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EMBEDDING 2-COMPLEXES IN $\mathbf{R}^4$

MARKO KRANJC

Using Freedman's results it is not very hard to see that every finite acyclic 2-complex embeds in $\mathbf{R}^4$ tamely. In the present paper a relative version of this fact is proved. We also study when a finite acyclic 2-complex with one extra 2-cell attached along its boundary can be tamely embedded in $\mathbf{R}^4$.

Introduction. In 1955 A. Shapiro found a necessary and sufficient condition for the existence of embeddings of finite n-complexes in $\mathbf{R}^{2n}$ if $n > 2$ (see **[14]**) by defining an obstruction using the ideas of H. Whitney (**[15]**). The obstruction is not homotopy invariant and is in general quite hard to compute. It is well-known that any finite acyclic n-complex embeds in $\mathbf{R}^{2n}$ if $n \neq 2$ (see for example **[8]**). Not long ago it was proved in **[16]** that any finite n-complex K with $H^n(K)$ cyclic embeds in $\mathbf{R}^{2n}$ if $n > 2$.

It is known that any finite acyclic 2-complex can be embedded in $\mathbf{R}^4$ (see **[9]**, compare also with **[11]**). In the present paper the following relative version is proved.

THEOREM 1. *Let K be a finite 2-complex obtained from a 2-complex L by adjoining one 2-cell e along its boundary. If $H^2(K) = 0$ then any π_1-negligible tame embedding of L into $\mathbf{R}^4$ can be extended to a π_1-negligible tame embedding of K into $\mathbf{R}^4$.*

REMARK. This result is the best possible in the following sense: there exists a π_1-negligible embedding of a finite acyclic 2-complex into $\mathbf{R}^4$ which cannot be extended over an additional 2-cell (see §3).

In §2 the following is proved:

THEOREM 2. *Let L be a finite acyclic 2-complex. Suppose K is obtained from L by attaching one additional 2-cell e_0 along its boundary. If a regular neighborhood of some complex $\tilde{K}$ which carries the second homology of K can be embedded in some orientable 3-manifold then K can be tamely embedded in $\mathbf{R}^4$.*

Note. $\tilde{K} \subset K$ carries the second homology of K if the inclusion $\tilde{K} \subset K$ induces an isomorphism $H_2(\tilde{K}) \approx H_2(K)$. A regular neighborhood of $\tilde{K}$ is the union of all simplices in the second barycentric subdivision of K which intersect $\tilde{K}$ (compare with **[13]**, page 33).

The author believes that this theorem is true without the condition on $\tilde{K}$.

The above results give only tame embeddings because the proofs use the disc embedding theorem (see **[6]**). To our best knowledge it is not even known if every finite contractible 2-complex embeds in $\mathbf{R}^4$ smoothly (i.e.: by an embedding which is smooth on the interior of each cell).

1. Embedding acyclic 2-complexes in $\mathbf{R}^4$. In what follows all 2-complexes will be finite simplicial or cell complexes. Everything will be smooth or PL except when the results of **[5]** will be used. All immersions will be regular (i.e.: self-intersections will be transverse and there will be no triple points). Familiarity with the basic work of Freedman and Quinn (**[6]**) is assumed. We are going to use the disc embedding theorem in the following form:

THEOREM (*Disc Embedding Theorem*). *Let M be a simply-connected 4-manifold with boundary, and let $f\colon (D^2, \partial D^2) \to (M, \partial M)$ be a framed regular immersion which restricts to an embedding on ∂D^2. Suppose there exists a transverse sphere S for $f(D^2)$ such that the homological intersection number $S \cdot S$ is even. Then there is a topologically framed disc in M with the same framed boundary as $f(\partial D^2)$; furthermore, the resulting tame disc has a transverse sphere.*

Note. If F is a connected surface immersed in a 4-manifold then a transverse sphere for F is an immersed sphere which intersects F transversely in a single point.

A proof of the disc embedding theorem can be found in **[5]**. However, since our formulation is slightly stronger, a Casson tower has to be constructed more carefully to ensure the existence of the transverse sphere. This can be achieved by using recent techniques of 4-dimensional topology which are described for example in **[2]** and in **[6]** (see **[11]**).

LEMMA 1. *If $f\colon K \to \mathbf{R}^4$ is a regular immersion of a 2-complex K then $H^2(f(K))$ is isomorphic to $H^2(K)$.*

Proof. Since f is a regular immersion, the singular set of f is finite, say $\{y_1, \ldots, y_t\}$ and so is each set $f^{-1}(y_i)$. Clearly $f(K)$ is homeomorphic to $K/f^{-1}(y_1)/\cdots/f^{-1}(y_t)$. Let K_i be the set $K/f^{-1}(y_1)/\cdots/f^{-1}(y_i)$. Then $K_i = K_{i-1}/f^{-1}(y_i)$. From the exact sequence of the pair $(K_{i-1}, f^{-1}(y_i))$ we get the isomorphism $H^2(K_{i-1}) = H^2(K_i)$, since $H^s(f^{-1}(y_i))$ is trivial for $s > 0$. It follows that $H^2(K) = H^2(K_0) = H^2(K_t) = H^2(f(K))$.

