin: $\text{Can}(n) = \{\varphi \in \text{Diff}(2n) : \varphi^* \text{ preserves Poisson brackets}\} = \{\varphi \in \text{Diff}(2n) : \varphi^* \in \text{Aut}(L)\}$. We readily see that $\varphi \to \varphi^*$ is an anti-isomorphism between $\text{Can}(n)$ and $\text{Aut}(A) \cap \text{Aut}(L)$.

6.2. THEOREM. *There is a split exact sequence of G-morphisms*

$$0 \to F_1(L) \to \mathrm{Can}(n) \overset{\mathrm{Jac}}{\to} \mathrm{Sp}(n) \to e, \tag{1}$$

where $F_1(L)$ *is the Campbell-Hausdorff group of* L *and the map* $F_1(L) \to \mathrm{Can}(n)$ *is given by* $l \to \exp(\mathrm{ad}(l)) \cdot x$.

Proof. Apply Theorem 4.5 to A twice: first as a ring and then as a Lie algebra. The result is an exact sequence $0 \to F_1(\mathrm{Der}(A)) \cap F_1(\mathrm{Der}(L)) \to \mathrm{Aut}(A) \cap \mathrm{Aut}(L) \to \mathrm{Gl}(K^n) \cap \mathrm{Aut}_0(L) \to e$, and the terms are immediately identified with those of (1). □

The meaning of Theorem 6.2 is that every formal canonical transformation u of K^{2n} preserving the origin can be uniquely decomposed as a product $u_0 \exp(\mathrm{ad}\, h)$, where $u_0 = \mathrm{Jac}(u)$ is a symplectic linear map and h is a formal hamiltonian vanishing to order three at 0.

As an application of this polar decomposition we offer a quick proof of a theorem which, at least when $n = 1$, is standard in the dynamical systems literature (compare [**9**, pp. 30–33]).

6.3. EXAMPLE. Let $u \in \mathrm{Can}(n)$ be in normal form, i.e. $uu_0 = u_0 u$, where $u_0 = \mathrm{Jac}(u)$ ([**9**, p. 31]). *Then u has an integral.* Indeed, if $u = u_0 \exp(\mathrm{ad}\, h)$ then $\exp(\mathrm{ad}\, h)$ commutes with u_0. But then $\exp(\mathrm{ad}\, h) = \exp(\mathrm{ad}\, h) u_0 u_0^{-1} = u_0 \exp(\mathrm{ad}\, h) u_0^{-1} = \exp(u_0\, \mathrm{ad}\, h u_0^{-1}) = \exp(\mathrm{ad}(u_0 h))$. Since $\exp \circ \mathrm{ad}$ is injective, $u_0 h = h$. But then $uh = h$ as h is certainly invariant under its own flow.

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