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EXTENSION OF FLOWS VIA DISCONTINUOUS FUNCTIONS

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We consider flows (X, T) with X compact Hausdorff, and suitable discontinuous functions $f : X \to W$ where W is an arbitrary compact Hausdorff space. A ring extension of the ring of all continuous complex valued functions on $X(C(X))$ is formed and equipped with a norm. The Gelfand-Naimark theorem is applied to the completion of this normed ring to produce an almost one-to-one extension $\rho : (X_f, T) \to (X, T)$.

The question of isomorphism of flows (X_f, T) and (X_g, T) corresponding to functions f and g is discussed, as well as the lifting of dynamical properties from (X, T) to (X_f, T). Extension of flows via classes of discontinuous functions is considered, showing that no new examples arise in this way. A characterization theorem for extensions is proved when T is locally compact Hausdorff, showing that every minimal almost one-to-one extension of (X, T) can be obtained using our construction.

Introduction. In this paper we are concerned with creating extensions of flows by means of discontinuous functions. In essence the device is to add a suitable discontinuous function to a ring of continuous functions, obtaining a new structure space, in such a way that a new flow is generated which is an extension of the original one. Markley investigated extensions involving splitting along a single orbit by a somewhat different approach in [6].

The motivation for this theory is two fold. One hopes to modify existing examples to introduce new desired properties. We are able to introduce any compact metric space as a fibre in the extension in a similar way to that used by N. G. Markley in his situation. The classical Sturmian discrete flows involving adding two point fibres to a minimal circle rotation, thus obtaining highly proximal flows from equicontinuous ones, are probably the best known examples of this type of extension.

Another use for this extension process is to build models of the original flow. The initial flow is replaced by one of a desired type which is still close to the original in the sense that appropriate dynamical properties lift and the lifting map is "almost" an isomorphism i.e. it is injective on a dense second category set. This type of modelling has been discussed by several authors (see for example H. B. Keynes and J. R. Robertson [3], R. Bowen and P. Walters [1], H. B. Keynes and M. Sears [4]). Although a common philosophy underlies each of these situations, each is approached in a different way. We specify conditions under which a model of this type will arise automatically, giving machinery for this type of analysis.

Section 1 discusses the construction of the extension in detail. Sections 2, 3, and 4 are concerned with lifting properties, when two extensions will be isomorphic, and extensions using more than one discontinuous function. In §5 we note that for a locally compact acting group, our construction is completely general for minimal flows in that any minimal almost automorphic (almost one to one) extension arises in this way.

1. The construction. Let (X, T) be a transformation group with compact Hausdorff phase space X, topological group T acting freely on X. Suppose that $f : X \to W$ is a function from X into another compact Hausdorff space W. Denote by $t \circ f$ the map of X into W defined by $t \circ f(x) = f(xt)$ for all $x \in X, t \in T$, and by f_x the maps of T into W given by $f_x(t) = f(xt)$ for all $x \in X$, $t \in T$.

1.1. DEFINITION. (i) $C(f) = \{x \in X : f \text{ is continuous at } x\}$

(ii) $Cf = \{x \in X : xT \subset C(f)\} = \bigcap\{C(t \circ f) : t \in T\}$

(iii) $\Omega(f, x) = \bigcap\{\mathrm{cl}(f(U \cap Cf)) : U \in \mathscr{U}(x)\}$, where $\mathscr{U}(x)$ is the neighbourhood filterbase at x, for all $x \in X$. We call $\Omega(f, x)$ the *variation* of f at x.

When there is no danger of ambiguity, Cf is written as C.

Call f *acceptable* if:

(a) C is dense in X.

(b) $\{f_x : x \in C\}$ is an equicontinuous set of maps of T into W.

(c) $f(x) \in \Omega(f, x)$ for all $x \in X$.

We will be concerned with the extension of (X, T) by acceptable functions. We remark that any continuous function is acceptable.

Let $\mathscr{R}$ be the smallest ring containing $C(X)$ (the continuous complex-valued functions on X) and all maps of the form $g \circ s \circ f$ where $g \in C(W)$ and $s \in T$, and identify functions $r, r' \in \mathscr{R}$ if $r(c) = r'(c)$ for all $c \in C$. If we define $\|\cdot\| : \mathscr{R} \to \mathbf{R}$ by $\|r\| = \sup\{|r(x)| : x \in C\}$, then $(\mathscr{R}, \|\cdot\|)$ is a normed ring. Denote its completion by $(\overline{\mathscr{R}}, \|\cdot\|)$. If $c \in C$ and $\bar{r} \in \overline{\mathscr{R}}$, then $\bar{r}(c)$ means $\lim r_n(c)$ where $r_n \in \mathscr{R}$ for all n and $r_n \to \bar{r}$.

1.2. DEFINITION.

$$\Omega(r,x) = \bigcap\{\mathrm{cl}(r(U \cap C)) : U \in \mathscr{U}(x)\} \quad \text{for all } r \in \overline{\mathscr{R}}, x \in X.$$

By the Gelfand-Naimark theorem there is a compact Hausdorff space $\hat{X}$ and an isometric isomorphism S of $\overline{\mathscr{R}}$ onto $C(\hat{X})$. We show that $(\hat{X}, T)$ is a transformation group (under a suitably defined action) which has (X, T) as a factor. First we have

1.3. THEOREM. *If X and W are metrizable and T is separable, then $\hat{X}$ is metrizable.*

Proof. Since $C(X)$ and $C(W)$ are separable, we need only find a countable dense subset of $\{s \circ f : s \in T\}$ with topology given by the metric $d(s \circ f, t \circ f) = \sup\{d_W(f(xs), f(xt)) : x \in C\}$. An obvious candidate is $\{s_n \circ f : n \in \mathbf{N}\}$ where $\{s_n : n \in \mathbf{N}\}$ is dense in T, and this is dense by the equicontinuity of $\{f_x : x \in X\}$. □

The Action of T on $\hat{X}$. Let $\hat{\rho} : C(X) \to C(\hat{X})$ be given by $\hat{\rho} = S \circ 1$ where 1 is the inclusion map of $C(X)$ into $\overline{\mathscr{R}}$. Then $\hat{\rho}$ is an isometry of $C(X)$ into $C(\hat{X})$. Accordingly there is a continuous map $\rho : \hat{X} \to X$ with $\rho(\hat{X})$ dense in X (so ρ is onto X), and such that $\hat{\rho}(h) = h \circ \rho$ for all $h \in C(X)$.