LEMMA 2. *If K is a 2-complex and if e is a 2-cell of K then any embedding of $\overline{K-e}$ in $\mathbf{R}^4$ can be extended to an embedding of $\overline{(K-e)} \cup$ (a collar of ∂e in e).*

Proof. Let $f\colon \overline{K-e} \to \mathbf{R}^4$ be an embedding. We can extend f to a regular immersion $g\colon K \to \mathbf{R}^4$. $g(e)$ intersects $g(\overline{K-e})$ in finitely many points $x_1, \dots, x_s$. Let X be the set $(\bigcup_{i=1}^{s} g^{-1}(x_i)) \cap e$. Then X is again a finite set and $g|K-X$ is an embedding. Since X is contained in the interior of e, there is a collar A of ∂e in e which does not contain any point of A. Therefore $g|\overline{(K-e)} \cup A$ is an embedding.

LEMMA 3. *Let K be a 2-complex obtained from a 2-complex L by adjoining a single 2-cell e to L along its boundary. Suppose $H^2(K) = 0$. If A is a collar of ∂e in e then any π_1-negligible embedding $f\colon L \cup A \to \mathbf{R}^4$ can be extended to a π_1-negligible embedding $g\colon K \to \mathbf{R}^4$.*

Proof. Let $\alpha = f(\partial A - \partial e)$. Let N be a regular neighborhood of $f(L)$ in $\mathbf{R}^4$ containing $f(L \cup A)$ and such that $\alpha = \partial N \cap f(L \cup A)$. Since the embedding f is π_1-negligible, $\mathbf{R}^4 - N$ is simply-connected and therefore α bounds a regularly immersed disc D such that $N \cap \operatorname{int}(D) = \varnothing$.

Since $N \cup D$ retracts to $L \cup A \cup D$, and since $L \cup A \cup D$ is the image of K by a regular immersion, $H^2(N \cup D)$ is isomorphic to $H^2(K)$, by Lemma 1. Therefore, by Alexander duality, $H_1(\mathbf{R}^4 - (N \cup D))$ is trivial. Let $M = \mathbf{R}^4 - N$. Since $H_1(M-D) = 0$, there is an orientable surface F embedded in M such that it intersects D transversely in one point (a meridian μ of D bounds an embedded orientable surface in $M-D$, because $H_1(M-D) = 0$; if we glue to it the disc lying in the fiber of a tubular neighborhood of D, and having μ for its boundary, we get F). Choose a collection of simple closed curves a_i, b_i on F such that $a_i \cap a_j = \varnothing$, $b_i \cap b_j = \varnothing$, for all i, j, and such that $a_i \cap b_j = \varnothing$, for $i \neq j$, and a single point if $i = j$, and which generate $H_1(F)$. Since each of these curves bounds an immersed disc in M (M is simply-connected), we can perform a sequence of double surgeries to change F to an immersed sphere S. Move $D - F$ off of S by finger moves of D to get a new immersed disc D which has S for its transverse sphere (see [2], page 226). Since $S \subset \mathbf{R}^4$, the intersection number $S \cdot S$ is zero; therefore we can apply the disc embedding theorem to replace D by a tamely embedded disc which still has a transverse sphere. This defines a π_1-negligible extension of f in the obvious way.

Theorem 1 clearly follows from the above lemma. We also get the following two corollaries.

COROLLARY 1. *If K is a 2-complex such that $H^2(K) = 0$ then there exists a π_1-negligible embedding of K in $\mathbf{R}^4$.*

Proof. Let $e_1, \dots, e_r$ be the 2-cells of K, and let

$$K_i = K^{(1)} \cup e_1 \cup \cdots \cup e_i.$$

Since $H^3(K, K_i) = 0$, it follows from the cohomology sequence of the pair (K, K_i) that $H^2(K_i) = 0$, for every i.

Let $f_0 \colon K^{(1)} \to \mathbf{R}^4$ be some embedding. Clearly f_0 is π_1-negligible. It is enough to show that any π_i-negligible embedding $f_{i-1} \colon K_{i-1} \to \mathbf{R}^4$ can be extended to a π_1-negligible embedding $f_i \colon K_i \to \mathbf{R}^4$, if $i < r+1$. By Lemma 2 it is possible to extend f_{i-1} over a collar of ∂e_i in e_i. Then use Lemma 3 to get f_i.

