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DIFFERENTIAL IDENTITIES, LIE IDEALS, AND POSNER'S THEOREMS

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Two well-known results of E. C. Posner state that the composition of two nonzero derivations of a prime ring cannot be a nonzero derivation, and that in a prime ring, if the commutator of each element and its image under a nonzero derivation is central, then the ring is commutative. Our purpose is to show how the theory of differential identities can be used to obtain these results and their generalizations to Lie ideals and to rings with involution.

A number of authors have generalized these theorems of Posner in several ways. To be more specific, let R be a prime ring with center Z, and let d and h be derivations of R. The specific statements of Posner's theorems, to which we shall refer frequently, are the following:

POSNER'S FIRST THEOREM [**25**; Theorem 1, p. 1094]. *If* $\operatorname{char} R \neq 2$ *and if the composition* dh *is a derivation of* R, *then either* $d = 0$ *or* $h = 0$.

POSNER'S SECOND THEOREM [**25**; Theorem 2, p. 1097]. *If* $xx^d - x^d x \in Z$ *for all* $x \in R$, *then either* $d = 0$ *or* R *is commutative.*

The proof of the first theorem is fairly easy and extends to ideals of R. For this theorem, the case when $\operatorname{char} R = 2$ was obtained in [**6**] and later in [**13**], which also gives some generalizations to the case when $\operatorname{char} R \neq 2$ and R is a semi-prime ring. No attempt seems to have been made to extend Posner's first theorem to a Lie ideal L of R, assuming that dh is a Lie derivation on L. Several authors (see [**5**], [**7**], [**8**], [**16**], and [**22**]) have shown that $d = 0$ or $h = 0$ when $L^{dh} = 0$ or $L^{dh} \subset Z$. The second theorem of Posner was much more difficult to prove than the first, although an easier proof has been found [**3**]. When $\operatorname{char} R = 2$, this result is easy to prove. One such proof appears in [**1**] and, although not stated, it holds for Lie ideals of R. Partial generalizations of Posner's second theorem to ideals [**10**] and to Lie ideals when $\operatorname{char} R \neq 2$ [**4**] have also been obtained. More recently, a full generalization to Lie ideals when $\operatorname{char} R \neq 2$ has been proved

([22] and [5]), and in [14] there is an extension to d-invariant ideals in d-semi-prime rings.

In the references cited above, the arguments are generally ad hoc computations, often lengthy and clever. Our purpose is to obtain and extend these results in a systematic way by using the theory of differential identities as developed by V. Kharchenko [17] and extended in [18]. We are able to prove Posner's second theorem fairly easily for Lie ideals in any characteristic. His first theorem is harder for us to prove, but our result gives the full generalization to the case when dh is a Lie derivation acting on a Lie ideal L of R in any characteristic. In addition, we obtain results corresponding to Posner's theorems for the (skew) symmetric elements in rings with involution. The statements of our main results are:

THEOREM. *If L is a noncommutative Lie ideal of R and d is a nonzero derivation of R so that $xx^d = x^d x \in Z$ for all $x \in L$, then either R is commutative, or* $\operatorname{char} R = 2$ *and R satisfies S_4;*

THEOREM. *If L is a noncommutative Lie ideal of R and d and h are nonzero derivations of R so that dh is a Lie derivation of L into R, then* $\operatorname{char} R = 2$ *and either R satisfies S_4 or $h = dc$ for c in the extended centroid of R;*

THEOREM. *If R has an involution, $*$, $J = J^*$ is a nonzero ideal of R, and d and h are nonzero derivations of R so that dh is a Lie derivation from the skew-symmetric elements of J to R, then R satisfies S_4, or d and h are inner and R must satisfy a nonzero generalized polynomial identity, unless* $\operatorname{char} R = 2$ *and $h = dc$ for c in the extended centroid of R.*

Differential identities and preliminary results. Our method of approaching these problems is to use results on differential identities to show that the derivations involved are inner, and then to conclude that R satisfies a generalized polynomial identity. This means that R embeds nicely in a primitive ring with nonzero socle [23] and by extending the base field we argue that one can assume that R is a matrix ring over a field. At this point, matrix computations yield the desired result. Unfortunately, even to state the result on differential identities which we need requires a considerable amount of terminology. We begin with a review of some important facts about the Martindale quotient ring and then discuss the notion of differential identity.

Throughout the paper, R will denote a prime ring with center Z, extended centroid C, and Martindale quotient ring Q (see **[23]** for details). One can view Q as equivalence classes of left R-module homomorphisms from ideals of R to R, so R embeds in Q as right multiplication on R. The center of Q is C, which is a field, and C is also the centralizer of R in Q. One can characterize C as those elements of Q which are R-bimodule mappings. For any $q \in Q$ there is a nonzero ideal I of R with $Iq \subset R$, and $q = 0$ if $Jq = 0$ for any nonzero ideal J of R. Using this, one can show easily that any subring of Q which is also an R-bimodule is a prime ring whose extended centroid is again C. One subring of Q of particular importance is $RC + C$, the central closure of R. Another subring arising in the theory of differential identities is $N(R) = N = \{q \in Q | Iq + qI \subset R$ for some nonzero ideal I of $R\}$ **[17]**. It is easy to see that $RC + C \subset N$. Let $\mathrm{Der}(R)$ denote the Lie ring of derivations of R. Any $d \in \mathrm{Der}(R)$ has a unique extension to Q, and this extension restricts to N (**[17]** or **[18]**). Thus, we may consider $\mathrm{Der}(R) \subset \mathrm{Der}(R)C \subset \mathrm{Der}(RC) \cap \mathrm{Der}(N)$, where dc for $d \in \mathrm{Der}(R)$ and $c \in C$ is given by $x^{dc} = x^d c$ for any $x \in N$. Now if $d \in \mathrm{Der}(R)$ extends to an inner derivation of Q, say $d = ad(y)$ for $q^d = qy - yq$, then $y \in N$ (**[17]** or **[18]**). The right C-subspace of $\mathrm{Der}(R)C$ consisting of those elements whose extensions to Q are inner is denoted by $\mathrm{Inn}(R)$. Finally, if R has an involution, *, then one can extend * to N by taking $q \in N$, J a nonzero ideal of R satisfying $Jq + qJ \subset R$, and defining q^* on J^* by $(j^*)q^* = (qj)^*$ (see **[24**; Theorem 4.1, p. 511**]**). In particular, $RC + C \subset N$ has an involution restricting to * on R, so we may assume that any involution of R is also defined on C.

Next, we review the notion of differential identity for R a prime ring with involution, *. Our discussion is a special case of the development in **[17]** and **[18]**. Let X be a set of indeterminates over C of the form $\{x_i\} \cup \{x_i^d\}$, where i ranges over the positive integers and d ranges over $\mathrm{Der}(R)C$. We shall say that x_i or x_i^d has subscript i, that x_i^d has exponent d, and that x_i has no exponent. Let $F(N, X, Y)$ denote the free product over C of N and $C\{X, Y\}$, where Y is another set of indeterminates $\{y_i\} \cup \{y_i^d\}$. One C-basis for $F(N, X, Y)$ is the set of all monomials $a_0 z_1 a_1 \cdots z_n a_n$, where the coefficients, $\{a_i\}$, belong to some C-basis of N, and $\{z_i\} \subset X \cup Y$. Any $f \in F(N, X, Y)$ involves only finitely many indeterminates, so for a suitable integer n, f defines a function from R^n to N. Specifically, for $(r_1, \ldots, r_n) \in R^n$ one substitutes, r_i for x_i, r_i^* for y_i, $(r_i)^d$ for x_i^d, and $(r_i^*)^d$ for y_i^d. If J

is a nonempty subset of R so that $f(J^n)$, the image of J^n under f, is zero, then f is called a *generalized *-differential identity* (G^*-DI) for J. A G^*-DI f for J which has all its indeterminates appearing without exponent, that is in $\{x_i\} \cup \{y_i\}$ is called a *generalized *-polynomial* identity (G*-PI) for J. When one ignores the fact that R has an involution, or does not assume an involution, the terminology above has its obvious parallels. Specifically, $f \in F(N, X)$ is a GDI (generalized differential identity) for J if $F(J^n) = 0$, and is a GPI for J if all indeterminates appearing in f are in $\{x_i\}$. In general, we regard $F(N, X) \subset F(N, X, Y)$ and consider any result for a G*-DI to hold for a GDI, with the obvious changes needed.

We note that our use of "GPI" is somewhat nonstandard because the coefficients of a GPI f are in N rather than in $RC + C$. This is a potential problem because we need to use Martindale's theorem [**23**; Theorem 3, p. 579] which asserts that if R satisfies a nonzero GPI with coefficients in $RC+C$, RC is a primitive ring with nonzero socle, and for a primitive idempotent $e \in RC$, the division ring $eRCe$ is finite dimensional over its center eC. This problem is resolved by [**19**; Theorem 2, p. 18] which shows that if $f \in F(N, X)$ is a nonzero GPI for an ideal J of R, then R satisfies a nonzero GPI with coefficients in R (also see [**18**; Proposition, p. 769]).