Next, choose $s \in T$. Then s determines an isometry $\hat{s}$ of $C(X)$ onto $C(X)$ given by $\hat{s}(g) = s \circ g$ for all $g \in C(X)$, where $s \circ g(x) = g(xs)$ for all $x \in X$. Similarly, s determines an isometry of $\mathscr{R}$ onto $\mathscr{R}$, which can easily be seen to extend to all of $\overline{\mathscr{R}}$, giving an isometry $\bar{s}$ of $\overline{\mathscr{R}}$ onto $\overline{\mathscr{R}}$. We use the notation $\bar{s}(r) = s \circ r$ (with the usual meaning on C) for all $r \in \overline{\mathscr{R}}$. Denote by $\hat{s}$ the corresponding isometry $S\bar{s}S^{-1}$ of $C(\hat{X})$ onto itself, and by s, again, the induced self-homeomorphism on $\hat{X}$. Writing xs for $s(x)$, we have $\hat{s}(h)(x) = h(s(x)) = s \circ h(x)$ for all $h \in C(\hat{X})$, $x \in \hat{X}$ and $s \in T$, and $\rho(xs) = \rho(x)s$ for all $x \in \hat{X}$ and $s \in T$.

If $s, t \in T$, clearly $x(st) = (xs)t$ for all $x \in \hat{X}$, so we can consider T to be a group acting freely on $\hat{X}$, and $\rho : \hat{X} \to X$ is equivariant. It remains to show that the map $\hat{X} \times T \to \hat{X}$ given by $(x, t) \to xt$ is continuous.

First note that since $\{f_x : x \in C\}$ is an equicontinuous family of maps, it follows that $\{r_x : x \in C\}$ is equicontinuous for all $r \in \overline{\mathscr{R}}$, where $r_x(t) = r(xt)$. Also if $x \in C$, $s, t \in T$ and $r \in \overline{\mathscr{R}}$ then $\|s \circ r - t \circ r\| = \sup\{|r_x(s) - r_x(t)| : x \in C\}$. Hence if $\varepsilon > 0$ and $t \in T$ there is an open neighbourhood $U(t)$ of t such that $\|s \circ r - t \circ r\| < \varepsilon$

whenever $s \in U(t)$. Thus if $\{s_\alpha\}$ is a net in T converging to t, then $s_\alpha \circ r \to t \circ r$ in $\overline{\mathscr{R}}$ for all $r \in \overline{\mathscr{R}}$. Now choose $g \in C(\hat{X})$. Then $g = Sr$ for some $r \in \overline{\mathscr{R}}$, and $s_\alpha \circ g = \hat{s}_\alpha(g) = S\bar{s}_\alpha S^{-1}(g) = S(s_\alpha \circ r) \to S(t \circ r) = S\bar{t}S^{-1}(g) = t \circ g$.

We have shown that if $\{s_\alpha\}$ is a net in T and $s_\alpha \to t$, then $s_\alpha \circ g \to t \circ g$ for all $g \in C(\hat{X})$. We can now prove the continuity of the action $(x, t) \to xt$ of T on $\hat{X}$.

Suppose that $\{(x_\alpha, s_\alpha)\}$ is a net in $\hat{X} \times T$ converging to (x, t), and suppose wlog that $x_\alpha s_\alpha \to z$. We show that $z = xt$. Choose any $g \in C(\hat{X})$. Then $s_\alpha \circ g \to t \circ g$ i.e. $\|s_\alpha \circ g - t \circ g\| \to 0$ i.e.

$$\sup\{|g(ys_\alpha) - g(yt)| : y \in \hat{X}\} \to 0.$$

Then $|g(x_\alpha s_\alpha) - g(x_\alpha t)| \to 0$ so $g(z) = g(xt)$. Since $g \in C(\hat{X})$ is arbitrary, $z = xt$. Hence $(\hat{X}, T)$ is a transformation group and $\rho : (\hat{X}, T) \to (X, T)$ is a homomorphism.

Each $r \in \overline{\mathscr{R}}$ defines a map $r : C \to \mathbf{C}$ which is bounded and continuous. Hence we can define the map $\gamma : \overline{\mathscr{R}} \to C(\beta C)$ to be the map which takes $r \in \overline{\mathscr{R}}$ to its unique continuous extension $\gamma r : \beta C \to \mathbf{C}$, where βC is the Stone-Čech compactification of C (and we regard C as a subset of βC). γ is clearly an isomorphism, so $\hat{h} = \gamma S^{-1}$ is an isomorphism of $C(\hat{X})$ into $C(\beta C)$. As usual there is an induced continuous onto map $h : \beta C \to \hat{X}$, say, such that $\hat{h}(g) = g \circ h$ for all $g \in C(\hat{X})$. Finally, let $\tau : \beta C \to X$ be the extension to βC of the identity $1 : C \to C$.

1.4. THEOREM. (i) $\tau = \rho h$

(ii) $(\hat{X}, h|C)$ *is a compactification of* C

(iii) $\rho : h(C) \to C$ *is a homeomorphism of* $h(C)$ *onto* C.

Proof. (i) If we let $\hat{\tau}(g) = g \circ \tau$ for all $g \in C(X)$, we obtain $\hat{\tau} = \hat{h}\hat{\rho}$.

(ii) Since C is dense in βC and h is onto, $h(C)$ is dense in $\hat{X}$. If $c_1, c_2 \in C$ and $h(c_1) = h(c_2)$, then $c_1 = c_2$ by (i). Clearly $h : C \to h(C)$ is continuous, so we need only show that $h : C \to h(C)$ is closed, say.

First note that if $B \subset X$, then $B \cap C \subset C \subset \beta C$ and $h(B \cap C) = \rho^{-1}B \cap h(C)$ from (i). Now, $\tau : C \to C$ is the identity homeomorphism of $C \subset \beta C$ onto $C \subset X$, so any closed subset of $C \subset \beta C$ has the form $B \cap C$ for some closed subset B of X. But $h(B \cap C) = \rho^{-1}B \cap h(C)$ is closed in $h(C)$, so $h : C \to h(C)$ is a closed map.

(iii) This follows since every closed subset of $h(C)$ has the form $h(B \cap C) = \rho^{-1}B \cap C$ where B ranges over the closed sets of X. □

Thus $\rho^{-1}C = h(C)$ is densely embedded as a copy of C in $\hat{X}$, so we do not distinguish between $C \subset X$ and $C \subset \hat{X}$; we will usually write $\rho^{-1}c = c$ for $c \in C$. $\rho : (\hat{X}, T) \to (X, T)$ is thus an almost automorphic extension (i.e. there are fibres of ρ which are points).