COROLLARY 2. *Any acyclic 2-complex can be embedded in $\mathbf{R}^4$.*

REMARK 1. Any contractible 2-complex K can be embedded in $\mathbf{R}^4$ so that the embedding is π_1-negligible and so that the transverse spheres are embedded: Let N be an abstract 4-dimensional regular neighborhood of K. Let D_i be a disc transverse to the 2-cell e_i of K such that $\partial D_i \subset \partial N$. By [5] the double $D(N)$ is homeomorphic to S^4. The double $D(D_i)$ is an embedded transverse sphere to e_i.

REMARK 2. Corollary 2 has a simple proof which was told to the author by Robert Edwards: If K is an acyclic 2-complex let N be an abstract 4-dimensional regular neighborhood of K. ∂N is a homology 3-sphere, therefore it bounds a contractible 4-manifold Δ (see [5]). Glue Δ to N along ∂N. The resulting manifold is homeomorphic to S^4, K is contained in it. (Compare with [9].)

2. Proof of Theorem 2.

LEMMA 1. *Suppose V is an orientable 3-manifold such that $H_1(V)$ is free and $H_2(V) = 0$. If a simple closed curve $C \subset \partial V$ is null-homologous in ∂V then a basis for $H_1(V)$ can be represented by disjoint simple closed curves $\alpha_1, \dots, \alpha_k$ contained in $\partial V - C$.*

Proof. Suppose we constructed disjoint simple closed curves $\alpha_1, \ldots, \alpha_{j-1} \subset \partial V - C$, $j \leq k$. We are going to define α_j. Let W be the manifold obtained by attaching 2-handles to V along the curves $\alpha_1, \ldots, \alpha_{j-1}$ so that the attaching annuli miss C. Thus $C \subset \partial W$. Clearly $H_1(W)$ is free and $H_2(W)$ is trivial. Since C is null-homologous in ∂V, it is also null-homologous in ∂W. Therefore it separates ∂W into two components with closures F_1 and F_2 (i.e.: $F_1 \cup F_2 = \partial W$, $F_1 \cap F_2 = C$).

Since C bounds in W, $H_1(W, C)$ is isomorphic to $H_1(W)$. The Mayer-Vietoris sequence of the pair $\{(W, F_1), (W, F_2)\}$ gives us the isomorphism $H_1(W, C) = H_1(W, F_1) \oplus H_1(W, F_2)$, because $H_2(W, \partial W) \to H_1(W, C)$ is the zero homomorphism and since $H_1(W, \partial W) = H^2(W) = 0$. Because $H_1(W, C)$ is free (being isomorphic to $H_1(W)$), so are $H_1(W, F_1)$ and $H_1(W, F_2)$.

Let $i_s \colon H_1(F_s) \to H_1(W)$ be the homomorphism induced by the inclusion $F_s \subset W$. Since C is zero in $H_1(\partial W)$, $H_1(W)$ is isomorphic to $\operatorname{im}(i_1) + \operatorname{im}(i_2)$. Without loss of generality we can assume that $\operatorname{im}(i_1) \neq 0$ (because $H_1(W) \neq 0$).

Let x be a non-zero element of $\operatorname{im}(i_1)$. Suppose that $x = nu$ for some primitive element $u \in H_1(W)$. Since $H_1(W, F_1)$ has no torsion, it follows from the short exact sequence

$$0 \to \operatorname{im}(i_1) \to H_1(W) \to H_1(W, F_1) \to 0$$

that u has to lie in $\operatorname{im}(i_1)$, for example $u = i_1(v)$, for some $v \in H_1(F_1)$. Since v is primitive and not homologous to ∂F_1 in F_1, it can be represented by a simple closed curve α_j in F_1 which can easily be made to lie in ∂V (see **[11]**, page 13 or **[12]**).

LEMMA 2. *Let V be an orientable 3-manifold such that $H_1(V)$ is free and $H_2(V) = 0$. Suppose $C_1, \ldots, C_k$ are disjoint simple closed curves in V representing a basis for $H_1(V)$.*

If C_0 is a simple closed curve in ∂V which separates ∂V then it is possible to choose framings of $C_1, \ldots, C_k$ so that C_0 is slice in the homology 3-sphere Σ obtained from the double $D(V)$ by surgery along the framed curves $C_1, \ldots, C_k$. More precisely: Σ bounds a contractible 4-manifold Δ such that C_0 bounds an embedded disc D in Δ.

Proof. By Lemma 1 it is possible to represent a basis of $H_1(V)$ by disjoint simple closed curves $A_1, \ldots, A_k$ in $\partial V - C$. Let W be the 3-manifold obtained by attaching 2-handles to V along the curves $A_1, \ldots, A_k$. Since $\partial W = S^2$ (W is acyclic), C_0 bounds a disc $\tilde{D}$ in ∂W.