The statement of the main result from [**18**] requires still more terminology. To say that $f \in F(N, X, Y)$ is *multilinear* means that f is multilinear and homogeneous in its *subscripts*; that is, no subscript appears twice in any single basis monomial appearing in f and all basis monomials in f have the same set of subscripts. Assume for simplicity that f is multilinear with subscript set $\{1, \dots, n\}$ and let $W \subset \mathrm{Der}(R)C$ be the set of all exponents appearing in f. Of course W is empty exactly when all variables in f are in $\{x_i\} \cup \{y_i\}$. To each monomial m in f we associate its exponent sequence $(h_1, \dots, h_n)$, where $h_i \in W$ if it is the exponent of the variable in m with subscript i, and $h_i = 1$ if the variable with subscript i is x_i or y_i. For example, $m = x_4^d y_2^d a x_3 b x_1^h$ has exponent sequence $(h, d, 1, d)$. For any such sequence $(h_1, \dots, h_n)$ we let $f_{(h_1, \dots, h_n)}$ be the sum of all monomials of f having this same exponent sequence, but with all exponents deleted. Thus, if

$$f = x_1^d y_2 x_3^d + x_3 y_1 x_2^d + y_3^d y_2 y_1^d + x_2^d x_3^d x_1,$$

then

$$f_{(d,1,d)} = x_1 y_2 x_3 + y_3 y_2 y_1, \quad f_{(1,d,1)} = x_3 y_1 x_2, \quad \text{and}$$
$$f_{(1,d,d)} = x_2 x_3 x_1.$$

Finally, the set $W = \{d_1, \dots, d_k\} \subset \text{Der}(R)C$ is called *independent modulo* $\text{Inn}(R)$ if $d_1c_1 + \cdots + d_kc_k \in \text{Inn}(R)$, for $c_i \in C$, implies that all $c_i = 0$. With all of these preliminaries we can now state the special case of [**18**; Theorem 7, p. 783] which we require.

THEOREM A. *Let R be a prime ring with involution, J a nonzero ideal of R, and $f \in F(N, X, Y)$ a G^*-DI for J which is multilinear with subscript set $\{1, \dots, n\}$ and exponent set $W \subset \text{Der}(R)C$ independent modulo* $\text{Inn}(R)$. *If $(h_1, \dots, h_n)$ is the exponent sequence for any basis monomial in f, and contains a maximal number of derivations among all such sequences, then $f_{(h_1,\dots,h_n)}$ is a G^*-PI for R, and R satisfies a nonzero GPI, if $f \neq 0$.*

By applying Theorem A we will be able to assume that R satisfies a nonzero multilinear GPI, say g. The multilinearity of g makes it clear that RC, and so, $R\overline{C} = RC \otimes_C \overline{C}$ satisfies g, where $\overline{C}$ is an algebraic closure of C, and with the identification of $R\overline{C} \subset N\overline{C} = N \otimes_C \overline{C}$. By Martindale's theorem [**23**; Theorem 3, p. 579] one concludes that $R\overline{C}$ is a primitive ring, that $H = \text{soc}(R\overline{C}) = \text{Soc}(RC)\overline{C} \neq 0$, and for any idempotent $e \in H$, $eHe \cong M_n(\overline{C})$ where n is the (uniform) dimension of eH, or of He. Therefore, the multilinear identities for R will be identities for H and the reduction to matrices depends on showing that H is finite dimensional over $\overline{C}$, in which case $R\overline{C} = H = M_n(\overline{C})$, $RC = M_n(C)$, and C is the quotient field of Z [**26**]. One technical problem which arises is whether a GPI g for R is a GPI for H; that is, can one consider the coefficients of g to be in $N(H)$? Our first lemma clarifies this matter and provides a related computation which will be useful in what follows.

LEMMA 1. *Let R satisfy a nonzero GPI, let $\overline{C}$ be an algebraic closure of C, and set $H = \text{Soc}(R\overline{C})$. Then $HN + NH \subset H$ and $N \subset N(H)$, where we consider $N, H \subset N\overline{C}$.*

Proof. For $q \in N$ let J be a nonzero ideal of R satisfying $Jq + qJ \subset R$. Now $J\overline{C} = JC \otimes \overline{C}$ is a nonzero ideal of $R\overline{C}$ and H is the unique minimal ideal of $R\overline{C}$, so

$$\begin{aligned} Hq + qH = H^2q + qH^2 &\subset H(J\overline{C})q + q(J\overline{C})H \\ &\subset H(R\overline{C}) + (R\overline{C})H \subset H. \end{aligned}$$

Therefore $HN + NH \subset H$, and since right multiplication by $q \in N$ is a left H-module homomorphism of H to itself, $N \subset N(H)$.

Since our main results concern Lie ideals, we collect some well-known facts about them in our next lemma. We say that R *satisfies* S_4 if R satisfies the standard polynomial identity of degree four; equivalently, R is an order in a simple algebra at most four dimensional over its center, the quotient field of the center of R [**26**], and so $R\overline{C} = M_2(\overline{C})$. The notation $[a, b] = ab - ba$ is used throughout, and recall that a Lie ideal of R is an additive subgroup L satisfying $[L, R] \subset L$.

LEMMA 2. *Let R be a prime ring, $d \in \mathrm{Der}(R)C$, L a noncommutative Lie ideal of R, and M the ideal of R generated by $[L, L]$. Then the following hold:*

(i) $M \subset L + L^2$;
(ii) $[M, M] \subset L$;
(iii) $[L, L]$ *is a noncommutative Lie ideal of R unless* $\mathrm{char}\, R = 2$ *and R satisfies S_4; and*
(iv) $[L, L]^d \subset Z$ *implies* $d = 0$ *unless* $\mathrm{char}\, R = 2$ *and R satisfies S_4.*

Proof. The proof of (i) is given in [**11**; proof of Lemma 1.3, p. 4]. Briefly, for $a, b \in L$ and $r \in R$, $[a, b]r = [ar, b] - a[r, b] \in L + L^2$ and then commutation with R gives the result. Now (ii) follows from (i) if $[L^2, M] \subset L$, and this holds using the identity $[xy, z] = [x, yz]+[y, zx]$. Next, (iii) is immediate from [**21**; Lemma 7, p. 120]. Finally, let A be the subring generated by $[U, U]$ for $U = [L, L]$. Unless $\mathrm{char}\, R = 2$ and R satisfies S_4, (iii) and (i) show that $J \subset A$ for J a nonzero ideal of R, and since $A^d \subset [U, U]^d = 0$, one has $J^d = 0$ which easily gives $d = 0$.

Our first theorem is the result which will enable us to show that $H = \mathrm{Soc}(R\overline{C})$ is finite dimensional. This theorem is of some independent interest because it shows that Lie ideals can satisfy nontrivial linear identities, whereas ideals cannot ([**23**], [**12**; Lemma 1.3.2, p. 22], or [**18**; Lemma 1, p. 766]), and further, that this can occur only when RC is finite dimensional. In the proof of the theorem, and in later proofs, we will need the fact that Litoff's theorem [**15**; Theorem 3, p. 90] holds in H.

THEOREM (*Litoff*). *For any $\{h_1, \ldots, h_n\} \subset H = \mathrm{Soc}(R\overline{C})$, there is an idempotent $e \in H$ so that $\{h_i\} \subset eHe$.*

THEOREM 1. *Let R be a prime ring, L a noncommutative Lie ideal of R, $\overline{C}$ an algebraic closure of C, and $f \in F(N, X)$ a multilinear GPI* *for L. Then either* $f = 0$, f *is a nonzero GPI for* $Q\overline{C}$, *or* $R\overline{C} \cong M_n(\overline{C})$ *and* f *is a GPI for* $[R\overline{C}, R\overline{C}]$.

Proof. We proceed by induction on the degree of f, and for the case $\deg(f) = 1$ let $f = f(x_1) = \sum a_i x_1 b_i$. There is a nonzero ideal M of R satisfying $[M, M] \subset L$, by Lemma 2, and it is clear that $\overline{f}(x_1, x_2) = f(x_1 x_2 - x_2 x_1)$ is a GPI for M. Assume that $f \neq 0$ in $F(N, X)$, so also $\overline{f} \neq 0$. We may conclude from ([**18**; Theorem 7, p. 783] or [**24**; Theorem 3.9, p. 510]) that $\overline{f}$ is a GPI for R. The multilinearity of $\overline{f}$ implies that $\overline{f}$ is also a GPI for $R\overline{C}$, and so for $H = \mathrm{Soc}(R\overline{C})$, using Lemma 1. In particular $f([H, H]) = 0$. Note also that $f(H) \subset H$ by Lemma 1. Employing Litoff's theorem, one can show that as $\overline{C}$-vector spaces, $H = [H, H] + \overline{C}e$ for any primitive idempotent $e \in H$ [**20**; proof of Theorem 4]. Therefore, $f(H) = \overline{C}f(e) \subset H$. Should $f(e) = 0$, then $f(H) = 0$ forcing $f = 0$ in $F(N(H), X)$ [**18**; Lemma 1, p. 766]. Consequently, we may assume that $f(e) \neq 0$ and use Litoff's theorem to find idempotents $g, g' \in H$ satisfying $f(e) \in gHg$ and $\{g, ga_i\} \subset g'Hg'$, where the a_i are the left coefficients of f. If H is infinite dimensional over $\overline{C}$, there is a primitive idempotent $e' \in H$ which is orthogonal to g'. As we have seen, $f = 0$ if $f(e') = 0$, so we may write $f(e') = cf(e)$ for $c \in \overline{C} - \{0\}$. Hence $cf(e) = cgf(e) = gf(e') = \sum ga_i e' b_i \subset \sum g'Hg'e'b_i = 0$, contradicting $f(e') \neq 0$. We are forced to conclude that either $f = 0$ or H is finite dimensional over $\overline{C}$. Since H is an ideal and simple subalgebra of $R\overline{C}$, the second possibility gives $R\overline{C} = H \cong M_n(\overline{C})$, completing the proof when f is linear.