The lift of f to $\hat{X}$. Let $t \in T$ and consider the map $(t \circ f)^{\frown}: C(W) \to \overline{\mathscr{R}}$ given by $(t \circ f)^{\frown}(g) = g \circ t \circ f$. Then $S(t \circ f)^{\frown}$ is a homomorphism of $C(W)$ into $C(\hat{X})$, so there is a corresponding continuous map $(t \circ f)'$: $\hat{X} \to W$ such that $S(t \circ f)^{\frown}(g) = g \circ (t \circ f)'$ for $g \in C(W)$. Let $f' = (e \circ f)'$ where e is the identity of T. It follows that $(t \circ f)' = t \circ f'$ for all $t \in T$, so $S(t \circ f)^{\frown}(g) = g \circ t \circ f'$ for all $g \in C(W)$. The usefulness of these functions $t \circ f'$ will be apparent after a short digression.

Suppose $\psi : \overline{\mathscr{R}} \to \mathbf{C}$ is a homomorphism of $\overline{\mathscr{R}}$ onto $\mathbf{C}$. Then ψS^{-1} is a homomorphism of $C(\hat{X})$ onto $\mathbf{C}$ and so there is a point $y \in \hat{X}$ such that $\psi S^{-1} = \theta_y$, where $\theta_y : C(\hat{X}) \to \mathbf{C}$ is defined by $\theta_y(g) = g(y)$ for all $g \in C(\hat{X})$. In particular if $x \in C$ and $\psi = \psi_x$, the evaluation homomorphism at x, then it can be shown that $\rho y = x$.

1.5. LEMMA. *If $r \in \overline{\mathscr{R}}$ and $c \in C$, then $(Sr)(c) = r(c)$.*

Proof. By the above there is $y \in \rho^{-1}c$ such that $\psi_c S^{-1} = \theta_y$. But $\rho^{-1}c = c$, so $\psi_c = \theta_c S$. Now if $r \in \overline{\mathscr{R}}, r(c) = \psi_c(r) = \theta_c(Sr) = (Sr)(c)$.

Denote the fibre $\rho^{-1}x$ over $x \in X$ by $\hat{X}_x$.

1.6. LEMMA. *For $r \in \overline{\mathscr{R}}$: (i) if $y \in \hat{X}$ then $(Sr)(y) \in \Omega(r, \rho y)$ i.e. $Sr : \hat{X}_x \to \Omega(r, x)$ for all $x \in X$.*

(ii) $Sr : \hat{X}_x \to \Omega(r, x)$ is onto for all $x \in X$.

Proof. (i) Suppose that $y \in \hat{X}_x$. Choose a net $\{c_\alpha\} \subset C \subset \hat{X}$ such that $c_\alpha \to y$. Then $Sr(c_\alpha) = r(c_\alpha)$ for all α, and $Sr(c_\alpha) \to Sr(y)$. Thus $r(c_\alpha) \to Sr(y)$. Finally, $c_\alpha = \rho c_\alpha \to \rho y = x$, so $Sr(y) \in \Omega(r, x)$.

(ii) Let $a \in \Omega(r, x)$. Then there is a net $\{c_\alpha\} \subset C \subset X$ such that $c_\alpha \to x$ and $r(c_\alpha) \to a$. In $\hat{X}$ we may assume wlog that $c_\alpha \to y$, say. Clearly $Sr(y) = a$ and $\rho y = x$. □

We now return to the main development.

1.7. THEOREM. *If $x \in X$, $t \in T$ and $g \in C(W)$: (i) $S(g \circ t \circ f) = g \circ t \circ f'$.*

(ii) $f'(c) = f(c)$ for all $c \in C$. Moreover, $t \circ f'(c) = t \circ f(c)$ for all $c \in C$.

(iii) $t \circ f' : \hat{X}_x \to \Omega(t \circ f, x)$ *and is onto.*
(iv) *if* $x \in C(t \circ f)$, *then* $f'(yt) = f(xt)$ *for all* $y \in \hat{X}_x$
(v) $\Omega(f, x) = \{f(x)\}$ *iff* $x \in C(f)$.

Proof. (i) $S(g \circ t \circ f) = (S(t \circ f)^\frown)(g) = (t \circ f)'(g) = g \circ t \circ f'$.

(ii) For any $g \in C(W)$, $g(f'(c)) = S(g \circ e \circ f)(c) = g \circ e \circ f(c) = g(f(c))$ by Lemma 1.5 and by (i) above. So $f'(c) = f(c)$ for all $c \in C$.

(iii) This follows by (ii) and the fact that C is dense in X and $\hat{X}$.

(iv) follows from (iii).

(v) Suppose $x \notin C(f)$. Then there is an open neighbourhood V of $f(x)$ in W such that $f(U) \cap (W \backslash \overline{V}) \neq \varnothing$ for each $U \in \mathscr{U}(x)$, so for each $U \in \mathscr{U}(x)$ there is a point $x_U \in U$ with $f(x_U) \in W \backslash \overline{V}$. Since by hypothesis $f(y) \in \Omega(f, y)$ for each $y \in X$, it follows that for each $U \in \mathscr{U}(x)$ there is $c_U \in C$ such that $c_U \in U$ and $f(c_U) \in W \backslash \overline{V}$. Suppose wlog that $f(c_U) \to a$. Then $a \in \Omega(f, x)$ and $a \neq f(x)$, so $\Omega(f, x) \not\supseteq \{f(x)\}$. The converse is trivial. □

1.8. COROLLARY. *The following are equivalent:*
(i) $\Omega(f, xt) = \{f(xt)\}$ *for all* $t \in T$.
(ii) $x \in C$.
(iii) $\hat{X}_x$ *is a singleton.*
So $\rho^{-1}x$ *is a point if and only if* $x \in C$.

An embedding of $\hat{X}$. For each $t \in T$, let $W_t = W$, and define $\mathscr{W} = \Pi\{W_t : t \in T\}$ and $F : \hat{X} \to \mathscr{W} \times X$ by $F = \Pi\{t \circ f' : t \in T\} \times \rho$.

1.9. THEOREM. *F is an embedding.*

Proof. F is clearly continuous and closed, so we need only show that F is injective. Suppose that $F(x_1) = F(x_2)$. Then $\rho x_1 = \rho x_2 = x$ say. We show that $(Sr)(x_1) = (Sr)(x_2)$ for all $r \in \overline{\mathscr{R}}$ i.e. $g(x_1) = g(x_2)$ for all $g \in C(\hat{X})$, whence $x_1 = x_2$.