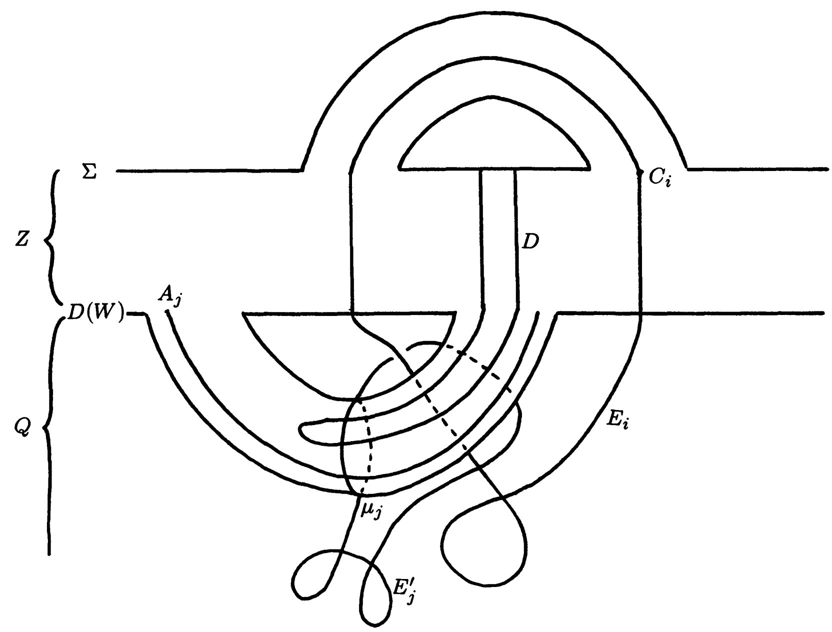


FIGURE 1

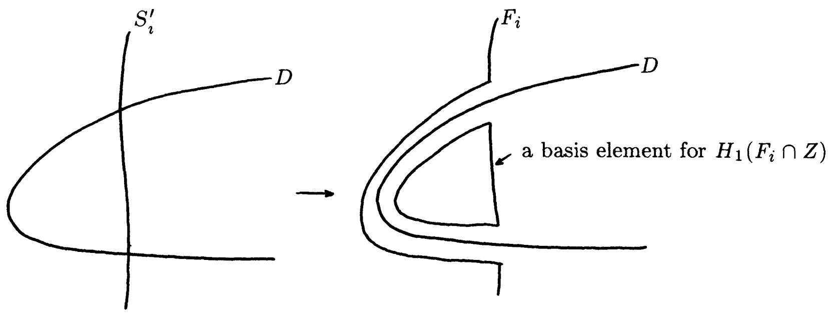
$D(W)$ is a homology 3-sphere. We can think of $D(W)$ as being gotten from $D(V)$ by a sequence of surgeries along the curves $A_1, \dots, A_k$.

Let Σ be a homology 3-sphere obtained from $D(V)$ by a sequence of surgeries along the framed curves $C_1, \dots, C_1$. The framings will be chosen later.

Since both Σ and $D(W)$ are obtained from $D(V)$ by surgery, there are cobordisms X and Y from $D(V)$ to Σ and to $D(W)$, respectively. We can construct X by attaching 2-handles to $D(V) \times I$ along $C_1, \dots, C_k \subset D(V) \times 1$ and Y by attaching 2-handles along $A_1, \dots, A_k$, respectively. Let $\mu_1, \dots, \mu_k$ be the meridians of $A_1, \dots, A_k$, respectively. If Y is turned "upside-down" it becomes a cobordism from $D(W)$ to $D(V)$. Y is constructed from $D(W) \times I$ by attaching 2-handles along $\mu_1, \dots, \mu_k \subset D(W) \times 1$. If X and Y are glued together along $D(V)$ we get a cobordism Z from $D(W)$ to Σ. To construct Z from $D(W) \times I$ we have to attach 2-handles to $D(W) \times I$ along the curves $C_1, \dots, C_k$, $\mu_1, \dots, \mu_k \subset D(W) \times 1$. $C_0 \subset D(V) \times 1 \subset \Sigma$ bounds a disc D in Z: D is the union of $C_0 \times I \subset D(V) \times I$ and $\tilde{D} \subset D(W) \times 0$.

Let Q be a contractible 4-manifold with boundary $D(W)$ (Q exists by Theorem 1.4′ of **[5]**). Let P be the 4-manifold obtained by gluing Z to Q along $D(W)$. The curves $C_1, \dots, C_k$, $\mu_1, \dots, \mu_k \subset D(W)$ bound immersed discs $E_1, \dots, E_k, E'_i, \dots, E'_k$, respectively, in Q (see Figure

1). These discs together with the cores of the 2-handles of Z form a collection of immersed spheres $S_1, \ldots, S_k$, $S'_i, \ldots, S'_k$ in P such that S_i corresponds to C_i and S'_i to μ_i, $i = 1, \ldots, k$. The spheres $S'_1, \ldots, S'_k$ intersect D with zero intersection numbers. All intersections arise from intersections of the meridians $\mu_1, \ldots, \mu_k$ with $\tilde{D}$. By a series of pipings along disjoint arcs in $\tilde{D}$ each S'_i can be changed to an immersed surface F_i disjoint from D, and such that F_i intersects F_j only in $\operatorname{int}(Q)$. $F_1, \ldots, F_k$ represent the same homology classes in $H_2(P)$ as $S'_1, \ldots, S'_k$. It is possible to represent half of symplectic generators of $H_1(\bigcup_{i=1}^k F_i)$ by simple closed curves lying in $D(W) = \partial Q$.