Now let $\deg f = k > 1$ and assume that $R\overline{C}$ is not finite dimensional over $\overline{C}$. Write $f = f(x_1, \ldots, x_k)$ and consider $\overline{f} = f(x_1, \ldots, x_{k-1}, y)$ for any $y \in L$. It is clear that $\overline{f}$ is multilinear and homogeneous of degree $k - 1$, and that $\overline{f}(L^{k-1}) = 0$, so by induction, $\overline{f}(R^{k-1}) = 0$. Hence, for any $\overline{r} \in R^{k-1}$, $f'(x) = f(\overline{r}, x)$ is linear and $f'(L) = 0$. The case $k = 1$ now forces $f' = 0$ and we have that f is a GPI for R. We observe that f is also a GPI for Q [**19**; Theorem 1, p. 17] and so for $Q\overline{C}$ by multilinearity. When $R\overline{C} \cong M_n(\overline{C})$, the multilinearity of f implies that f is a GPI for $L\overline{C} = LC \otimes_C \overline{C}$. Since $[R\overline{C}, R\overline{C}] \subset L\overline{C}$ by Lemma 1, the proof is complete.

Posner's second theorem for Lie ideals and involutions. We have now assembled what we need to prove our first main result, which is

Posner's second theorem for Lie ideals. As we indicated in the introduction, this result appears in [22] and [5]. Recall that Z denotes the center of R.

THEOREM 2. *Let R be a prime ring, L a noncommutative Lie ideal of R, and $d \in \mathrm{Der}(R) - \{0\}$. If $[x, x^d] \in Z$ for all $x \in L$, then either R is commutative, or* $\mathrm{char}\, R = 2$ *and R satisfies S_4.*

Proof. Suppose that $d \notin \mathrm{Inn}(R)$ and linearize the expression $[x, x^d]$ to obtain the multilinear $g = [x_1, x_2^d] + [x_2, x_1^d] \in F(N, X)$. When $\mathrm{char}\, R = 2$ and $t, y \in L$, $[t, y]^d = g(t, y) \in Z$, or equivalently, $[L, L]^d \subset Z$. Thus, R satisfies S_4 by Lemma 2, proving the theorem. We may assume henceforth that $\mathrm{char}\, R \neq 2$. In g, replace the variables with commutators to get

$$f = [[x_1, x_2], [x_3^d, x_4] + [x_3, x_4^d]] + [[x_3, x_4], [x_1^d, x_2] + [x_1, x_2^d]],$$

and set $\overline{f} = [f, x_5]$. Then $\overline{f}$ is a GDI for M, the ideal of R given in Lemma 2 and satisfying $[M, M] \subset L$. Now if $\overline{f} = 0$ in $F(N, X)$, then $f = 0$ also, it would follow that $[[x_1, x_2], [x_3, x_4]]$ is a GPI for R. But then R would be commutative, using Lemma 2. Therefore, we may assume $\overline{f} \neq 0$ and apply Theorem A with $W = \{d\}$ and exponent sequence $(d, 1, 1, 1, 1)$ to conclude that $[[[x_3, x_4], [x_1, x_2]], x_5]$ is a GPI for R. Again, R must be commutative, finishing the proof when $d \notin \mathrm{Inn}(R)$.

Now assume $d = \mathrm{ad}(A)$, so $h = [g, x_3]$ is a GPI for L, where each x_i^d is replaced with $[x_i, A]$. Note that h is written as a sum of distinct basis monomials of $F(N, X)$ if $A \notin C$, and that $A \in C$ means $d = 0$. Hence $h \neq 0$ and so from Theorem 1 either h is a GPI for H, or $H = R\overline{C} \cong M_n(\overline{C})$ and h is a GPI for $[H, H]$. In the first case take $e^2 = e \in H$ and replace each x_i with e to obtain $0 = [e, g(e, e)] = [e, A]$; that is, A centralizes all idempotents in H. but when H is infinite dimensional, it is generated by its idempotents, forcing $A \in C$ and $d = 0$. Consequently, we may assume that $H = R\overline{C} \cong M_n(\overline{C})$ for $n > 1$, and that h is a GPI for $[H, H]$.

Let $\{e_{ij}\}$ be the usual matrix units for H. If $i \neq j$, then $e_{ij} = [e_{ii}, e_{ij}] \in [H, H]$, so $g(e_{ij}, e_{ij})$ is a scalar matrix. This shows that $A_{ji} = 0$, so A is a diagonal matrix. For any invertible $P \in H$, $P^{-1}hP$ is still a GPI for $[H, H]$, and it follows that $P^{-1}AP$ is also a diagonal matrix. This is possible only if A is scalar, resulting again in $d = 0$ and completing the proof of the theorem.

COROLLARY. *Let R be a prime ring, I a nonzero ideal of R, and $d \in \mathrm{Der}(R) - \{0\}$. If $[x, x^d] \in Z$ for each $x \in I$ then R is commutative.*

Proof. From Theorem 2 we may assume $\mathrm{char}\, R = 2$ and that R satisfies S_4. The proof of Theorem 2 shows that we may take $d = \mathrm{ad}(A)$. Since $RC \cong M_2(C)$ and C is the quotient field of Z **[26]**, it follows that $[x, [x, A]] \in C$ for any $x \in RC = IC$. In particular, the choice of $x = e_{11}$ shows that A is diagonal, and then taking $x = e_{11} + e_{12}$ yields the contradiction $A \in C$.

In Theorem 2, the exception given when $\mathrm{char}\, R = 2$ is necessary because if $R = M_2(C)$, for C a field with $\mathrm{char}\, C = 2$, then $L = [R, R]$ is noncommutative, but $[L, L] \subset C$. Hence if $d = \mathrm{ad}(A)$ for $A \in L$, then $[x, x^d] \in C$ holds for all $x \in L$.

Our next theorem is the version of Theorem 2 for rings with involution. When R has an involution, $*$, and $J^* = J$ is an ideal of R, set $S(J) = \{y \in J | y^* = y\}$, $T(J) = \{y + y^* | y \in J\}$, and $K(J) = \{y - y^* | y \in J\}$. We shall consider the situation when $[x, x^d] \in Z$ for all $x \in T(J)$, or for all $x \in K(J)$. As one might expect, the example mentioned just above shows that one must again exclude the case when R satisfies S_4.

EXAMPLE. Let $R = M_2(C)$ for C a field and assume first that $\mathrm{char}\, R \neq 2$. When R has the symplectic involution $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, $S = C \cdot I_2$ so $[S, S^D] = 0$ for any $D \in \mathrm{Der}(R)$. With the usual transpose involution, $K = C(e_{12} - e_{21})$, so $[K, K] = 0$ and $[K, K^D] = 0$ whenever $K^D \subset K$.

Now, if $\mathrm{char}\, C = 2$, $T = K$ and for the transpose involution $T = C(e_{12} + e_{21})$ is commutative, so again $[T, T^D] = 0$ if $T^D \subset T$; for example, $D = \mathrm{ad}(A)$ with $A_{12} = A_{21}$. For the symplectic involution on $R, T = C \cdot I_2$ and $S = [R, R]$. As we have seen $[S, S] \subset C \cdot I_2$, so for any $D \in \mathrm{Der}(R)$ $[S, S^D] \subset C \cdot I_2$ and $[T, T^D] = 0$.

Recall that an involution, $*$ in R is of the *first kind* if $S(C) = C$, and of the *second kind* otherwise **[24]**. If $*$ is of the second kind, let $c \neq c^*$ and choose a nonzero ideal $J = J^*$ of R with $cJ + c^*J \subset R$. Then for $y \in J$, $(c - c^*)y = (cy + c^*y^*) - c^*(y + y^*)$ and $(c - c^*)y = (cy - c^*y^*) + c^*(y^* - y)$, so $J \subset CT(R) \cap CK(R)$. Consequently, each of CT and CK contains a nonzero ideal of RC. When $*$ is of the first kind, then it can be extended to $R\overline{C}$ by setting $(a \otimes \overline{c}) = a^* \otimes \overline{c}$ for $a \in RC$.

Our next lemma states an important fact which we need in Theorem 3. It is certainly known, but there is no apparent reference, particularly when $\operatorname{char} R = 2$. The statement that it holds for K when $\operatorname{char} R \neq 2$ is given in [**9**; p. 529]. We provide an argument for completeness.

LEMMA 3. *Let R be a prime ring with involution, $*$, and $I = I^*$ a nonzero ideal of R. If either $[T(I), T(I)] \subset Z$ or $[K(I), K(I)] \subset Z$ then R satisfies S_4.*

Proof. Suppose that $[T(I), T(I)] \subset Z$, let $f = [[x_1+y_1, x_2+y_2], x_3]$, and note that f is a nonzero multilinear G*-PI for I. By Theorem A, f is a G*-PI for R, so R satisfies S_6 [**2**; Theorem 1, p. 63]. Localizing R at $S(Z)$ [**26**] and using the multilinearity of f and S_6 enable us to assume that R is simple Artinian and satisfies both f and S_6. If $*$ is of the second kind, then as we indicated above $R = CT$, resulting in $[R, R] \subset C$, which forces R to be commutative [**11**; Corollary, p. 9]. When $*$ is of the first kind, $R\overline{C}$ satisfies f and S_6, so unless R satisfies $S_4, R\overline{C} \cong M_3(\overline{C})$ and $*$ is of transpose type. But for $i \neq j$, $t_{ij} = c_i e_{ij} + c_j e_{ji} \in T$, where $c_i c_j \neq 0$ and $[t_{12}, t_{23}] \notin \overline{C} \cdot I_3$. Thus R must satisfy S_4. The proof when $[K(I), K(I)] \subset Z$ proceeds in exactly the same fashion, starting with $g = [[x_1 - y_1, x_2 - y_2], x_3]$.

