PACIFIC JOURNAL OF MATHEMATICS
Vol. 135, No. 1, 1988

PROPERTIES OF UPPERS TO ZERO IN $R[x]$

ELOISE HAMANN, EVAN HOUSTON AND JON L. JOHNSON

If k is a field, $k[x]$ is a principal ideal domain and the ideal structure of $k[x]$ is well understood. For example, a nonzero ideal is prime if and only if its generator is irreducible. If R is an integral domain with quotient field k, it is natural to ask if the set of ideals I of $R[x]$ such that $Ik[x]$ is proper can be equally well described. Since such ideals can contain no nonzero elements of R, one hopes that the structure will be dominated by the structure of $k[x]$. While such ideals need not be principal, we define a notion of almost principal which does hold for a large class of rings R. We study this class and give examples where ideals are not almost principal. The almost principal property is related to the following questions:

(1) When is $(ax-b)k[x] \cap R[x]$ generated by linear elements? (Ratliff)

(2) When is $(f(x))k[x] \cap R[x]$ divisorial? (Houston) and

(3) When is an ideal I, which is its own extension-contraction from $R[x]$ to $R[[x]]$ and back, equal to $\mathrm{cl}(I)$ in the x-adic topology? (Arnold)

We now list the main concepts we shall use. Recall that R denotes an integral domain and k denotes its quotient field.

DEFINITION 1. An ideal I of $R[x]$ is called *almost principal* if there exists an $f(x) \in I$ of positive degree and a nonzero $s \in R$ such that

$$sI \subseteq (f(x))R[x].$$

It is easy to see that for an ideal I of the form $I = Ik[x] \cap R[x]$, I almost principal is equivalent to the existence of a polynomial $f(x) \in I$ and a nonzero $s \in R$ such that for each $h(x) \in k[x]$ with $h(x)f(x) \in R[x]$, we have $sh(x) \in R[x]$. (The element s is independent of $h(x)$.) This characterization of almost principal for contracted ideals I is the basis for our proofs that I is almost principal. In fact, to prove that a noncontracted ideal J is almost principal, the usual approach is to show that $I = Jk[x] \cap R[x] \supseteq J$ is almost principal. If I is almost principal, then J is almost principal ($f(x) \in I$ may be chosen as an element of J and $sJ \subseteq sI \subseteq (f(x))R[x]$).

DEFINITION 2. $R[x]$ is called an *almost principal ideal domain* if all ideals of $R[x]$ with proper extensions to $k[x]$ are almost principal.

By the remarks following Definition 1, $R[x]$ is an almost principal ideal domain if and only if all the contracted ideals $I = (f(x)k[x]) \cap R[x]$ are almost principal.

We recall the following definitions for the reader.

DEFINITION 3. The *content ideal* of $f(x) \in R[x]$, denoted $c(f(x))$, is the ideal of R generated by the coefficients of f.

It is well known that the content operator satisfies the *content formula*: There exists an integer m such that

$$c(f(x))^{m+1}c(g(x)) = c(f(x))^m c(f(x)g(x)). \qquad \text{[G, Theorem 28.1]}$$

DEFINITION 4. For a fractional ideal J of R, set $J^{-1} = [R\colon J]_k$ and $J_v = (J^{-1})^{-1}$. The fractional ideal I of R is *divisorial* (*or a v-ideal*) if $I = I_v$.

DEFINITION 5. A nonzero ideal J of $R[x]$ is an *upper to zero* if $J \cap R = 0$ and J is prime.

Note that all uppers to zero extend to proper ideals of $k[x]$.

We use the following notation throughout the remainder of the paper: R is an integral domain with quotient field k, $f(x)$ is a polynomial of positive degree in $R[x]$, and $I = f(x)k[x] \cap R[x]$. We repeat the assumption on I in some of the main results for emphasis.

1. Almost principal ideals and related questions. The following theorem is known [J, Proposition 3.3].

THEOREM 1.1. *If R is either Noetherian or integrally closed ($R = \overline{R}$), then $R[x]$ is an almost principal ideal domain.*

We now present some further conditions for an integral domain $R[x]$ to be an almost principal ideal domain.