Since Q is contractible, each of these curves bounds an immersed disc in Q, missing D. Using these discs we can change each F_i into a new immersed sphere S'_i by performing a sequence of surgeries. Clearly the intersection numbers were not affected by going from the old S'_i's to the new ones. The spheres $S_1, \ldots, S_k$, $S'_1, \ldots, S'_k$ represent a basis for $H_2(P)$.

Choice of framings for $C_1, \ldots, C_k$: Choose them in such a way that $S_i \cdot S_i = 0$, for all $i = 1, \ldots, k$.

Finding the intersection numbers $S_i \cdot S'_j$: Suppose $C_i = \sum_{j=1}^k x_{ij} A_j$ in $H_1(V)$. Let G_i be an oriented surface in V such that $\partial G_i = C_i - \sum x_{ij} A_j$ in $H_1(V)$. In $D(W)$ each A_j bounds a disc D_j such that $D_j \cdot \mu_s = \delta_{js}$. Capping off the boundary components of G_i corresponding to the curves A_j, we get a surface $\hat{G}_i$ with boundary C_i. Obviously $\hat{G}_i \cdot \mu_j = x_{ij}$. Therefore $\operatorname{lk}(C_i, \mu_j) = x_{ij}$, and thus $S_i \cdot S'_j = x_{ij}$.

We are going to show next that $S'_i \cdot S'_j = 0$ for all $1 \leq i, j \leq k$. By Poincaré duality $H_1(D(V))$ is isomorphic to $H^2(D(V))$. Let $F_1, \ldots, F_k$ be closed surfaces dual to $A_1, \ldots, A_k$, respectively, i.e.: $F_i \cdot A_j = \delta_{ij}$. By a series of pipings on each F_j along the curves $A_1, \ldots, A_k$ we can achieve that $A_i \cap F_j = \varnothing$ for $i \neq j$, and $A_i \cap F_i = \{\text{point}\}$. Each F_i

defines a null-homology of μ_i in $D(V)-N(\bigcup_{i=1}^k A_i)$, where $N(\bigcup_{i=1}^k A_i)$ is a regular neighborhood of $\bigcup_{i=1}^k A_i$ in $D(V)$. Since μ_j can be made to miss F_i, the linking numbers $\mathrm{lk}(\mu_i, \mu_j)$ are all zero. Therefore $S_i' \cdot S_j' = 0$ for all $1 \leq 1, j \leq k$.

Let now Δ' be a regular neighborhood of $Q \cup S_1 \cup \cdots \cup S_k \cup S_1' \cup \cdots \cup S_k'$ in $P - D$. Since all the singularities of $\bigcup_{i=1}^k S_i \cup (\bigcup_{j=1}^k S_j')$ lie inside Q, Δ' is simply-connected.

Let (y_{ij}) be the inverse of the matrix (x_{ij}). Pipe together (in Δ') copies of the spheres S_i with suitable orientations to get for each i a 2-sphere $\tilde{S}_i$ realizing the element $\sum_{j=1}^k y_{ij} S_j$ of $H_2(\Delta')$. Then $\tilde{S}_i \cdot S_s' = \sum_{j=1}^k y_{ij} x_{js} = \delta_{is}$, and $\tilde{S}_i \cdot \tilde{S}_j = 2 \sum_{j<s} y_{ij} y_{is} S_j \cdot S_s$. Thus $\tilde{S}_i \cdot \tilde{S}_j$ is even for all $1 \leq i, j \leq k$. Let S_i'' be the immersed sphere representing the element $\tilde{S}_i - (1/2)(\tilde{S}_i \cdot \tilde{S}_i) S_i' - \sum_{s>i} (\tilde{S}_i \cdot \tilde{S}_s) S_s'$. Then $S_i'' \cdot S_j' = \tilde{S}_i \cdot S_j' = \delta_{ij}$, and also $S_i'' \cdot S_j'' = 0$, for every i, j. Therefore the conditions of Theorem 1.2 of [5] are satisfied and Δ' can be changed into a contractible manifold Δ'' by a series of surgeries which do not affect $\partial \Delta'$. By gluing Δ'' to $P - \Delta'$ along $\partial \Delta' = \partial \Delta''$ we get a contractible manifold Δ with boundary Σ. D is the desired slice.