THEOREM 3. *Let R be a prime ring with involution, $*$, $I = I^*$ a nonzero ideal of R, and $d \in \operatorname{Der}(R) - \{0\}$. If $[x, x^d] \in Z$ for all $x \in T(I)$, or for all $x \in K(I)$, then R satisfies S_4.*

Proof. Assume first that $d \notin \operatorname{Inn}(R)$ and set

$$g = [[x_1 + y_1, x_2^d + y_2^d], x_3] + [[x_2 + y_2, x_1^d + y_1^d], x_3] \quad \text{and}$$

$$f = [[x_1 - y_1, x_2^d - y_2^d], x_3] + [[x_2 - y_2, x_1^d - y_1^d], x_3].$$

Then if $[x, x^d] \in Z$ holds for $T(I), g \in F(N, X, Y)$ is a multilinear G*-DI for I, and if it holds for $K(I)$, f is a multilinear G*-DI for I. We proceed assuming that the hypothesis holds for $T(I)$ and indicate what changes, other than obvious ones, are necessary at each step if one assumes instead that the hypothesis holds for $K(I)$. Now $g \neq 0$ since it is a sum of distinct basis monomials, so by Theorem A $g_{(1,d,1)}$ is a G*-PI for R. But then $[T(R), T(R)] \subset Z$ and R satisfies S_4 by Lemma 3. Therefore, we may assume $d = \operatorname{ad}(A)$ and replacing each expression t^d in g with $[t, A]$ enables us to view g as a G*-PI for I.

Next, observe that $g \neq 0$, unless $A \in C$ and $d = 0$, since the basis monomial $x_1x_2Ax_3$ in g is not canceled. By Theorem A, g is a G*-PI for R and R satisfies a nonzero GPI. Suppose that * is of the second kind. Our comments before Lemma 3 show there is an ideal J of R satisfying $J \subset CT(R)$. Since $d = \mathrm{ad}(A)$ and g is an identity for R, it follows that for $a, b \in J$, $[a, b^d] + [b, a^d] \in C \cap R = Z$. If $\mathrm{char}\, R = 2$, then $[J, J]^d \subset Z$ and R must satisfy S_4 by Lemma 2. If char $\neq 2$ taking $a = b = x \in J$ shows that $[x, x^d] \in Z$ and R must be commutative by the Corollary to Theorem 2. Consequently, we may now assume that * is of the first kind, so * extends to $R\overline{C} = RC \otimes_C \overline{C}$. Furthermore by multilinearity of g and Lemma 1, g is G*-PI for $H = \mathrm{Soc}(R\overline{C}) \neq 0$, from which it also follows that $[x, [x, A]] \in \overline{C}$ for all $x \in T(H) \subset T(I\overline{C})$.

Suppose for now that $\mathrm{char}\, R = 2$ and that H is infinite dimensional over $\overline{C}$. If $a, b \in T(H)$ then $[a, b]^d = [a, b^d] + [b, a^d] \in \overline{C} \cap H = 0$, using Lemma 1 again. Thus $[[T, T], A] = 0$ and since the subring generated by $[T, T]$ is H [**21**; Theorem 25, p. 129], we must have $A \in \overline{C}$ and $d = 0$. This contradiction means that when $\mathrm{char}\, R = 2$, $H = R\overline{C} \cong M_n(\overline{C})$, and R satisfies S_4 unless $n > 2$. If * is of transpose type on H, then $\{t_{ij} = c_i e_{ij} + c_j e_{ji} | i \neq j$ and certain $c_i \in C$ with $c_i c_j \neq 0\}$ span T over $\overline{C}$, and it is easy to see that $[T, T] = T$. As above, for $a, b \in T$, $[a, b]^d \in \overline{C}$, so $[T, A] = [[T, T], A] \subset \overline{C} \cdot I_n$. Consequently, $[T, A] = [[T, T], A] = 0$, A centralizes T, and so, A centralizes $e_{ik} \in \overline{C} t_{ij} t_{jk}$ for i, j, k distinct. This clearly gives the contradiction $A \in \overline{C} \cdot I_n$.

Consider now $\mathrm{char}\, R = 2$, $H \cong M_n(\overline{C})$ and assume that * is of symplectic type. This means that $n = 2m$ and if $B \in H$ is written as $B = \sum B_{ij} E_{ij}$ for $1 \leq i, j \leq m$, where $B_{ij} \in M_2(\overline{C})$ and $E_{ij} = I_2 \in M_2(\overline{C})$ in the "i-j" position, then $B^* = \sum B^*_{ji} E_{ij}$ with $\left(\begin{smallmatrix} a & b \\ c & t \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} t & -b \\ -c & a \end{smallmatrix}\right)$. In particular $E_{ii} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) E_{ii} + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) E_{ii} \in T$. Taking $x = E_{ii}$ and using E_{ii} is an idempotent shows that $[E_{ii}, [E_{ii}, A]] = [E_{ii}, A] \in \overline{C} \cdot I_n$ which yields $A_{ij} = 0$ if $i \neq j$; that is, $A = \sum A_{ii} E_{ii}$. Now, for $B \in M_2(\overline{C})$ set $Y = BE_{ij} + B^* E_{ji} + E_{ii} \in T$ and compute $[Y, [Y, A]] = [Y^2, A] = (BA_{jj} + A_{ii}B)E_{ij} + (B^* A_{ii} + A_{jj} B^*) E_{ji} \in \overline{C} \cdot I_n$ which forces $BA_{jj} = A_{ii}B$. When $B = I_2$ one gets $A_{ii} = A_{jj}$, resulting in $BA_{ii} = A_{ii}B$, so $A \in \overline{C}$ and $d = 0$. This contradiction finishes the proof when $\mathrm{char}\, R = 2$, so we may now assume that char $\neq 2$.

Just as in the case $\mathrm{char}\, R = 2$, we want to show that H is finite dimensional, so assume for now that H is infinite dimensional over $\overline{C}$. Thus for $t_1, t_2 \in T(H)$, $[t_1, t_2^d] + [t_2, t_1^d] \in \overline{C} \cap H = 0$, using the fact that g is a G*-PI for H, and Lemma 1. For $y \in T(H)$ and $h \in H$, $hy + yh^* \in T(H)$. The computation above, with $t_1 = t_2 = y$ shows $[y, y^d] = 0$, and then taking $t_1 = hy + yh^*$ and $t_2 = y$ gives

$$\begin{aligned} 0 &= [hy + yh^*, y^d] + [y, (hy + yh^*)^d] \\ &= [h, y^d]y + y[h^*, y^d] + y(hy + yh^*)^d \\ &\quad - h^d y^2 - hy^d y - y^d h^* y - y(h^*)^d y. \end{aligned}$$

Consequently, $h^d y^2 = hy^d y - y^d hy + yh_1 - hy^d y - y^d h^* y$ for $h_1 \in H$, and we may conclude from this that for $y \in T(H)$, $H^d y^2 \subset yH + y^d H$. If also $x \in T(H)$ then $(hx^2)^d = h^d x^2 + h(x^2)^d$, and it follows that $H(x^2)^d y^2 \subset H^d x^2 y^2 + H^d y^2 \subset xH + x^d H + yH + y^d H$. Suppose that $(x^2)^d y^2 \neq 0$ for some choice of x and y. Now we have $H = H(x^2)^d y^2 H = xH + x^d H + yH + y^d H$, which forces H to have finite uniform dimension, since $x, y, x^d, y^d \in H$, contradicting the infinite dimensionality of H over $\overline{C}$. Therefore, we may assume $(x^2)^d y^2 = 0$. Since char $\overline{C} \neq 2$, $T(H) = S(H)$, and the span over $\overline{C}$ of $\{t^2 | t \in S(H)\}$ is a Jordan ideal of $S(H)$ $(t^2 s + st^2 = (t^2 + s)^2 - (t^2)^2 - s^2)$; this span is $S(H)$ **[11**; Theorem 2.6, p. 32]. Also, the span of $\{t^2 | t \in K(H)\}$ contains $S(H)$, by a theorem of Baxter [**11**; Theorem 2.3, p. 29]. Consequently, under either possible original hypothesis, $S(H)^d S(H) = 0$, and it follows from the fact that $S(H)$ generates H as a ring [**12**; Theorem 2.1.6, p. 61], that $d = 0$. In summary, we may now assume that $H = R\overline{C} \cong M_n(\overline{C})$ where $n > 2$ and char $\overline{C} \neq 2$.