Clearly $(Sr)(x_1) = (Sr)(x_2)$ for all $r \in \mathscr{R}$, by (i) of Theorem 1.7, since $f'(x_1 t) = f'(x_2 t)$ for all $t \in T$ by hypothesis. Now let $r \in \overline{\mathscr{R}}$. Then there is a sequence $(r_n) \subset \mathscr{R}$ such that $\|r_n - r\| \to 0$, so that $\|Sr_n - Sr\| \to 0$. But

$$\begin{aligned} |Sr(x_1) - Sr(x_2)| &\leq |Sr(x_1) - Sr_n(x_1)| + |Sr_n(x_1) - Sr_n(x_2)| \\ &\quad + |Sr_n(x_2) - Sr(x_2)| \\ &= |Sr(x_1) - Sr_n(x_1)| + |Sr_n(x_2) - Sr(x_2)| \quad \text{for any } n. \end{aligned}$$

For $\varepsilon > 0$ choose n such that $\|Sr_n - Sr\| < \varepsilon/2$. Then $|Sr(x_1) - Sr(x_2)| < \varepsilon$. As $\varepsilon > 0$ is arbitrary, $Sr(x_1) = Sr(x_2)$. □

1.10. COROLLARY. *If Y is a space and $G : Y \to \hat{X}$ is a map, then G is continuous if and only if ρG is continuous and $t \circ f' \circ G$ is continuous for all $t \in T$.*

For all $x \in X$ let $D(x) = \{t \in T : xt \notin C(f)\}$, and let $\mathscr{D}(x) = \Pi\{\Omega(f, xs) : s \in D(x)\}$. Define $F_x : \hat{X}_x \to \mathscr{D}(x)$ by $F_x = \Pi\{s \circ f' : s \in D(x)\}$.

1.11. THEOREM. (i) *If $D(x) \neq \varnothing$, then $F_x : \hat{X}_x \to \mathscr{D}(x)$ is an embedding.*

(ii) *$D(x) = \varnothing$ iff $x \in C$ iff $\hat{X}_x$ is a singleton.*

Proof. (i) F_x is continuous and closed. Suppose $F_x(x_1) = F_x(x_2)$ for some $x_1, x_2 \in \hat{X}_x$. Then $t \circ f'(x_1) = t \circ f'(x_2)$ for all $t \in D(x)$, and the same is true of all $t \notin D(x)$ by Theorem 1.7 (iv), so that $F(x_1) = F(x_2)$. Now by 1.9, $x_1 = x_2$.

(ii) is clear by 1.8. □

An interesting special case of this theorem is

1.12. COROLLARY. *Suppose that $D(x)$ is at most one point for all $x \in X$. If $xt \notin C(f)$, then $t \circ f' : \hat{X}_x \to \Omega(f, xt)$ is a homeomorphism of $\hat{X}_x$ onto $\Omega(f, xt)$.*

Proof. By 1.11 and 1.7(iii). □

REMARKS. (i) It is not true in general that F_x of 1.11 is onto $\mathscr{D}(x)$, as the examples below demonstrate.

(ii) Corollary 1.12 provides a way of "building in" precisely the fibres that we want. It is just necessary to define appropriate functions f with a single discontinuity along any particular orbit.

1.13. COROLLARY. *If $D(x)$ is at most one point for each $x \in X$, if each $\Omega(f, x)$ is connected, then for an open or closed $A \subset X$, $\rho^{-1}A$ is connected if and only if A is connected. In particular, $\hat{X}$ is connected if X is connected.*

1.14. EXAMPLES. Let $X = [0, 1)$ where addition in X is modulo 1 i.e. X is the circle group, and let $\phi : X \to X$ be a nontrivial group rotation. Then X and ϕ determine a discrete transformation group

$(X, \mathbf{Z})$. Let $I = [-1, 1]$ and choose any $a, b \in I$. Choose $x \in X$ such that $0 < x < \phi(x) < 1$, and define $f : X \to I$ by

$$f(t) = \begin{cases} at/x, & 0 \leq t \leq x, \\ \sin(1/(t - x)), & x < t \leq \frac{1}{2}(x + \phi(x)), \\ \sin(1/(\phi(x) - t)), & \frac{1}{2}(x + \phi(x)) \leq t < \phi(x), \\ (1 - t)b/(1 - \phi(x)), & \phi(x) \leq t < 1. \end{cases}$$

Then f is acceptable with

$$C(f) = X \backslash \{x, \phi(x)\}, \qquad Cf = X \backslash \{\phi^n x : n \in Z\},$$
$$D(x) = \{0, 1\} \quad \text{and} \quad \Omega(f, x) = I = \Omega(f, \phi x).$$

The extension $(\hat{X}, \mathbf{Z})$ of $(X, \mathbf{Z})$ by f is determined by the space $\hat{X}$ and a surjective homeomorphism $\hat{\phi} : \hat{X} \to \hat{X}$ with $\rho\hat{\phi} = \phi\rho$.

If $F_x : \hat{X}_x \to \Omega(f, x) \times \Omega(f, \phi(x))$ is the embedding of Theorem 1.11, then $F_x = f' \times 1 \circ f' = f' \times f'\hat{\phi}$, and it is easy to see that $F_x(\hat{X}_x) = \{a\} \times [-1, 1] \cup [-1, 1] \times \{b\}$, so that the fibre $\hat{X}_x$ over x is isomorphic to this space. Clearly F_x is not onto $\mathscr{D}(x)$. □

Definition. The extension $(\hat{X}, T)$ of (X, T) via an acceptable function f is denoted by (X_f, T) and is called the f-extension of (X, T).

A metric for X_f. We know that when X and W are metrizable and T is separable, then X_f is metrizable. The following corollary of 1.9 allows us to write down a convenient metric for X_f.

1.15. Theorem. *Suppose $\{t_n : n \in \mathbf{N}\}$ is a dense subset of T, and let $F' = \Pi\{t_n \circ f' : n \in \mathbf{N}\} \times \rho$. If $W_n = W$ for all $n \in \mathbf{N}$, then $F' : X_f \to \Pi\{W_n : n \in \mathbf{N}\} \times X$ is an embedding.*

Now suppose X and W are metrizable with metrics d_X and d_W respectively. In view of 1.15 it is clear that the map $d_f : X_f \times X_f \to [0, \infty)$ given by

$$d_f(x, y) = \max\left[\sup_{n\in\mathbf{N}}\{\min(d_W(f'(xt_n), f'(yt_n))/2^n, 1)\}, d_X(\rho x, \rho y)\right]$$

is a metric for X_f.

Another embedding of X_f. To end this section we state yet another corollary of 1.9. Let W^T be the space of all continuous maps from T into W with compact-open topology.