THEOREM 1.2. *If the conductor of R in the integral closure of R (denoted by $\overline{R}$), is nonzero, then $R[x]$ is an almost principal ideal domain.*

Proof. Let $I = f(x)k[x] \cap R[x]$, $h(x) \in k[x]$, $f(x)h(x) \in R[x]$, and a_n the leading coefficient of $f(x)$. Then $(1/a_n)f(x)$ is monic in $k[x]$ so $(1/a_n)f(x)a_nh(x) \in R[x]$. By [G, Theorem 10.4], $a_nh(x)$ is in the integral closure of $R[x]$. Since the conductor C is nonzero, then there exists a $t \in C$ so that $t\overline{R} \subseteq R$. Thus $ta_nh(x) \in R[x]$ and the proof is complete.

COROLLARY 1.3. *If $\overline{R}$ is a finite R-module then $R[x]$ is an almost principal ideal domain.*

PROPOSITION 1.4. *Let J be an ideal in $R[x]$ generated by elements of bounded degree and whose extension to $k[x]$ is proper. Then J is almost principal.*

Proof. Let

$$Jk[x] = f(x)k[x], \quad \text{and} \quad f(x) = x^t(a_nx^n + \cdots + a_1x + a_0) \in J,$$

$n + t > 0$, a_0 nonzero. Assume that J can be generated by elements (possibly an infinite number of elements) of degree $\leq t + n + m$. We claim $sJ \subseteq (f(x))$ for $s = (a_0)^{m+1}$. To see this let $h(x) = f(x)g(x) \in J$ with $\deg g \leq m$, $g(x) \in k[x]$. Writing $g(x) = b_mx^m + \cdots + b_0$, $f(x) \cdot g(x) \in R[x]$ implies that

$$(a_nx^n + \cdots + a_1x + a_0)(b_mx^m + \cdots + b_0) \in R[x].$$

The coefficient of x^l for $l \leq m$ is

$$c_l = \sum_{i=0}^{l} a_ib_{l-i}.$$

We claim that $a_0^{l+1}b_l \in R$. We proceed by induction noting that $a_0b_0 \in R$ and assuming that $a_0^{k+1}b_k \in R$ for $k < l$. Since $c_l \in R$, then

$$a_0^lc_l = \sum a_0^la_ib_{l-i} = \sum a_i(a_0^l)b_{l-i} \in R.$$

For $j < l$, $a_0^{j+1}b_j \in R$ by the inductive hypothesis and so $a_ia_0^lb_j \in R$ for $i + j = l$. Thus the term $a_0a_0^lb_l = a_0^{l+1}b_l \in R$.

Letting $l = m$, we have $(a_0^{l+1})(b_mx^m + \cdots + b_0) = (a_0^{l+1})g(x) \in R[x]$. Thus J is almost principal.

To establish further results, we prove the following lemmas.

LEMMA 1.5. *Let $f_1(x)$ and $f_2(x)$ be polynomials in $R[x]$, $f(x) = f_1(x)f_2(x)$, and $I_j = f_j(x)k[x] \cap R[x]$ for $j = 1, 2$. Then I is almost principal if and only if I_1 and I_2 are almost principal.*

Proof. Assume that I is almost principal. If $h(x)f_1(x) \in R[x]$, then $h(x)f_1(x)f_2(x) \in R[x]$ since $f_2(x) \in R[x]$. Since $f_1(x)f_2(x) = f(x)$ then $h(x)f(x) \in R[x]$. By assumption there exists a nonzero $s \in R$

so that $sh(x) \in R[x]$ for all such $h(x)$. Hence I_1 is almost principal. Similarly, I_2 is almost principal.

Assume that I_1 and I_2 are almost principal. Let $h(x) \in k[x]$ so that $h(x)f(x) \in R[x]$. Since $f(x) = f_1(x)f_2(x)$, then $(h(x)f_1(x))f_2(x) \in R[x]$. By assumption there exist nonzero s_1 and s_2 in R so that for all $g_1(x), g_2(x)$ in $k[x]$ with $g_1(x)f_1(x)$ and $g_2(x)f_2(x)$ in $R[x]$, $s_1g_1(x) \in R[x]$ and $s_2g_2(x) \in R[x]$. Thus $s_2(h(x)f_1(x)) \in R[x]$ and so $s_1s_2(h(x)) \in R[x]$. For $s = s_1s_2$, we have $sh(x) \in R[x]$ for all $h(x) \in k[x]$ with $h(x)f(x)$ in $R[x]$.