If * is of transpose type on H, then as in the char $R = 2$ case $\{t_{ij} = c_i e_{ij} + c_j e_{ji} | i \neq j$ and $c_i c_j \neq 0\}$ spans $T(H)$. Since $Y = [t_{ij}, [t_{ij}, A]] \in \overline{C} \cdot I_n$ and $n > 2$, for $k \neq i, j$ we have $Y_{ik} = -c_i c_j A_{ik} = 0$, and it follows that A is diagonal. Thus $Y = 2c_i c_j (A_{ii} - A_{jj})(e_{ii} - e_{jj}) \in \overline{C} \cdot I_n$, and so $A_{ii} = A_{jj}$ forcing $A \in \overline{C}$ and again $d = 0$. Finally, if * is of symplectic type on H, then $n = 2m$ and $B \in H$ can be written $B = \sum B_{ij} E_{ij}$ as in the char $R = 2$ case. Now $E_{ii} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) E_{ii} + \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)^* E_{ii} \in T(H)$, so $Y = [E_{ii}, [E_{ii}, A]] \in \overline{C} \cdot I_n$ and this yields $0 = Y_{ij} = A_{ij}$ when $i \neq j$. Now set $t = BE_{ij} + B^* E_{ji}$ for any $B \in M_2(\overline{C})$ and let

$$X = [t, [t, A]] = 2BB^*(A_{ii}E_{ii} + A_{jj}E_{jj}) - 2(BA_{jj}B^*E_{ii} + B^*A_{ii}BE_{jj}),$$

using $A = \sum A_{ii} E_{ii}$ and $BB^* \in \overline{C} \cdot I_2$. Since X is scalar, for $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $BB^* = 0$ and it follows that $X = 0$, the "1-2" entry of A_{jj} is zero and the "2-1" entry of A_{ii} is zero. Using $B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ shows that A_{ii} and A_{jj} are diagonal. Lastly setting $B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ yields the fact that $A_{ii} = A_{jj} \in \overline{C} \cdot I_2$, giving the contradiction $A \in \overline{C} \cdot I_n$. Therefore $n \leq 2$ and R satisfies S_4, finishing the proof of the theorem.

Posner's first theorem for Lie ideals. We turn now to Posner's first theorem. An additive mapping d of R which satisfies $[x,y]^d = [x^d, y] + [x, y^d]$ for all $x, y \in A \subset R$ is called a *Lie derivation* of A into R, and the set of all such will be denoted Lie-Der(A, R). If $(xy)^d = x^d y + xy^d$ for all $x, y \in A$ we write $d \in \mathrm{Der}(A, R)$. Clearly, $\mathrm{Der}(A, R) \subset$ Lie-Der(A, R). Suppose now that $d, h \in \mathrm{Der}(R)$, L is a Lie ideal of R, and $dh \in$ Lie-Der(L, R). As Posner observed [**25**, p. 1094], for $x, y \in L$, $[x,y]^{dh} = [x^{dh}, y] + [x^d, y^h] + [x^h, y^d] + [x, y^{dh}]$ since $d, h \in \mathrm{Der}(R)$, and also $[x, y]^{dh} = [x^{dh}, y] + [x, y^{dh}]$ since $dh \in$ Lie-Der(R). Together these equations give $[x^d, y^h] + [x^h, y^d] = 0$, and this exhibits the GDI in which we are interested. Working with the commutator of this expression with another variable will enable us to obtain all of the results mentioned in the introduction. In our results about Lie ideals it is necessary to exclude the case when $\mathrm{char}\, R = 2$ and R satisfies S_4. For example, when $R = M_2(C)$ and $\mathrm{char}\, C = 2$, we have seen that $[L, L] \subset C$ for $L = [R, R]$, so taking $d = \mathrm{ad}(A)$ and $h = \mathrm{ad}(B)$ for $A, B \in L$ and C-independent results in $dh \in$ Lie-Der(L, R), since $L^{dh} = 0$, and also $R^{dh} \subset C$, although dh need not be zero. However, a direct extension of Posner's first theorem to ideals does not require this exception. As we mentioned earlier, Posner's proof [**25**; Theorem 1, p. 1094] actually holds for ideals when $\mathrm{char}\, R \neq 2$, and when $\mathrm{char}\, R = 2$ a (characteristic free) proof is given in [**6**], and another in [**13**]. The proof in [**13**] is not obviously adaptable to ideals, and while the proof in [**6**] does work for ideals, it has never appeared in print. For the sake of completeness of our results, and as an easy illustration of our approach, we provide a proof of this theorem for ideals. First we state a lemma from [**18**] which we will need to use a number of times in the results which follow. Special cases whose proofs are essentially the same are [**23**; Theorem 1, p. 577] and [**12**; Lemma 1.3.2, p. 22].

LEMMA 4 [**18**; *Lemma* 1, *p.* 766]. *Let R be a prime ring and let $f = \sum a_i x_1 b_i \in F(N, X, Y)$ be a GPI for a nonzero ideal I of R. If $\{a_i\} \subset N - \{0\}$ then $\{b_i\}$ is C-dependent and if $\{b_i\} \subset N - \{0\}$ then $\{a_i\}$ is C-dependent.*

THEOREM 4. *Let R be a prime ring, $d, h \in \mathrm{Der}(R)$ and J a nonzero ideal of R. if $dh \in \mathrm{Der}(J, R)$ then either $d = 0$, $h = 0$, or $\mathrm{char}\, R = 2$ and $d = hc$ for $c \in C$.*

Proof. Set $f = x_1^d x_2^h + x_1^h x_2^d$, so that for $x, y \in J$, $f(x, y) = (xy)^{dh} - x^{dh}y - xy^{dh} = 0$. If $\{d, h\}$ is independent modulo $\mathrm{Inn}(R)$, then f is a multilinear GDI to which we may apply Theorem A to get the identity $f_{(d,h)} = x_1x_2$. This contradiction shows that we may assume $h = dc + \mathrm{ad}(A)$ for $c \in C$, and $d \notin \mathrm{Inn}(R)$. Replacing, in f, each $(x_i)^h$ with $x_i^d c + [x_i, A]$ gives a GDI $\overline{f}$ for J, and if $\mathrm{char}\, R = 2$, by Theorem A $\overline{f}_{(d,1)} = x_1[x_2, A]$ is an identity for R. Thus $A \in C$, and $h = dc$ results, finishing the proof if $\mathrm{char}\, R = 2$. If $\mathrm{char}\, R \neq 2$ and $c \neq 0$ we now get the identity $\overline{f}_{(d,d)} = 2cx_1x_2$ for R, another contradiction. Thus we may assume $d = \mathrm{ad}(A)$, and by a similar argument, $h = \mathrm{ad}(B)$. If $h \neq 0$ or $d \neq 0$, there is $y \in J$ with either $y^h \neq 0$ or $y^d \neq 0$. Write out $f(x_1, y) = x_1(Ay^h + By^d) - Ax_1y^h - Bx_1y^d$ and use Lemma 4 to conclude that $\{1, A, B\}$ is C-dependent, again completing the proof if $\mathrm{char}\, R = 2$. When $\mathrm{char}\, R \neq 2, d \neq 0$, and $h \neq 0$, $f = 2c[x_1, A][x_2, A]$, when $B = cA + c_1$, and substituting x for x_1 and yx for x_2 shows that $A \in C$, giving the contradiction $d = 0$.

We return now to the consideration of Lie ideals. As in our generalization of Posner's second theorem, much of the work in generalizing the first theorem occurs after applying Theorem A. The computations for matrix rings are more involved than for the earlier result and we present them in a separate theorem which gives Posner's first theorem for inner derivations of matrix rings.

THEOREM 5. *Let $R = M_n(C)$ for C an algebraically closed field, and for $A, B \in R$ set $g(x_1, x_2) = [[x_1, A], [x_2, B]] + [[x_1, B], [x_2, A]]$. If $[g, x_3]$ is a GPI for $[R, R]$, then $\{I_n, A, B\}$ is C-dependent, unless* $\mathrm{char}\, R = 2$ *and $n = 2$, and either $A \in C \cdot I_n$ or $B \in C \cdot I_n$ if* $\mathrm{char}\, R \neq 2$.

Proof. Let $\{e_{ij}\}$ be the standard matrix units in R. It is clear that it suffices to prove the theorem if either A or B is replaced by itself plus a scalar matrix, or if both A and B are replaced by $P^{-1}AP$ and $P^{-1}BP$, respectively, for any $P \in \mathrm{GL}_n(C)$. Assume throughout that $A \notin C \cdot I_n$. Note first that for $i \neq j, e_{ij} \in [R, R]$ and $e_{ii} - e_{jj} \in [R, R]$, so $g(e_{ij}, e_{ii} - e_{jj}) \in C \cdot I_n$. Computing the j-i entry of this element yields $8A_{ji}B_{ji} = 0$. Assume for now that $\mathrm{char}\, C \neq 2$, so that

$$A_{ji}B_{ji} = 0 \quad \text{if } i \neq j. \tag{1}$$

Replacing A and B with $\overline{A}$ and $\overline{B}$, their conjugates by $P = I_n + ce_{ij}$ for $c \in C$, and applying equation (1) yields

$$0 = \overline{A}_{ij}\overline{B}_{ij} = (A_{ij} + c(A_{ii} - A_{jj}) - c^2A_{ji})(B_{ij} + c(B_{ii} - B_{jj}) - c^2B_{ji}).$$

If $A_{ij} \neq 0$, then $B_{ij} = 0$ by (1). Since C is infinite, there is $c \in C$ with $\overline{A}_{ij} \neq 0$ and $\overline{B}_{ij} \neq 0$ unless $B_{ji} = B_{ii} - B_{jj} = 0$. But $\overline{A}_{ij}\overline{B}_{ij} = 0$ by (1), so we may conclude

(2) $\quad$ if $A_{ij} \neq 0$, then $B_{ij} = B_{ji} = 0$ and $B_{ii} = B_{jj}$.