1.16. THEOREM. *If T is a locally compact Hausdorff group, then the map $x \to (f'_x, \rho x)$ is an embedding of X_f in $W^T \times X$.*

Proof. Let $\pi : X_f \times T \to X_f$ be given by $\pi(x,t) = xt$, and let $G : X_f \times T \to W$ be given by $G = f' \circ \pi$. Then G is continuous, so the map $G' : X_f \to W^T$ defined by $G'(x) = f'_x$ is also continuous. Finally, $G' \times \rho : X_f \to W^T \times X$ is continuous, closed and, by 1.9, injective. □

2. Lifting some dynamical properties.

2.1. LEMMA. (i) *If x is a transitive point of (X,T), then every point of $\rho^{-1}x$ is transitive in (X_f, T).*

(ii) *If (x,x') is a transitive point of $(X \times X, T)$, then every point of $\rho^{-1}x \times \rho^{-1}x'$ is transitive in $(X_f \times X_f, T)$.*

Proof. (i) Choose $y \in \rho^{-1}x$, and let $B = \overline{yT}$. Then ρB is closed in X and $\rho B \supset xT$, so $\rho B = X$. Thus B meets every fibre of ρ, so $B \supset C$. Therefore $B = X_f$.

(ii) Similar.

2.2. THEOREM. (i) *(X_f, T) is topologically transitive iff (X,T) is topologically transitive.*

(ii) *(X_f, T) is minimal iff (X,T) is minimal.*

(iii) *(X_f, T) is weak mixing iff (X,T) is weak mixing.*

2.3. THEOREM. *If (X,T) is minimal, then (X_f, T) is a minimal proximal extension.*

Proof. Suppose $y, y' \in X_f$ are distinct and $\rho y = \rho y' = x$. If $c \in C$ we can choose a net $(t_\alpha) \subset T$ such that $xt_\alpha \to c$. Hence $\rho(yt_\alpha) \to c$ and $\rho(y't_\alpha) \to c$, so $yt_\alpha \to c$ and $y't_\alpha \to c$. □

This result is clearly true for any almost automorphic extension of a minimal flow (Brönstein 3.12.7).

2.4. DEFINITION. (i) In the group T, write $t_\alpha \to \infty$ if the net $(t_\alpha) \subset T$ has no convergent subnets. (If T is locally compact Hausdorff, this means that $t_\alpha \to \infty$ in the one-point compactification $T^* = T \cup \{\infty\}$.)

(ii) If (X,T) is a transformation group and $A \subset X$, then A is called asymptotic if whenever $(t_\alpha) \subset T$ is a net with $t_\alpha \to \infty$ and there is $x \in A$ such that $xt_\alpha \to y$ say, then $zt_\alpha \to y$ for all $z \in A$.

(iii) An extension $\pi : (Y,T) \to (X,T)$ is called an asymptotic extension if $\pi^{-1}x$ is asymptotic for all $x \in X$.

2.5. THEOREM. (i) *If* $x \in X$ *and* $\overline{D(x)}$ *is compact in* T, *then* $\rho^{-1}x$ *is asymptotic.*

(ii) *If* $\overline{D(x)}$ *is compact for all* $x \in X$, *then* $\rho : (X_f, T) \to (X, T)$ *is an asymptotic extension.*

Proof. (i) Suppose $x \in X$ with $\overline{D(x)}$ compact. Let $y \in \rho^{-1}x$ and let (t_α) be a net in T such that $t_\alpha \to \infty$ and $yt_\alpha \to z$ say. Choose $y' \in \rho^{-1}x$ and suppose wlog that $y't_\alpha \to z'$. If $t \in T$ then $t_\alpha t \to \infty$, so there is β such that $t_\alpha t \notin \overline{D(x)}$ for all $\alpha \geq \beta$, and since $\rho y' = \rho y = x, f'(y't_\alpha t) = f(xt_\alpha t) = f'(yt_\alpha t)$ for all $\alpha \geq \beta$, by 1.7. Therefore $f'(z't) = f'(zt)$. Since t was arbitrary in T, 1.9 implies $z' = z$.

(ii) is immediate, by (i). □

2.6. COROLLARY. *If* $D(x)$ *is finite for all* $x \in X, \rho : (X_f, T) \to (X, T)$ *is an asymptotic extension.*

As an application of the metric derived from 1.15 when $T = \mathbf{Z}$ and (X, d_X) and (W, d_W) are metric space, we examine expansiveness. An acceptable function $f : X \to W$ is said to satisfy property $(*)$ if there exists $\varepsilon > 0$ such that whenever $\omega_1, \omega_2 \in \Omega(f, x)$ for some $x \in X$ and $\omega_1 \neq \omega_2$, then $d_W(\omega_1, \omega_2) > \varepsilon$.

2.7. THEOREM. *If* f *satisfies property* $(*)$ *on an expansive transformation group* $(X, \mathbf{Z})$ *with metric phase space, then* $(X_f, \mathbf{Z})$ *is expansive.*

Proof. Suppose $x_1, x_2 \in X_f$ are distinct. If $\rho x_1 = \rho x_2 = x$ say, then by Theorems 1.9 and 1.7(iii) there is an $n \in \mathbf{Z}$ such that $n \circ f'(x_1) \neq n \circ f'(x_2)$ and both belong to $\Omega(f, xn)$. By hypothesis $d_W(f'(x_1 n), f'(x_2 n)) > \varepsilon$. Now for any $m \in \mathbf{Z}$ we have

$$d_f(x_1 m, x_2 m) = \sup\{\min(d_W(f'(x_1 i), f'(x_2 i))/2^{|i-m|}, 1) : i \in \mathbf{Z}\}$$

since $\rho x_1 = \rho x_2$. Hence $d_f(x_1 n, x_2 n) > \min(\varepsilon, 1)$.

If on the other hand $\rho x_1 \neq \rho x_2$, there is $n \in Z$ such that $d_X(\rho x_1 n, \rho x_2 n) > \delta$, where δ is an expansive constant for (X, Z). Hence $\min(\varepsilon, \delta, 1)$ is an expansive constant for $(X_f, \mathbf{Z})$. □

REMARK. In the case where $D(x)$ is a singleton or empty for each $x \in X$, the expansiveness of $(X_f, \mathbf{Z})$ implies that f satisfies property $(*)$.

2.8. PROPOSITION. *Let $f : X \to W$ be acceptable on (X, T), and suppose that T is separable with $\{t_n : n \in \mathbf{N}\}$ dense in T. Then*

(i) $C = \bigcap\{C(t_n \circ f) : n \in \mathbf{N}\}$.