Let L be a simplicial 2-complex, and let L'' be its second barycentric subdivision. If v is a vertex of L let $\tilde{f}_v$ be a regular immersion of the link $\mathrm{lk}(v)$ of v in L'' into S^2. Thus for every vertex $\overline{v}$ of $\mathrm{lk}(v) \cap L^{(1)}$, $\tilde{f}_v(\overline{v})$ has disc neighborhood $D_{\overline{v}}$ in S^2 such that $\tilde{f}_v | \tilde{f}_v^{-1}(D_{\overline{v}})$ is one-to-one. Since the star $\mathrm{st}(v)$ of v in L'' has a natural cone structure over $\mathrm{lk}(v)$, and since B^3 is also a cone over S^2, $\tilde{f}_v$ can be extended to a map $f_v \colon \mathrm{st}(v) \to B_v \approx B^3$ in a natural way.

For each edge s of L with vertices v_0 and v_1 attach a 1-handle h_s along $D_{\overline{v}_0} \cup D_{\overline{v}_1}$, where $\overline{v}_i = \mathrm{st}(v_i) \cap L^{(1)}$, to get (an orientable) handlebody H. The mapping $f' = \coprod_{V \in L^{(0)}} f_v \colon \coprod_{V \in L^{(0)}} B_v \to H$ can be extended over the 1-handles as follows:

If s is an edge of L with vertices v_0 and v_1, let Z_s be the star of its barycenter in L'', and let $X_i = Z_s \cap \mathrm{st}(v_i)$. There exists a homeomorphism $\varphi_s \colon X_0 \times I \to Z_s$ which is identity on $X_0 \times 0$ and which carries $X_0 \times 1$ onto X_1. Let $\psi_s \colon D^2 \times I \to h_s$ be a homeomorphism such that $\psi_s^{-1} f'(X_i)$ is a union of straight rays from the origin to the boundary of $D^2 \times i$. f' can be extended over Z_s. For example, if $\psi_s^{-1} f' \varphi_s(z, i) = (\chi_i(z), i)$, $z \in X_0$, define a map $f_s \colon Z_s \to h_s$ by

$$f_s \varphi_s(z, t) = \psi_s(\exp(i\alpha(z)t) \cdot \chi_0(z), t)$$

where $t \in I$, $z \in X_0$, and $\chi_1(z)/|\chi_1(z)| = \exp(i\alpha(z)) \cdot \chi_0(z)/|\chi_0(z)|$. f_s is an extension of f' over Z_s.

Any such family of maps $\{f_v\}_{v\in L^{(0)}}$ and $\{f_s\}_{s\in L^{(1)}-L^{(0)}}$ defines a mapping f of a regular neighborhood U of $L^{(1)}$ to a handlebody H such that $f|\mathrm{Fr}(U)\colon \mathrm{Fr}(U) \to \partial H$ is a regular immersion and such that $f|L^{(1)}$ is an embedding ($\mathrm{Fr}(V)$ denotes the frontier of U in K). Furthermore, by slight adjustments, $\mathrm{Fr}(U)$ can be made a union of smooth circles, and $f|\mathrm{Fr}(U)$ a smooth regular immersion. f can also be made smooth on $U - L^{(1)}$ and on the interior of each edge of L.

Both U and H have a natural mapping cylinder structure over $L^{(1)}$ (i.e.: U and H are homeomorphic to mapping cylinders of natural projections $\mathrm{Fr}(U) \to L^{(1)}$ and $\partial H \to L^{(1)} = f(L^{(1)})$, respectively). These structures can be made compatible with f in the following sense. If $p\colon \mathrm{Fr}(U)\times I \to U$ and $q\colon \partial H \times I \to H$ are the projections induced by the two mapping cylinder structures such that $p(\mathrm{Fr}(U)\times 0) = q(\partial H \times 0) = L^{(1)}$ then $q(f(x),t) = f(p(x,t))$, for $x \in \mathrm{Fr}(U)$.

Let $e_1,\dots,e_g$ be the 2-cells of L. Denote by α_i the intersection $e_i \cap \mathrm{Fr}(U)$. Thus L is obtained from U by attaching discs along $\bigcup_{i=1}^g \alpha_i$ via homeomorphisms.

The immersion $f|\mathrm{Fr}(U)\colon \mathrm{Fr}(U) \to H$ can be changed to an embedding $\tilde{F}\colon \mathrm{Fr}(U) \to H$, by pushing parts of $f(\mathrm{Fr}(U))$ slightly inside H near the intersections. $\tilde{F}$ in turn defines an embedding $F\colon U \to H\times I$ as follows:

$$F(p(x,t)) = (q(\tilde{F}(x),t), (t+1)/2), \quad t\in I, x\in \mathrm{Fr}(U).$$

Clearly $F(\mathrm{Fr}(U)) \subset H\times 1$, and $F(\mathrm{int}(U)) \subset \mathrm{int}(H)\times [1/2,1) \subset \mathrm{int}(H\times I)$. Denote by C_i the curve $F(\alpha_i) \subset H\times 1$. If we choose a framing for each C_i, and attach 2-handles along $C_1,\dots,C_g$ we get a 4-manifold N. F can be extended to an embedding $\hat{F}\colon L \to N$ by mapping $e_i \cap (\overline{L-U})$ onto the core of the corresponding 2-handle.