Next, if $k \neq i, j$, then using (1) again gives

$$0 = \overline{A}_{ik}\overline{B}_{ik} = (A_{ik} - cA_{jk})(B_{ik} - cB_{jk}).$$

As above, if $A_{ik} \neq 0$ we could choose $c \in C$ so $\overline{A}_{ik}\overline{B}_{ik} \neq 0$ unless $B_{jk} = 0$. That is,

(3) $\quad$ if $A_{ik} \neq 0$ then $B_{jk} = 0$ for i, j, k distinct.

Conjugate A and B by $P \in \mathrm{GL}_n(C)$ so that A is in Jordan canonical form. If A is diagonal we may assume that $A_{11} \neq A_{22}$, since $A \notin C \cdot I_n$, and so, conjugation of A and B with $I_n + ce_{12}$ changes A to an upper triangular matrix with $e_{11}A = c_1e_{11} + c_2e_{12}$ for $c_i \in C$ and $c_2 \neq 0$. We may assume A has this form, since its Jordan form does if it is not diagonal. Now, if $n = 2$ then since $A_{12} \neq 0$ we have $B_{12} = B_{21} = 0$ and $B_{11} = B_{22}$ by (2), which is to say, $B \in C \cdot I_2$, finishing the proof. If $n > 2$, conjugate A and B by $P = I_n + (e_{23} + \cdots + e_{2n})$ to obtain $\overline{A}$ and $\overline{B}$, where $e_{11}\overline{A} = c_1e_{11} + c_2(e_{12} + \cdots + e_{1n})$. Since $\overline{A}_{1j} \neq 0$ for $j > 1$, from (2) $\overline{B}_{1j} = \overline{B}_{j1} = 0$ and $\overline{B}_{11} = \overline{B}_{jj}$, and using (3) gives $\overline{B}_{jk} = 0$ if $j \neq k$ and $k > 1$. Thus $\overline{B} \in C \cdot I_n$, proving the theorem when $\operatorname{char} C \neq 2$.

Assume now that $\operatorname{char} C = 2$ and $n > 2$. As above, $g(e_{ij}, e_{ii} - e_{jj}) \in C \cdot I_n$, so computing the j-k entry gives

(4) $\quad A_{ji}B_{jk} + A_{jk}B_{ji} = 0$ for i, j, k distinct

and computing the k-i entry yields

(5) $\quad A_{ji}B_{ki} + A_{ki}B_{ji} = 0$ for i, j, k distinct.

Just as in the $\operatorname{char} C \neq 2$ case, we may assume that A is upper triangular and that $e_{11}A = A_{11}e_{11} + c(e_{12} + \cdots + e_{1n})$ with $c \neq 0$. We can also assume that $A_{11} = B_{11} = 0$ by replacing A with $A + A_{11}I_n$ and B with $B + B_{11}I_n$. Using (5) for $j > i > 1$ yields $0 = A_{ji}B_{1i} + A_{1i}B_{ji} = cB_{ji}$, since A is upper triangular, and because $c \neq 0$ we conclude

(6) $\quad B_{ji} = 0$ if $j > i > 1$.

Now use (5) with $j, k > 1$ and $j \neq k$ to obtain $0 = A_{kj}B_{1j} + A_{1j}B_{kj}$, or equivalently,

(7) $\quad cB_{kj} = B_{1j}A_{kj}$ for $j \neq k$ and $j, k > 1$.

Still assuming $j, k > 1$ and $j \neq k$, by (4) $0 = A_{1k}B_{1j} + A_{1j}B_{1k}$, and since $A_{1k} = A_{1j} = c \neq 0$ we have $B_{1j} = B_{1k} = z$ for $j, k > 1$. Substituting in (7) gives

$$B_{kj} = c^{-1}zA_{kj} \quad \text{if } j > 1 \text{ and } k \neq j. \tag{8}$$

Equations (4) and (5) hold for the conjugates $\overline{A} = P^{-1}AP$ and $\overline{B} = P^{-1}BP$ for $P = I_n + ye_{1j}$, where $y \in C$ and $j \neq 1$. In particular for $1, j, k$ distinct, (5) yields

$$\begin{aligned} 0 &= \overline{A}_{kj}\overline{B}_{1j} + \overline{A}_{1j}\overline{B}_{kj} \\ &= A_{kj}(B_{1j} + y(B_{11} + B_{jj}) + y^2B_{j1}) \\ &\quad + (c + y(A_{11} + A_{jj}))(B_{kj} + yB_{k1}). \end{aligned}$$

This relation holds for all $y \in C$, which is infinite, so the coefficient of y must be zero. Now $A_{11} = B_{11} = 0$ and $B_{kj} = c^{-1}zA_{kj}$ by (8), so we may conclude that

$$A_{kj}B_{jj} + cB_{k1} + c^{-1}zA_{kj}A_{jj} = 0 \quad \text{if } 1, j, k \text{ are distinct.} \tag{9}$$

Since A is upper triangular, $A_{kj} = 0$ if $k > j$, so if $k > 2$, (9) implies that $B_{k1} = 0$. When $k = 2$, take $j = 3$, multiply (9) by B_{21}, and use $A_{23}B_{21} = 0$ from (4) to get $cB_{21}^2 = 0$. Consequently

$$B_{k1} = 0 \quad \text{if } k \geq 1. \tag{10}$$

Finally, using (4) with $1, j, k$ distinct gives

$$\begin{aligned} 0 &= \overline{A}_{1k}\overline{B}_{1j} + \overline{A}_{1j}\overline{B}_{1k} \\ &= (c + yA_{jk})(B_{1j} + y(B_{11} + B_{jj})) \\ &\quad + (c + y(A_{11} + A_{jj}))(B_{1k} + yB_{jk}). \end{aligned}$$

As above, the coefficient of y must be zero, which shows that $cB_{jj} + zA_{jk} + cB_{jk} + zA_{jj} = 0$, and so by (8), $B_{jj} = c^{-1}zA_{jj}$ for $j > 1$. This computation, together with (10) and (8) yields $B = c^{-1}zA$, so that $\{I_n, A, B\}$ is C-dependent, completing the proof of theorem.

We come now to our last main technical result, from which our generalization of Posner's first theorem and the related results mentioned in the introduction will follow easily. Our approach is like that in Theorem 2 or Theorem 3. The reduction to rings satisfying a GPI using Theorem A is fairly easy, but the argument from that point to the matrix case is considerably more involved. For convenience, we first isolate a special case of Lemma 4 which will be useful to have.

LEMMA 5. *Let R be a prime ring and let $f = [[x_1, A], B] \in F(N, X)$. If f is a GPI for a nonzero ideal J of R, then $\{1, A, B\}$ is C-dependent, and either $A \in C$ or $B \in C$ unless* $\operatorname{char} R = 2$ *and $A^2, B^2 \in C$.*

Proof. From Lemma 4, the set of left coefficients $\{1, A, B, BA\}$ of f is C-dependent. If $\{1, A, B\}$ is independent, then writing BA as a linear combination of these and using Lemma 4 again gives the contradiction $\{A, B\} \subset C$. Thus $\{1, A, B\}$ must be C-dependent. If neither $A \in C$ nor $B \in C$, then $[[x, A], A]$ is a GPI for J. When $\operatorname{char} R = 2$, this is equivalent to $[x, A^2]$ forcing $A^2 \in C$, and so $B^2 \in C$ also. When $\operatorname{char} R \neq 2$ replacing x with uv, for $u, v \in J$ shows $[u, A][v, A] = 0$, and now replacing v with vu leads to $[u, A]J[u, A] = 0$ resulting in the contradiction $A \in C$, and so, proves the lemma.

THEOREM 6. *Let R be a prime ring, L a noncommutative Lie ideal of R, and $d, h \in \operatorname{Der}(R)$. Set $g = [x_1^d, x_2^h] + [x_1^h, x_2^d]$ and $f = [g, x_3]$. If f is a GDI for L then either $d = 0$, $h = 0$, or* $\operatorname{char} R = 2$ *and either R satisfies S_4 or $h = dc$ for $c \in C$.*

Proof. We assume throughout that $d \neq 0, h \neq 0$, and if $\operatorname{char} R = 2$ then R does not satisfy S_4. Let M be the nonzero ideal of R satisfying $[M, M] \subset L$ given by Lemma 2. For any $y, t \in L$ use f to define a multilinear GDI for M by setting

$$f(y,t) = [[[x_1^d, x_2], y^h] + [[x_1, x_2^d], y^h] + [[x_1^h, x_2], y^d] + [[x_1, x_2^h], y^d], t].$$

If $\{d, h\} = W$ is independent modulo $\operatorname{Inn}(R)$, apply Theorem A to conclude that $f(y,t)_{(d,1)} = [[[x_1, x_2], y^h], t]$ is a GPI for R. Using Lemma 2, we have first that $[[R, R], L^h] \subset Z$, and then that $h = 0$, since $[[L, L], [L, L]]^h = 0$. Thus, we may now assume that $d \notin \operatorname{Inn}(R)$ and $h = dc + \operatorname{ad}(A)$ for $c \in C$.