LEMMA 1.6. *If I is almost principal and $I \subset I_1 = f_1(x)k[x] \cap R[x]$, then I_1 is almost principal.*

Proof. Since $f(x) \in I \subset I_1$, there exists an $l(x) \in k[x]$ such that $f(x) = l(x)f_1(x)$. For $s_0 \in R$ with $s_0l(x) \in R[x], (f(x))k[x] = (s_0f(x))k[x]$. We may therefore assume that $l(x) \in R[x]$. By Lemma 1.5, I almost principal implies that I_1 is almost principal.

The contracted ideal $I = f(x)k[x] \cap R[x]$ is an upper to zero precisely when f is irreducible in $k[x]$. By Lemma 1.5 and earlier remarks, it is clear that $R[x]$ is an almost principal ideal domain if and only if all the uppers to zero are almost principal.

The following results relate the almost principal property to content ideals of polynomials which occur in I.

LEMMA 1.7. *If $g(x) \in R[x]$ and the content of g satisfies*

$$c(g)^{-1} = R,$$

then for any $a \neq 0$ in R, $(a, g(x))_v = R[x]$.

The lemma follows immediately from [H, Lemma 4.4].

PROPOSITION 1.8. *Let $I = (f(x)k[x]) \cap R[x]$ for some $f(x)$ irreducible in $k[x]$. If there exists a $g \in I$ with $c(g)^{-1} = R$, then I is almost principal.*

Proof. With the hypothesis as given, we may write $g(x) = f(x)l(x)$ and pick $d \neq 0$ in R so that $dl(x) \in R[x]$. Let $f(x)h(x)$ be an arbitrary element in I. For some $a \neq 0$ in R (dependent upon $h(x)$) $ah(x) \in R[x]$. Since $d \in R$, $adh(x) \in R[x]$ as well. But $g(x)dh(x) = dl(x)h(x)f(x) \in R[x]$. Thus we have

$$(a, g(x))dh(x)f(x) \subseteq (f(x))$$

and

$$(a, g(x))_v dh(x)f(x) \subseteq (f(x)).$$

Now, by Lemma 1.7, $(a, g(x))_v = ((a, g(x))^{-1})^{-1} = R[x]$, so $dh(x)f(x)$ is in the ideal $(f(x))$. It follows that $dh(x) \in R[x]$ and that I is almost principal.

PROPOSITION 1.9 (ARNOLD). *If J is an ideal of $R[x]$ such that $Jk[x]$ is proper and J contains an element $l(x)$ such that $\bigcap_{n=1}^{\infty} c(l(x))^n \neq 0$, then J is almost principal.*

Proof. We prove that the contracted ideal $I_1 = Jk[x] \cap R[x]$ is almost principal since this implies that J is almost principal. By Lemma 1.6, it suffices to prove that $I = l(x)k[x] \cap R[x]$ is almost principal. Let $g(x) \in I$. Then $g(x) = h(x)l(x)$ for some $h(x) \in k[x]$, and by the content formula there exists an integer m with

$$c(l(x))^{m+1}c(h(x)) = c(l(x))^m c(g(x)) \subseteq R.$$

Thus if $s \in \bigcap (l(x))^n$, then $sh(x) \in R[x]$ for all $h(x)$ with $h(x)l(x) \in k[x]$. It follows that I whence J is almost principal.

Let I be as usual $f(x)k[x] \cap R[x]$ and define

$$C = [f(x)R[x]\colon I]_{R[x]}.$$

Observe that the ideal I is almost principal if and only if $C \cap R$ is nonzero. The following claims provide information about C and the closely related ideal I^{-1}. (The proofs of Claims 1.10 through 1.12 actually require only that $Ik[x] = f(x)k[x]$ for some $f(x)$ in I.)

Claim 1.10.

$$I^{-1} = C/f(x) \quad \text{(or equivalently } C = f(x)I^{-1}\text{)}.$$

Proof. Since $I^{-1}I \subset R[x]$, then $f(x)I^{-1}I \subset f(x)R[x]$. The polynomial $f(x)$ is in I and hence $f(x)I^{-1}$ is in $R[x]$. From our definition of C we see that $f(x)I^{-1} \subset C$.