∂N is obtained from $\partial(H\times I) = D(H)$ (=the double of H) by a sequence of surgeries along the framed curves $C_1,\dots,C_g$.

Proof of Theorem 2. Let e_0, $e_{k+1},\dots,e_g$ be the 2-cells of $\tilde{K}$, and let $e_1,\dots,e_k$ be the remaining 2-cells of K. Let $\tilde{U}$ be a regular neighborhood of $\tilde{K}$ in K. Suppose $\tilde{U}$ is contained in an orientable 3-manifold M. Let $\tilde{H}$ be a regular neighborhood of $\tilde{K}^{(1)}$ in M such that $\tilde{H}\cap\tilde{U}$ is a regular neighborhood of $\tilde{K}^{(1)}$ in K. The inclusion $\tilde{H}\cap\tilde{U} \subset \tilde{H}$ defines mappings f_v, $v\in\tilde{K}^{(0)}$ and f_s, $s\in \tilde{K}^{(1)} - \tilde{K}^{(0)}$. For the rest of the vertices and edges of K define maps f_v and f_s in any way as described above.

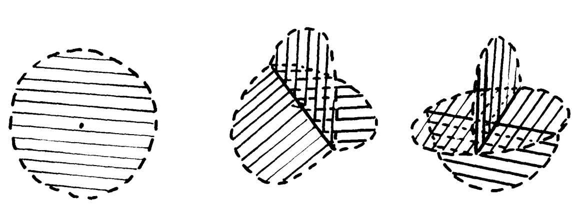
As above, these maps define a mapping $f\colon U \to H$ of a regular neighborhood U of $K^{(1)}$ in K into a handlebody H. f restricts to an embedding on $\alpha_0 \cup (\bigcup_{i=k+1}^{g} \alpha_i)$ such that

$$f\left(\alpha_0 \cup \left(\bigcup_{i=k+1}^{g} \alpha_i\right)\right) \cap f\left(\bigcup_{i=1}^{k} \alpha_i\right) = \varnothing$$

(α_i are as above). As above f induces an embedding $F\colon U \to H \times I$. Clearly $C_i = f(\alpha_i)$, for $i = 0, k+1, \dots, g$. Since L is acyclic, and since $\tilde{K}$ carries $H^2(K)$, $C_0 = \sum_{i=k+1}^{g} \alpha_i C_i$ in $H_1(H)$. We want to show now that $C_0 = \sum_{i=k+1}^{g} \alpha_i C_i$ also in $H_1(\partial H)$. Suppose $B_1, \dots, B_g$ is a basis of $\ker(i)$ (where $i\colon H_1(\partial H) \to H_1(H)$ is induced by the inclusion $\partial H \subset H$) dual to $C_1, \dots, C_g$, i.e.: $C_i \cdot B_j = \delta_{ij}$ in $H_1(\partial H)$. If $C_0 = \sum_{i=k+1}^{g} \alpha_i C_i + \sum_{i=1}^{g} \beta_i B_i$ in $H_1(\partial H)$ then $C_0 \cdot C_j = -\beta_j = 0$ which proves the claim. Attach 2-handles to framed curves $C_1, \dots, C_g$ in $H \times 1$ to get a 4-manifold N and an extension of the embedding F to an embedding $\tilde{F}\colon L \cup U \to N$, $\tilde{F}(\alpha_0) = C_0 \subset \partial N$. ∂N is a homology 3-sphere Σ. It is obtained from $D(H)$ by surgeries along $C_1, \dots, C_g$. Let V be the 3-manifold gotten from H by attaching 2-handles along the simple closed curves $C_{k+1}, \dots, C_g \subset \partial H$. Since $C_0 = \sum_{i=k+1}^{g} \alpha_i C_i$ in $H_1(\partial H)$, C_0 separates ∂V. Clearly $H_1(V)$ is free and $H_2(V) = 0$. Let $W = D(V)$. W can be obtained from $D(H)$ by surgeries along $C_{k+1}, \dots, C_g$. Therefore Σ can be obtained from W by surgeries along $C_1, \dots, C_k$. By Lemma 2, the framings of $C_1, \dots, C_k$ can be chosen so that C_0 is slice in Σ.

Let Δ be a contractible 4-manifold with boundary Σ, such that C_0 bounds an embedded disc D in Δ. If we glue N to Δ along Σ we get a homotopy 4-sphere which is therefore an S^4 (see [5]). The embedding F can be extended to an embedding of K by sending $\overline{e_0 - U}$ onto D.