Let $\overline{f}(y, t)$ be $f(y, t)$ with each x_i^h replaced with $x_i^d c + [x_i, A]$. Now $\overline{f}(y, t)$ is a multilinear GDI for M, so Theorem A shows that $\overline{f}(y, t)_{(d,1)} = [[[x_1, x_2], y^h + y^d c], t]$ is a GPI for R. As above, it follows from Lemma 2 that $[[R, R], L^{h+dc}] \subset C$ and then that $h + dc = 0$. But $h = dc + \operatorname{ad}(A)$, so $2dc + \operatorname{ad}(A) = 0$ results and either $\operatorname{char} R = 2$ and $h = dc$ as claimed, or $\operatorname{char} R \neq 2$ and either $d \in \operatorname{Inn}(R)$, a contradiction, or $c = 0$ forcing $0 = h + dc = h$, another contradiction. Therefore, the theorem is proved unless $d = \operatorname{ad}(A)$, and by a similar argument, $h = \operatorname{ad}(B)$.

In f, replace x_i^d with $[x_i, A]$ and x_i^h with $[x_i, B]$ so that f is a multilinear GPI for L. By Theorem 1, either $R\overline{C} \cong M_n(\overline{C})$ and

f is a GPI for $[R\overline{C}, R\overline{C}]$, or f is a GPI for $Q\overline{C}$. In the first case $N = RC = M_n(C)$, so $A, B \in R\overline{C}$ and using Theorem 5 finishes the proof: if $\operatorname{char} R = 2$ then $\{1, A, B\}$ are C-dependent so $d = 0, h = 0$, or $h = dc$, and if $\operatorname{char} R \neq 2$ then $A \in C$ or $B \in C$. Consequently, we may assume henceforth that f is a GPI for $Q\overline{C}$, and that $R\overline{C}$ is not finite dimensional over $\overline{C}$.

We shall finish the proof by showing that $R\overline{C} = H = \operatorname{Soc}(R\overline{C})$ is finite dimensional, but first we need to know that $H \neq 0$; that is, that $f \neq 0$. We claim that we may assume $\{1, A, B\}$ is C-independent, and so, $\overline{C}$-independent. Otherwise, $B = c_1 + c_2 A$ for $c_i \in C$ and $c_2 \neq 0$, since we are assuming $A \notin C$ and $B \notin C$. If $\operatorname{char} R = 2$, then $h = \operatorname{ad}(B) = dc_2$, finishing the proof, so assume $\operatorname{char} R \neq 2$. Substituting in f, one gets the GPI $f' = 2c_2[[[x_1, A], [x_2, A]], x_3]$ for R. Now $f' \neq 0$ in $F(N, X)$ because $A \notin C$ implies that $x_1 A x_2 A x_3$ is a basis monomial of f' which cannot be canceled. Thus $H \neq 0$, f' is a GPI for H, and $g' = [[x_1, A], [x_2, A]]$ satisfies $g'(H, H) \subset \overline{C} \cap H = 0$, using Lemma 2 and our assumption that H is infinite dimensional. In particular, for any $y \in H$ $[[x_1, A], [y, A]]$ is a GPI for H, so from Lemma 5 either $A \in \overline{C}$, forcing $d = 0$, or $H^d \subset \overline{C}$. Finally, if $H^d \subset \overline{C}$ then $[H, H]^d = 0$ and $d = 0$ follows from Lemma 2. This contradiction establishes our claim that $\{1, A, B\}$ can be assumed to be $\overline{C}$-independent.

Observe next that $f \neq 0$ since $x_1 A x_2 B x_3$ is a basis monomial of $F(N, X)$ appearing in f. Hence R does satisfy a nonzero GPI, so $H \neq 0$ and $f(H^3) = 0$. As in the last paragraph, using Lemma 2, $g(H, H) \subset H \cap \overline{C} = 0$, so g is a GPI for H. We claim next that for some $h \in H$, $[h, A]$ and $[h, B]$ are $\overline{C}$-independent. If not, then when $[h, B] \neq 0$, $[h, A] = c[h, B]$ for some $c = c(h) \in \overline{C}$. Rewriting g gives $g(x_1, h) = [[x_1, A + cB], [h, B]]$, a GPI for H, and from Lemma 5 $[h, B] \in \overline{C} + \overline{C}A + \overline{C}B$. Thus $[H, B]$ is at most 3-dimensional over $\overline{C}$. Clearly, $[H, B] \subset He$ for an idempotent $e \in H$, so the infinite dimensionality of H and Litoff's theorem enable us to find an idempotent $e' \in H - \{0\}$ orthogonal to e. Therefore, $0 = [H, B]e' = [H^2, B]e' = [H, B]He'$, forcing $B \in \overline{C}$ and contradicting $h \neq 0$. Our claim is established.

Fix $y \in H$ so that y^d and y^h are $\overline{C}$-independent and note that $y^d, y^h \in H$ by Lemma 1, so for some idempotent $e \in H, y^d, y^h \in eH$. By our assumption on $H, H(1-e) = \{t \in H | te = 0\} \neq 0$. For all $t \in H$ and $v \in H(1-e) - \{0\}$, $0 = vg(t, y) = vt(Ay^h + By^d) - vAty^h - vBty^d$, so $vg(x_1, y)$ is a GPI for H and we may conclude from Lemma 4 that $\{y^h, y^d, Ay^h + By^d\}$ is $\overline{C}$-dependent. From our choice of y, it follows that $Ay^h + By^d = c_1 y^h + c_2 y^d$, if either $vA \neq 0$ or $vB \neq 0$. Setting

$\overline{A} = A - c_1$ and $\overline{B} = B - c_2$, it is clear that $d = \text{ad}(\overline{A})$ and $h = \text{ad}(\overline{B})$, so our choice of y is unaffected by replacing A and B by $\overline{A}$ and $\overline{B}$. However, now $\overline{A}y^h + \overline{B}y^d = 0$, so computing $vg(t, y)$ again shows that $v\overline{A}x_1y^h + v\overline{B}x_1y^d$ is a GPI for H and we obtain $v\overline{A} = 0 = v\overline{B}$ from Lemma 4. Hence, there is no loss of generality in assuming $vA = vB = 0$ for all $v \in H(1-e)$. Since from Lemma 1 $Ax \in H$ for all $x \in H$, we get from $H(1-e)Ax = 0$, that $Ax = eAx \in H$, resulting in $(A - eA)H = 0$, and so $A = eA \in H$. Similarly, $B \in H$ and by Litoff's theorem $A, B \in \overline{e}H\overline{e}$ for some idempotent $\overline{e} \in H$. Using the infinite dimensionality of H, choose $\overline{e}'$, a nonzero idempotent orthogonal to $\overline{e}$, and consider the identity $-\overline{e}'g(x_1, \overline{e}'x_2) = \overline{e}'x_2(B[x_1, A] + A[x_1, B])$ for H. Since H is a simple ring, $B[x_1, A] + A[x_1, B] = Bx_1A + Ax_1B - (BA + AB)x_1$, is a GPI for H, which is impossible by Lemma 4 and the independence of $\{1, A, B\}$. This contradiction completes the proof of the theorem.

As an immediate consequence of Theorem 6 one can obtain the generalization of Posner's first theorem to Lie ideals in any characteristic, as well as the related results mentioned earlier.

THEOREM 7. *Let R be a prime ring, L a noncommutative Lie ideal of R, and $d, h \in \text{Der}(R) - \{0\}$. If R does not satisfy S_4 when* $\text{char}\, R = 2$, *then*:

(i) *if $dh \in$ Lie-Der(L, R) then* $\text{char}\, R = 2$ *and $h = dc$ for $c \in C$;*
(ii) *if $dh \in \text{Der}(L, R)$ then* $\text{char}\, R = 2$ *and $h = dc$ for $c \in C$;*
(iii) *if $L^{dh} \subset Z$ then* $\text{char}\, R = 2$, *$h = dc$ for $c \in C$ and $dh = 0$;*
(iv) *if $[[L, A], B] \subset C$ for $A, B \in N$, then either $A \in C, B \in C$, or* $\text{char}\, R = 2$, *$\{1, A, B\}$ is C-dependent, and $A^2, B^2 \in C$; and*
(v) *if $[A, L^d] \subset C$ for $A \in N$, then either $A \in C$ or* $\text{char}\, R = 2$, *$d = \text{ad}(A)c$ for $c \in C$, and $A^2 \in C$.*

Proof. (i) As we indicated before Lemma 4, this assumption implies that $[x_1^d, x_2^h] + [x_1^h, x_2^d]$ is a GDI for R, so the conclusion is immediate from Theorem 6.

(ii) This follows from (i) since $\text{Der}(L, R) \subset$ Lie-Der(L, R).

(iii) For all $x, y \in L$, $[x^d, y^h] + [x^h, y^d] = [x, y]^{dh} - [x^{dh}, y] - [x, y^{dh}] = [x, y]^{dh} \in Z$, so the conclusion follows from Theorem 6, except for $dh = 0$ when $\text{char}\, R = 2$ and $h = dc$. But in this case $dh = d^2c = D \in \text{Der}(R)$, because $\text{char}\, R = 2$, so $L^D \subset Z$, forcing $[L, L]^D = 0$ and $D = 0$ results from Lemma 2.

(iv) Clearly $f = [[[x_1, A], B], x_2]$ is a GPI for L, so by Theorem 1, either f is a GPI for N, or $R\overline{C} \cong M_n(\overline{C})$ and f is a GPI for $[R\overline{C}, R\overline{C}]$. Set $d = \mathrm{ad}(A)$ and $h = \mathrm{ad}(B)$. In the first case, when f is a GPI for N, we are done using $R = L = N$ in (iii) above. In the second case, $A, B \in R\overline{C}$ so the result follows again from (iii) with $R\overline{C}$ replacing R and $[R\overline{C}, R\overline{C}]$ replacing L.