Let $t \in C$, then $tI \subset (f)$ so $th(x)f(x) \in (f(x))$ for all $h(x)f(x)$ in I. Thus $(t/f(x))I \subset R[x]$ and so $(t/f(x)) \in I^{-1}$. It follows that $C \subset f(x)I^{-1}$.

Claim 1.11. If $I^{-n} = (I^{-1})^n$ for each n, then $T_{R[x]}(I) = R[C/f(x)]$, where T denotes the ideal transform.

Proof. We use the definition of the ideal transform

$$T_{R[x]}(I) = \bigcup_{n=1}^{\infty} I^{-n}.$$

The claim follows easily since $I^{-n} = (I^{-1})^n$, and $I^{-1} = C/f(x)$.

Claim 1.12. If f is irreducible in $k[x]$ and C contains an element which is not a multiple of $f(x)$ in $k[x]$, then C contains a nonzero constant and hence I is almost principal.

Proof. Let $g(x)$ be an element of C which is not a multiple of $f(x)$. The polynomials $f(x)$ and $g(x)$ are relatively prime in $k[x]$, hence there exist elements $h_1(x)$ and $h_2(x)$ in $k[x]$ such that

$$1 = f(x)h_1(x) + g(x)h_2(x).$$

Let $0 \neq s \in R$ such that $sh_1(x), sh_2(x) \in R[x]$, then

$$s = f(x)sh_1(x) + g(x)sh_2(x).$$

Since $g(x) \in C$, $f(x) \in C$, and $sh_1(x)$ and $sh_2(x)$ are in $R[x]$, then $s \in C$.

Claim 1.13. If $f(x)$ is irreducible in $k[x]$ and I is not almost principal, then $C \subseteq I$.

Proof. Since I is not almost principal, Claim 1.12 implies that each element of C is a multiple of $f(x)$ and hence in I.

LEMMA 1.14. $I^{-1} \cap k[x] = [I : I]$.

Proof. Clearly $[I : I] \subseteq I^{-1} \cap k[x]$. Suppose $h(x) \in I^{-1} \cap k[x]$. Then $h(x)I \subseteq R[x]$ and $h(x)I \subseteq h(x)f(x)k[x] \subseteq fk[x]$. Thus $hI \subseteq I$ because of the assumption that $I = f(x)k[x] \cap R[x]$.

The results 1.8, 1.12, and 1.13 give information about uppers to zero since $f(x)$ was assumed to be irreducible in $k[x]$. Under the assumption that I is an upper to zero, we now obtain a number of conditions in terms of I^{-1} and $[I : I]$ equivalent to the almost principal property.

PROPOSITION 1.15. *The following are equivalent for I an upper to zero with $I = f(x)k[x] \cap R[x]$.*

(1) *I is almost principal.*

(2) $I^{-1}k[x] = (1/f)k[x]$ (*or* $Ck[x] = k[x]$).

(3) $I^{-1} \not\subseteq k[x]$.

(4) *I^{-1} is not a ring.*

(5) $I^{-1} \neq [I : I]$.

(6) *There exists a $g(x) \in R[x] \backslash I$ with $g(x)I \subseteq f(x)R[x]$.*

Proof. (1) $\Rightarrow$ (2) That $I^{-1}k[x] \subseteq (1/f)k[x]$ is clear. Since I is almost principal, $sI \subseteq (f)$ for some nonzero s in R. Then $s/f \in I^{-1}$which implies that $1/f \in I^{-1}k[x]$.

(2) $\Rightarrow$ (3) We may write $1/f = \sum_{i=1}^{m} u_i h_i$ with $u_i \in I^{-1}$ and $h_i \in k[x]$. There exists a nonzero $s \in R$ so that $sh_i(x) \in R[x]$ for each i. Then $s/f = \sum u_i s h_i \in I^{-1}$, but $s/f \notin k[x]$.

(3) $\Rightarrow$ (4) Choose $u \in I^{-1} \backslash k[x]$. We write $u = g(x)/f(x)$, where $g(x) \in R[x] \backslash I$. The element $u^2 \notin I^{-1}$ since $u^2 f = g^2/f$, f is irreducible and $g/f \notin k[x]$. Hence I^{-1} is not closed under multiplication.