(ii) If $C(t_n \circ f)$ is a G_δ set for each $n \in \mathbf{N}$, then $C \subset X$ and $\rho^{-1}C \subset X_f$ are G_δ sets, so C and $\rho^{-1}C$ are dense second category sets in their respective spaces.

Proof. (i) By definition $C \subset \bigcap\{C(t_n \circ f) : n \in \mathbf{N}\}$. Now suppose $x \in \bigcap\{C(t_n \circ f) : n \in \mathbf{N}\}$, and let $y, y' \in \rho^{-1}x$. If $t \in T$ is arbitrary, there is a sequence (t_k) from $\{t_n : n \in \mathbf{N}\}$ such that $t_k \to t$. Now $f'(yt_k) = f'(y't_k)$ for all k by 1.7(iv), so $f'(yt) = f'(y't)$ by continuity. 1.9 now implies that $y = y'$, so $\rho^{-1}x$ is a singleton. Now by 1.11(ii), $x \in C$.

(ii) follows from (i). □

The results of this section enable us to construct models of flows in the sense described in the introduction. All that is required (provided T is separable) is that $C(t_n \circ f)$ is a dense G_δ set for each t_n (usually we can arrange that these sets are actually open). Dynamical properties lift as in the above theorems.

In the case of discrete flows, the process is immediate by choosing a suitable function (e.g. a characteristic function defined on the circle with minimal rotation immediately produces a minimal extension and the lift is injective except on a set of 1° Category). When handling continuous flows we need to construct our functions more carefully using sections. As an example we outline the construction of a suspension model for real flows using our approach (see **[4]**).

DEFINITION. Let $(X, \mathbf{R})$ be a real flow. A section of a point $x \in X$ is a closed set $S \subset X$ with $x \in S$ and such that for some $\delta > 0$ (a section time), $S \cap S(0, \delta] = S \cap S[-\delta, 0) = \emptyset$.

We define S^* as the relative interior of S i.e. $S^* = \mathrm{Int}(S(-\delta, \delta)) \cap S$. Then S^* is open in S, $S^*(-\delta, \delta)$ is open in X, and we can choose sections in such a way that S^* is dense in S. Furthermore we have the following result (Lemma 7 of **[1]**):

2.9. THEOREM. *There is a $\zeta > 0$ so that the following holds: For each $\alpha > 0$ there is a finite family $\mathscr{S}$ of pairwise disjoint sections of time ζ and diameter at most α such that $X = Y[-\alpha, 0] = Y[0, \alpha]$ where $Y = \bigcup_{S \in \mathscr{S}} S^*$.*

2.10. THEOREM. *Every real flow is modelled by a real suspension.*

Proof. We will construct an extension which has a global section i.e. a section Γ with the property that every orbit intersects Γ and that $\Gamma(-\delta,\delta)$ is open for δ sufficiently small. It is well known that such a flow can be realised as a suspension over Γ.

Let $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ be chosen as above with ζ small enough so that $S_i[-\zeta,\zeta] \cap S_j[-\zeta,\zeta] = \varnothing$ for $i \neq j$. Define f on X by $f(x \cdot t) = 1 - |t|/\zeta$ if $x \in S_i$ (for some i) and $|t| < \zeta$, and $f(x) = 0$ otherwise. It is clear that f is an acceptable function which satisfies the conditions of Proposition 2.8. Form the extension $(X_f, \mathbf{R})$. We will show that $\Gamma = f'^{-1}(1)$ is a global section for this flow. First note that Γ will be a section with time ζ.

Now if $x \in X_f$, then for some $t \in \mathbf{R}$ and S_i, $\rho(xt) \in S_i^*$. Since $S_i^* \subset C(f)$, $f'(xt) = f(\rho(x)t) = 1$. Thus every orbit intersects Γ. Suppose that $\Gamma(-\zeta,\zeta)$ is not open. Then we can find a point $x \in \Gamma(-\zeta,\zeta)$ and a sequence $c_n \in \rho^{-1}(C)$ with $c_n \notin \Gamma(-\zeta,\zeta)$ and $c_n \to x$. Equivalently, we can find $x \in \Gamma$ and a sequence $c_n \notin \Gamma(-\delta,\delta)$ and $c_n \to x$ for some $\delta > 0$. Now $f(c_n) = f'(c_n) \to 1$. We deduce $c_n = s_n t_n$ where $c_n \in S_i$ for some i and $t_n \in \mathbf{R}$ with $t_n \to 0$. Thus $c_n(-t_n) \in S_i$, so $f'(c_n(-t_n)) = 1$ and for n sufficiently large $c_n \in \Gamma(-\delta,\delta)$ which is a contradiction.

REMARK. Since $\mathbf{R}^n$ parallels can be obtained for Theorem 2.9, the same process can be used to obtain models for R^n flows which have global sections. A difficult open question is which of those flows are $\mathbf{R}^n$ suspensions (see [5]).

3. An isomorphism theorem. Let f and g be acceptable functions on a flow (X,T) producing extensions $\rho_f : (X_f,T) \to (X,T)$ and $\rho_g : (X_g,T) \to (X,T)$ respectively. For each $x \in X$ let $\omega_f(x) = \Pi\{\Omega(f,xt) : t \in T\} \times \{x\}$, and let $\Omega(f) = \bigcup\{\omega_f(x) : x \in X\}$. Referring to Theorem 1.9 and its notation, regard $\Omega(f)$ as a subspace of $\mathscr{W}_f \times X$; then $F : X_f \to \Omega(f)$ is an embedding. Let $G : X_g \to \Omega(g)$ be the corresponding embedding for X_g.

One might imagine that if f and g have homeomorphic variations at each point, the resulting extensions (X_f,T) and (X_g,T) would be isomorphic. In fact, one needs some sort of uniformity across the variations in the sense that not only can we map $\omega_f(x)$ homeomorphically onto $\omega_g(x)$, but that the resulting collection of homeomorphisms acts in a continuous way from $\Omega(f)$ to $\Omega(g)$. In this case the resulting isomorphism maps fibres to fibres. This is made concrete in the next theorem.

3.1. THEOREM. *Suppose that f and g are acceptable on (X, T) and let $m : (X, T) \to (X, T)$ be a homomorphism. If there is a continuous surjection $m' : \Omega(f) \to \Omega(g)$ such that $m' : \omega_f(x) \to \omega_g(mx)$ and is onto for all $x \in X$, then there is a homomorphism $\pi : (X_f, T) \to (X_g, T)$ such that $\rho_g \pi = m \rho_f$. Moreover, π is an isomorphism if m' is injective.*

Proof. If $c \in Cf$ then $\omega_f(c)$ is a point. Therefore $\omega_g(mc)$ is a point, so $mc \in Cg$. Thus $m(Cf) \subset Cg$. Also, $m'(\{f'(ct)\}_t \times \{c\}) = (\{g'(mct)\}_t \times \{mc\})$ so $m'F = Gm$ on Cf.