(v) Consider the GDI $f = [[A, x_1^d], x_2]$ for L. If $d \notin \mathrm{Inn}(R)$ then $f = 0$ forces $A \in C$, and if $f \neq 0$ then using Theorem A forces $A \in C$ again. Hence $d = \mathrm{ad}(B)$ and the result follows from (iv).

Posner's first theorem for rings with involution. Unlike the situation for Posner's second theorem, there is no full generalization of the first theorem for $T(I)$ or $K(I)$ when R has an involution. Of course, from our earlier example one must expect to exclude $R = M_2(C)$, but things do not work well in general either, as our next example shows.

EXAMPLE. Let C be a field with $\mathrm{char}\, C \neq 2$ and let R be the ring of all countable by countable matrices over C having only finitely many nonzero entries. One can consider each element of R to be $A \in M_{2n}(C)$, in the upper left corner, and so one can define an extended symplectic involution $*$ on R. In particular, if $t \in T(R)$ then $t_{12} = t_{21} = 0$, so if $d = \mathrm{ad}(e_{12})$, then $T^{d^2} = 0$ although $d^2 \neq 0$. A similar example holds for K when T contains a nilpotent element. For C, the complex numbers, let $*$ be transpose on R. Then for $d = \mathrm{ad}(t)$ where $t = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ in the upper left corner, it follows that $K^{d^2} = 0$, although $d^2 \neq 0$.

These examples are typical in that if dh is a derivation on T or K then d and h must be inner and R must satisfy a GPI. The justification of this statement comprises the remainder of the paper.

Our next result is like Theorem 6 in that the hypothesis is implied by different, but related, conditions on composites of derivations.

THEOREM 8. *Let R be a prime ring with involution $*$, let $J = J^*$ be a nonzero ideal of R, and assume that R does not satisfy S_4. Let $d, h \in \mathrm{Der}(R) - \{0\}$, $g = [x_1^d, x_2^h] + [x_1^h, x_2^d]$, and $f = [g, x_3]$. If f is a GDI for $K(J)$, then $d, h \in \mathrm{Inn}(R)$ and R satisfies a nonzero GPI, or* $\mathrm{char}\, R = 2$ *and $h = dc$ for $c \in C$.*

Proof. Set $g^* = [x_1^d - y_1^d, x_2^h - y_2^h] + [x_1^h - y_1^h, x_2^d - y_2^d]$ and $f^* = [g^*, x_3]$. Suppose that $\{d, h\} = W$ is independent modulo $\mathrm{Inn}(R)$ and apply Theorem A to the G*-DI f^* for J to get the G*-PI $f^*_{(d,h,1)} = [[x_1 - y_1, x_2 - y_2], x_3]$ for R. This implies that $[K, K] \subset Z$, forcing R to satisfy S_4 by Lemma 3, so that d and h cannot be independent. Assume next that $d \notin \mathrm{Inn}(R)$ and $h = dc + \mathrm{ad}(A)$ for $c \in C$. In f^* replace expressions t^h with $t^d c + [t, A]$ and when $\mathrm{char}\, R = 2$ use Theorem A again to conclude that $f^*_{(d,1,1)} = [[x_1 - y_1, [x_2 - y_2, A]], x_3]$ is a G*-PI for R. Thus $[K, [K, A]] \subset C$, which yields $[[V, V], A] = 0$ for $V = [K, K]$. The subring generated by $[V, V]$ contains a nonzero ideal of R (**[9]** and **[21]**), unless R satisfies a polynomial identity. In the first case, $A \in C$, and in the second case $RC = M_n(C)$. If $RC = M_n(C)$ and * is of the second kind then $KC = RC$, so using $f^*_{(d,1,1)}$ one sees that $[RC, [RC, A]] \subset C$. But now $[L, A] \subset C$ for the Lie ideal $L = [RC, RC]$ and it follows that $[[L, L], A] = 0$ so $A \in C$ by Lemma 2. When * is of the first kind, $K(RC) = K(R)C$, so using $f^*_{(d,1,1)}$ we obtain $[k, [k, A]] \in C$ for all $k \in K(RC)$, and since $A \in RC$ we may apply Theorem 3 to conclude again that $A \in C$. Thus in all cases $A \in C$, so $h = dc$ and we are finished if $\mathrm{char}\, R = 2$. Should $\mathrm{char}\, R \neq 2$ then applying Theorem A again gives the G*-PI $f^*_{(d,d,1)} = 2c[[x_1 - y_1, x_2 - y_2], x_3]$ for R, resulting in $[K, K] \subset Z$. This gives the contradiction that R satisfies S_4, using Lemma 3. Therefore, we may assume that $d = \mathrm{ad}(A)$ and, by a similar argument, $h = \mathrm{ad}(B)$. With the obvious substitutions f^* becomes a G*-PI for J. By Theorem A, R satisfies a nonzero GPI, completing the proof, unless $f^* = 0$. But $f^* = 0$ implies $g^* = 0$ which in turn gives $g = 0$, so applying Theorem 6 finishes the proof.

Using Theorem 8 we can obtain a partial extension of Theorem 7 to rings with involution.

THEOREM 9. *Let R be a prime ring with involution, $J = J^*$ a nonzero ideal of R, and assume that R does not satisfy S_4. If $d, h \in \mathrm{Der}(R) - \{0\}$ and either $dh \in \text{Lie-Der}(K(J), R)$, $dh \in \mathrm{Der}(K(J), R)$, or $K(J)^{dh} \subset Z$, then $d, h \in \mathrm{Inn}(R)$ and R satisfies a nonzero GPI, unless $\mathrm{char}\, R = 2$ and $h = dc$. When $K(J)^{dh} \subset Z$, $\mathrm{char}\, R = 2$ and $h = dc$, then $dh = 0$.*

Proof. Each condition gives the identity of Theorem 8, so that theorem proves this one, except for the last statement. If $\mathrm{char}\, R = 2$, then $dh = d^2c = D \in \mathrm{Der}(R)$, so $K(J)^D \subset C$ and $D = 0$ follows from Theorem 3.

Our last result is a version of Theorem 9 for $T(J)$. Now $T(J)$ is a Jordan ideal of $S(R)$, which means that $t \circ s = ts + st \in T(J)$ for all

$t \in T(J)$ and $s \in S(R)$. For $d \in \mathrm{End}(R)$ we write $d \in \text{Jor-Der}(A, R)$ for $A \subset R$, if $(x \circ y)^d = x^d \circ y + x \circ y^d$ for all $x, y \in A$. Of course, $\mathrm{Der}(A, R) \subset \text{Jor-Der}(A, R)$.

THEOREM 10. *Let R be a prime ring with involution which does not satisfy S_4, let $J = J^*$ be a nonzero ideal of R, and let $d, h \in \mathrm{Der}(R) - \{0\}$. If either $dh \in \text{Jor-Der}(T(J), R)$, $dh \in \mathrm{Der}(T(J), R)$, or $T(J)^{dh} = 0$, then $d, h \in \mathrm{Inn}(R)$ and R satisfies a nonzero GPI unless* $\mathrm{char}\, R = 2$ *and $h = dc$, in which case $T(J)^{dh} = 0$ implies $dh = 0$.*

Proof. We may assume $\mathrm{char}\, R \neq 2$ since otherwise $T(J) = K(J)$ and Theorem 9 applies. Each condition implies that

$$dh \in \text{Jor-Der}(T(J), R),$$

and this yields $x^d \circ y^h + x^h \circ y^d = 0$ for all $x, y \in T(J)$. Set

$$f = (x_1^d + y_1^d) \circ (x_2^h + y_2^h) + (x_1^h + y_1^h) \circ (x_2^d + y_2^d),$$

and proceed as in Theorem 8. If $\{d, h\}$ are independent modulo $\mathrm{Inn}(R)$ then $f_{(d,h)} = (x_1 + y_1) \circ (x_2 + y_2)$ is a G*-PI for R, so $t^2 = 0$ for all $t \in S(R)$ which easily gives a contradiction to R being prime. Next, if $d \notin \mathrm{Inn}(R)$ and $h = dc + \mathrm{ad}(A)$ then the identity $f_{(d,d)} = 2c(x_1 + y_1) \circ (x_2 + y_2)$ for R forces $c = 0$ and $h = \mathrm{ad}(A)$. But now $f_{(d,1)} = (x_1 + y_1) \circ ([x_2 + y_2, A])$ is an identity for R, and so for $RC_S = R(C \cap T(C)) \subset RC$. In particular $[t^2, A] = 0$ for all $t \in RC_S$. Since the span over C_S of $\{t^2 | t \in RC_S\}$ is a Jordan ideal of $S(RC_S) = S(R)C_S$, it follows that this span contains $T(I)$ for an ideal I of RC_S [**12**; Theorem 21.12, p. 71] and this forces $A \in C$ [**12**; p. 59]. Consequently, $d = \mathrm{ad}(A)$ and $h = \mathrm{ad}(B)$, so f is now a G*-PI for J. If $f \neq 0$ then R satisfies a nonzero GPI by Theorem A, whereas if $f = 0$ then $x_1^d x_2^h + x_1^h x_2^d$ is an identity for R and we are finished by applying Theorem 6.

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