(4) $\Rightarrow$ (5) Since I^{-1} is not a ring, I^{-1} cannot equal $[I : I]$.

(5) $\Rightarrow$ (6) Let $u \in I^{-1} \backslash [I : I]$. Then $uf \in R[x]$, and so there exists a $g(x) \in R[x]$ with $u = g/f$. Hence $g(x)I \subseteq (f)$. By Lemma 1.14, $u \notin k[x]$ so $g(x) \notin I$.

(6) $\Rightarrow$ (1) Follows immediately from Claim 1.12.

If I is principal ($I = fR[x]$), then $[I : I]$ is easy to find. In fact, $[I : I]$ determines whether I is principal in $R[x]$. The following proposition shows this and a bit more.

PROPOSITION 1.16. $[I : I] \neq R[x]$ *if and only if there exists a* $g(x) \in I \backslash (f)$ *with* $g(x)I \subseteq (f)$.

Proof. Suppose $g(x) = f(x)h(x) \in I \backslash f(x)R[x]$ with $g(x)I \subseteq (f(x))$. Then $hI \subseteq R[x]$. Since $hI \subseteq fk[x]$ also, we have $hI \subseteq I$ with $h \notin R[x]$. Conversely, if $h \in [I : I] \backslash R[x]$, then $g = fh \in I \backslash (f)$ with $gI \subseteq (f)$.

We now wish to relate the almost principal property to questions (2) and (3) mentioned in the introduction. We need two preliminary lemmas, both from unpublished work of Arnold.

LEMMA 1.17. *Let R be a domain, $I = f(x)k[x] \cap R[x]$, then $(f) = (f)^{ec}$ if and only if $I = I^{ec}$, where ec means extend to $R[[x]]$ and contract to $R[x]$.*

Proof. If $I = I^{ec}$, then $(f)^{ec} \subseteq I^{ec} = I \subseteq f(x)k[x]$. Thus

$$(f)^{ec} \subseteq f(x)k[x] \cap f(x)R[[x]] = f(x)R[x] \equiv (f).$$

Conversely, suppose that $(f) = (f)^{ec}$. If $g \in I^{ec}$, then $g = g_1h_1 + \cdots + g_nh_n$ with $g_i \in I$, and $h_i \in R[[x]]$ for $i = 1, \cdots, n$. There exists a nonzero $a \in R$ with $ag_i \in (f)$ for all i. Thus $ag = ag_1h_1 + \cdots + ag_nh_n \in fR[[x]] \cap R[x] = (f)^{ec}$. It follows that $ag \in (f)$, whence $g \in fk[x] \cap R[x] = I$, as desired.

LEMMA 1.18. $(f)^{ec} = \mathrm{cl}(f)$. *(Here* "cl" *denotes closure in the x-adic topology, so* $\mathrm{cl}(f) = \bigcap_{n=1}^{\infty}((f) + x^n bR[x])$.)

Proof. If $g \in (f)^{ec}$, then $g = f(x)(b_0 + b_1x + \cdots), b_i \in R$ for $i = 1, 2, \ldots$. Let $g_n = b_0 + \cdots + b_{n-1}x^{n-1}$ and $g'_n = b_n + b_{n+1}x + \cdots$. Then we write $g = fg_n + x^n fg'_n$. Since $g - fg_n \in R[x]$, we must have $fg'_n \in R[x]$. Hence $g \in \mathrm{cl}(f)$.

For the other inclusion we may assume $f(0) \neq 0$. Let $h \in \mathrm{cl}(f)$. For each $n = 0, 1, 2, \ldots$, write $h = fh_n + x^n l_n$, where $h_n, l_n \in R[x]$. For $m > n$ we have $fh_n + x^n l_n = fh_m + x^m l_m$, so that $f(h_n - h_m) \in x^n R[x]$. Since $f(0) \neq 0$, this gives $h_n - h_m \in x^n R[x]$. Therefore, all h_m for $m > n$, have the same coefficient of x^n. Denote that coefficient by a_n. It is then straightforward to show that $h(x) = f(x)(a_0 + a_1x + \cdots) \in (f)^{ec}$.