If $x \in X_f$, choose a net $(c_\alpha) \subset Cf \subset X_f$ converging to x. We may assume that $mc_\alpha \to y$ say, in X_g. But $m'F(c_\alpha) \to m'F(x)$ and $Gm(c_\alpha) \to G(y)$, so $m'F(x) = G(y)$. This shows that $m'F(X_f) \subset G(X_g)$, so $\pi : X_f \to X_g$ given by $\pi = G^{-1}m'F$ is well-defined and continuous.

π is equivariant. For if $c \in Cf$ and $t \in T$, then $\pi(ct) = m(ct) = m(c)t = \pi(c)t$ and continuity does the rest. Similarly, if $c \in Cf$ then $\rho_g \pi(c) = \rho_g m(c) = m(c) = m\rho_f(c)$, so again by continuity, $\rho_g \pi = m\rho_f$.

π is also surjective: $\rho_g \pi(X_f) = m\rho_f(X_f) = mX = X$ as m is surjective, so $\pi(X_f)$ meets every fibre of X_g. Thus $\pi(X_f) \supset Cg$, and, being closed, $\pi(X_f) = X_g$. Clearly π is injective if m' is injective. □

3.2. COROLLARY. *Let $f : X \to W$ be acceptable on (X, T) and let W' be a compact Hausdorff space and $H : W \to W'$ continuous. Then $Hf : X \to W'$ is acceptable and there is a homomorphism $\pi : (X_f, T) \to (X_{Hf}, T)$ such that $\rho_{Hf} \pi = \rho_f$. π is injective if H is injective.*

3.3. COROLLARY. *If $f : X \to W$ is acceptable on (X, T), then there is an acceptable map $g : X \to W \times X$ such that $\Omega(g, x) \cap \Omega(g, y) = \varnothing$ for all distinct $x, y \in X$, and an isomorphism $\pi : (X_g, T) \to (X_f, T)$ such that $\rho_f \pi = \rho_g$.*

Proof. Let $g : X \to W \times X$ be defined by $g(x) = (f(x), x)$ and apply 3.1. □

In view of 3.3, we remark that any acceptable map f on a transformation group (X, T) can be replaced by an injective acceptable map producing the same extension of (X, T).

4. Class extensions. So far we have considered the extension of flows by means of a single function. A natural question is whether we could obtain different examples by using classes of functions. We will show below that this is not the case. We just sketch the development which parallels §1.

Let $\mathcal{F} = \{f_i : i \in \Lambda\}$ be a set of functions on the phase space X of a transformation group (X, T), the range space W_i of f_i being compact Hausdorff for each $i \in \Lambda$. For each i let $C_i = Cf_i$, and let f_{ix} $(x \in X)$, $t \circ f_i$ $(t \in T)$ and $\Omega(f_i, x)$ (wrt C_i) have their usual meanings.

4.1. DEFINITION. $\mathcal{F}$ is called acceptable on (X, T) if:

(i) $C = \bigcap\{C_i : i \in \Lambda\}$ is dense in X.

(ii) $\{f_{ix} : x \in C_i\}$ is equicontinuous on T for each $i \in \Lambda$.

(iii) $f_i(x) \in \Omega(f_i, x)$ for each $x \in X, i \in \Lambda$.

(iv) $\{f_{ix} : i \in \Lambda, x \in C\}$ is equicontinuous.

Define $\mathcal{R}\mathcal{F}$ to be the smallest ring containing $C(X)$ and all compositions $g \circ t \circ f_i$ where $t \in T, i \in \Lambda$ and $g \in C(W_i)$. Proceeding in a way similar to that of §1, one obtains a transformation group $(X\mathcal{F}, T)$ with associated homomorphism $\rho : (X\mathcal{F}, T) \to (X, T)$, and the set C is densely embedded in $X\mathcal{F}$ with $\rho^{-1}(x)$ a singleton iff $x \in C$.

If we let $f : X \to \Pi\{W_i : i \in \Lambda\}$ be defined by $f = \Pi f_i$, then f is acceptable on (X, T) and

4.2. THEOREM. *There is an isomorphism* $\pi : (X_f, T) \to (X\mathcal{F}, T)$ *such that* $\rho\pi = \rho_f$.

On the other hand, construct the extension $\rho_i : (X_i, T) \to (X, T)$ of (X, T) via f_i for each $i \in \Lambda$. Let $C' \subset \Pi\{X_i : i \in \Lambda\}$ be defined by $C' = \{\{c_i\} : c_i = c \forall i \text{ for some } c \in C\}$. Let $Y = \mathrm{cl}(C)$ in $\Pi\{X_i : i \in \Lambda\}$. Then (Y, T) is a transformation group and if $\{y_i\} \in Y$, $\rho_i y_i = \rho_j y_j$ for each $i, j \in \Lambda$. Let $p : (Y, T) \to (X, T)$ be any of the maps ρ_i.

4.3. THEOREM. *There is an isomorphism* $\pi : (X\mathcal{F}, T) \to (Y, T)$ *such that* $p\pi = \rho$.

4.4. THEOREM. *Let* (Z, T) *be a transformation group and* $\mathcal{F}$ *an acceptable set of functions on* (X, T), *and suppose that for each* $i \in \Lambda$ *there is a homomorphism* $\pi_i : (Z, T) \to (X_i, T)$ *such that* $\rho_i \pi_i = \rho_j \pi_j$ *for each* $i, j \in \Lambda$. *Then there is a homomorphism* $\pi : (Z, T) \to (X\mathcal{F}, T)$. *If* $\{\pi_i : i \in \Lambda\}$ *separates points of* Z, π *is an isomorphism.*

4.4. DEFINITION. The fibered product $\#\{(X_i, T) : i \in \Lambda\}$ of a set of extensions $p_i : (X_i, T) \to (X, T)$ is the transformation group $(\#X_i, T)$ where

$$\#X_i = \{\{x_i\} \in \Pi X_i : p_i x_i = p_j x_j \ \forall i, j \in \Lambda\}.$$

It is clear that with notation as above, (Y, T) is a sub-transformation group of $\#(X_i, T)$. In the case where $C_i \cup C_j = X$ whenever $i, j \in \Lambda$ are distinct, it is easy to show that $Y = \#X_i$.