Remark 1. If K is a generic 2-complex, i.e.: if it is locally homeomorphic to one of the following spaces

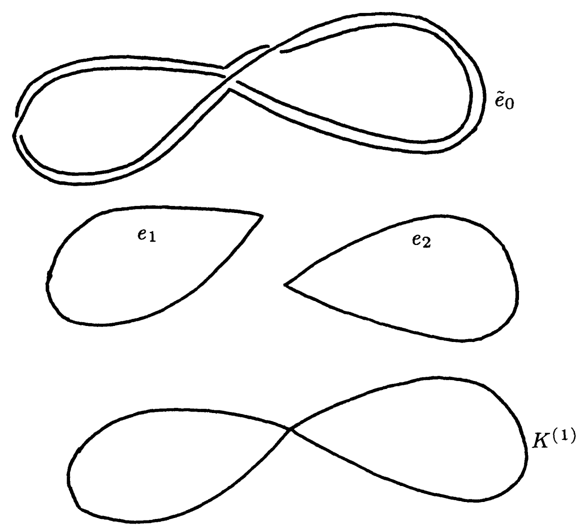


then it is possible to determine whether it can be embedded in some 3-manifold as follows: It is easy to embed a closed regular neighborhood U of the intrinsic 1-skeleton G of K (i.e.: the set of non-manifold points of K—compare with [7]) in a (possibly nonorientable) handlebody $\overline{H}$ so that $\mathrm{Fr}(U) \subset \partial\overline{H}$, and so that G is a spine of $\overline{H}$. K is obtained from U by attaching connected surfaces $F_1, \dots, F_t$ to $\mathrm{Fr}(U)$ along $\partial F_1 \cup \cdots \cup \partial F_t = U \cap (\overline{K - U})$. Let $w_1 \in H^1(\overline{H})$ be the orientation class: $w_1(C)$ is equal to 1 if C passes through nonorientable 1-handles of $\overline{H}$ an odd number of times, otherwise it is 0. K can be embedded in some 3-manifold if and only if $w_1(\partial F_1) = \cdots = w_1(\partial F_t) = 0$.

Remark 2. It is known that any finite 2-complex K such that its intrinsic 1-skeleton embeds in $\mathbf{R}^2$ can be embedded in $\mathbf{R}^4$. A discussion in this direction can be found in [7].

3. An example. In this section we give an example of a 2-complex K obtained from an acyclic 2-complex L by adjoining one 2-cell e_0, and a π_1-negligible embedding $f\colon L \to \mathbf{R}^4$ which cannot be extended to an embedding of K.

Let K be the complex obtained from a wedge of two circles by attaching three 2-cells $\tilde{e}_0$, e_1, and e_2 via immersions as follows:



Let U be a regular neighborhood of $K^{(1)}$ in K, and let $L = U \cup e_1 \cup e_2$. If $\alpha_0 = \mathrm{Fr}(U) \cap \tilde{e}_0$ then K is obtained from L by attaching a 2-cell e_0 along its boundary to α_0. Define an embedding of $\mathrm{Fr}(U)$ in a handlebody H with spine $S^1 \vee S^1$ $(= K^{(1)})$ as follows:

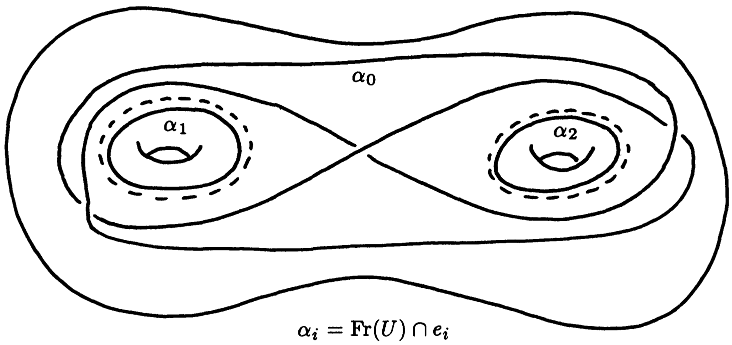


FIGURE 2

As in §2 this defines an embedding $\tilde{f}\colon U \to H \times I$. Attach 2-handles to $\tilde{f}(\alpha_1)$ and $\tilde{f}(\alpha_2)$ with framings indicated in Figure 2 by the dotted circles to get B^4. The cores of the two 2-handles can be used to extend $\tilde{f}$ to a π_1-negligible embedding $f\colon L \to B^4 \subset \mathbf{R}^4$. $f(\alpha_0)$ is the trefoil knot in the boundary of B^4. Therefore it is not slice and thus f cannot be extended to an embedding of K.

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WESTERN ILLINOIS UNIVERSITY
MACOMB, IL 61455