4.5. COROLLARY (*of* 4.3). *If $C_i \cup C_j = X$ for all distinct $i, j \in \Lambda$, then there is an isomorphism $\pi : (X\mathscr{F}, T) \to \#(X_i, T)$ such that $p\pi = \rho$.*

4.6. THEOREM. *Let (X, T) be minimal and let f, g be acceptable functions on (X, T) such that $Cf \cap Cg$ is non-empty and $Cf \cup Cg = X$. Then the extensions ρ_f and ρ_g are disjoint.*

Proof. By the preceding results, $(X_f, T)\#(X_g, T)$ is the $\{f, g\}$ extension of (X, T), so is minimal. The result follows as in [2]. □

5. Characterization of f-extensions. The characterization of f-extensions presented here shows that every minimal almost automorphic extension of a transformation group (X, T), with compact Hausdorff phase space and locally compact Hausdorff group T, is an f-extension of (X, T). This gives an alternative proof of Theorem 4.2 in this situation.

5.1. LEMMA. *Let (X, T) be a transformation group with T locally compact Hausdorff. Consider X as a set of maps of T into X. Then X is equicontinuous on T.*

Proof. $\pi : X \times T \to X$ given by $\pi(x, t) = xt$ is continuous, so $\hat{\pi} : X \to X^T$ is continuous and injective, where $\hat{\pi}(x)(t) = \pi(x, t)$, and where X^T is the set of continuous maps from T into X with compact-open topology. As X^T is Hausdorff and X is compact, $\hat{\pi}$ is a closed map, so $\hat{\pi}X$ is homeomorphic to X. But $\hat{\pi}X \subset X^T$ is closed and compact, so is equicontinuous. □

5.2. THEOREM. *Let $\pi : (Y, T) \to (X, T)$ be a homomorphism, where T is locally compact and Hausdorff. Suppose that the set of all singleton fibres of π is dense in Y. Then there is an acceptable $f : X \to Y$ such that $f' : (X_f, T) \to (Y, T)$ is an isomorphism and $\pi f' = \rho_f$.*

This immediately shows that every minimal almost automorphic extension is an f-extension:

5.3. COROLLARY. *If $\pi : (Y, T) \to (X, T)$ is a minimal almost automorphic extension and T is locally compact Hausdorff, then there is an* *acceptable $f : X \to Y$ such that $f' : (X_f, T) \to (Y, T)$ is an isomorphism and $\pi f' = \rho_f$.*

Proof of 5.2. Define $C' = \{y : \pi^{-1}\pi(y) = \{y\}, y \in Y\}$. Then C' is dense in Y and $C = \pi C'$ is dense in X. Next we define $f : X \to Y$ as follows. If $c \in C$, let $f(c) = \pi^{-1}c$. On the other hand, let O be any orbit in X with $O \cap C = \varnothing$. Choose $x_0 \in O$ and $f(x_0) \in \pi^{-1}x_0$, and define $f(x_0 t) = f(x_0)t$ for all $t \in T$. So $f : X \to Y$ is defined (using the axiom of choice).

(i) $C(f) \subset C$. Let $x \in C(f)$ and suppose $y, y' \in \pi^{-1}x$. As C' is dense in Y, suppose $c_\alpha \to y, c'_\alpha \to y'$ where (c_α) and (c'_α) are nets in C'. Then $\pi c_\alpha \to x$ and $\pi c'_\alpha \to x$. Now as $x \in C(f), f(\pi c_\alpha) \to f(x)$ and $f(\pi c'_\alpha) \to f(x)$. But $f(\pi c) = c$ for all $c \in C$, so $c_\alpha \to f(x)$ and $c'_\alpha \to f(x)$ i.e. $y = y'$. Thus $\pi^{-1}x$ is a singleton, so $x \in C$.

(ii) $C \subset C(f)$. Suppose $c \in C$. Choose a net $x_\alpha \to c$ in X such that $f(x_\alpha) \to l$. As $\pi f(x_\alpha) = x_\alpha, \pi(l) = c$. Hence $l = f(c)$ as $l \in C'$.

Hence $C(f) = C$, so $Cf = C$.

(iii) $\Omega(f, x) = \pi^{-1}x$ *for all* $x \in X$. If $y \in \Omega(f, x)$, there is a net $c_\alpha \to x$ from C such that $f(c_\alpha) \to y$. Thus $\pi f(c_\alpha) \to \pi y$, i.e., $c_\alpha \to \pi y$. Thus $\pi y = x$, i.e., $y \in \pi^{-1}x$. On the other hand, let $y \in \pi^{-1}x$. Then there is a net $(c'_\alpha) \subset C'$ such that $c'_\alpha \to y$, i.e., $f(\pi c'_\alpha) \to y$. But $\pi c'_\alpha \to \pi y = x$, so $y \in \Omega(f, x)$.

Hence $f(x) \in \Omega(f, x)$ for all $x \in X$.

(iv) $\{f_x : x \in C\}$ *is equicontinuous on* T. This follows by Lemma 5.1 applied to (Y, T).

Hence $f : X \to Y$ is acceptable on (X, T), so we can construct the f-extension $\rho : (X_f, T) \to (X, T)$. If f' is the usual lift of f to X_f, then $f' : X_f \to Y$ is onto as $f'(X_f) \supset f'(C) = f(C) = C'$ and f' is closed. f' is also clearly equivariant, and since $\pi f'(c) = \pi f(c) = c = \rho c$ $\forall c \in C; \pi f' = \rho$. Lastly, we show that f' is injective.

Suppose $x, x' \in X_f$ and $f'(x) = f'(x')$. Then $\rho x = \pi f'(x) = \pi f'(x') = \rho x'$. Now the map F of Theorem 1.9 is an embedding. But for each $t \in T, f'(xt) = f'(x)t = f'(x')t = f'(x't)$, so $F(x) = F(x')$. Hence $x = x'$.

Thus $f' : (X_f T) \to (Y, T)$ is an isomorphism and $\pi f' = \rho$. □

5.4. Corollary. *Suppose (X, T) and (Y, T) are transformation groups with T locally compact Hausdorff, and let (X_f, T) be an f-extension of (X, T). If there are homomorphisms $p : (X_f T) \to (Y, T)$*

and $\pi : (Y, T) \to (X, T)$ *such that* $\pi p = \rho_f$, *then there is an acceptable* $g : X \to Y$ *and an isomorphism* $G : (Y, T) \to (X_g, T)$ *such that* $\rho_g G = \pi$.

Proof. We need only show that $A = \{y : y \in Y \text{ and } \pi^{-1}\pi y = \{y\}\}$ is dense in Y. But $A \supset pCf$ (where $Cf \subset X_f$). □

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