THEOREM 1.19. *Let R be an integral domain with quotient field k, $I = (f(x)k[x]) \cap R[x]$ an upper to zero in $R[x]$, then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) where*

(1) *I is almost principal*

(2) *I is divisorial*

(3) *$I = I^{ec}$ implies $I = \mathrm{cl}(I)$.*

Proof. The proof that we give for (2) $\Rightarrow$ (3) is due to Arnold.

(1) $\Rightarrow$ (2) Assume that I is almost principal, then there exists a nonzero $s \in R$ so that $sI \subseteq f(x)R[x]$. With $sI \subseteq f(x)R[x]$,

$$s(I)_v = (sI)_v \subseteq (f)_v = (f) \subseteq I.$$

Thus $I_v \subseteq I$ and I is divisorial.

(2) $\Rightarrow$ (3) Assume that I is divisorial. Let $u(x) \in I^{-1}$ with $u(x) \in k(x)$. Then $u(x)f(x) \in R[x]$ and so we may write $u(x) = h(x)/f(x)$ for some $h(x) \in R[x]$. For any $g(x) \in I$,

$$h(x)g(x) = (f(x)u(x))g(x) = (u(x)g(x))f(x) \in (f).$$

Thus $h(x)I \subseteq (f)$. From the preceding lemmas it follows that

$$h(x)\mathrm{cl}(I) \subseteq \mathrm{cl}(f) = (f).$$

Hence $u(x)\mathrm{cl}(I) \subseteq R[x]$. Thus

$$I^{-1}\mathrm{cl}(I) \subseteq R[x].$$

We have

$$\mathrm{cl}(I) \subseteq I_v = I.$$

2. Examples and counterexamples. In this section we present examples of ideals which are not almost principal, not divisorial, and do not satisfy (3) of Theorem 1.19. We show that condition (3) of Theorem 1.19 does not imply (2). All the examples involve a linear $f(x)$, and this case is thoroughly discussed.

Arnold's Example. Let $R = F[t, \{ty^{2^n}\}_{n=0}^{\infty}]$ and $f(x) = tyx - t$ for F a field. Then the ideal $I = (f(x)k[x]) \cap R[x]$ is not almost principal, and satisfies neither (2) nor (3) of Theorem 1.19.

To show that Arnold's Example is not almost principal, we show a more general class of polynomials in certain domains lead to ideals which are not almost principal. We begin with a definition.

DEFINITION. Let R be a domain with quotient field k. An element $t \in k$ is *spotty over* R (or just plain spotty) if $\bigcap_{i=1}^{\infty}[R\colon t^{k_i}] \neq 0$ for some infinite subset of integers $\{k_i\}$, but $\bigcap_{i=1}^{\infty}[R\colon t^i] = 0$.

Recall that an element t in the quotient field k of an integral domain R is *almost integral* over R if $\bigcap_{i=1}^{\infty}[R\colon t^i] \neq 0$. Thus t almost integral over R implies that t is not spotty over R.

THEOREM 2.1. *Let c and d be elements of R with $f(x) = cx - d$. If $y = c/d$ is spotty over R, then $I = (f(x)k[x]) \cap R[x]$ is not almost principal, not divisorial, and $I = I^{ec}$, but $I \neq \mathrm{cl}(I)$.*

By Theorem 1.19, it suffices to show that $I = I^{ec}$ with $I \neq \mathrm{cl}(I)$. We need the following lemma from unpublished work of Arnold.

LEMMA 2.2. *If $y \in k$ is not almost integral over R, then $l(x) = yx-1$ satisfies $(l) = (l)^{ec}$.*

Proof. Let $g = lh \in R[x]$, $h \in R[[x]]$, and write $h(x) = a_0 + a_1x + \cdots$. Let n be the degree of g. We have

$$\begin{gathered} a_n y - a_{n+1} = 0, \\ a_{n+1} y - a_{n+2} = 0, \\ \vdots \end{gathered}$$

which implies that $a_n y^k \in R$ for all k. Hence $a_n = 0$ and $a_{n+i} = 0$ for all i. Thus $g \in (l)$.

Proof of Theorem 2.1. Since $y = c/d$, note that$(cx - d)k[x] = (yx-1)k[x]$, so by Lemmas 1.17 and 2.2, $I = I^{ec}$. However, if $s \in R$ and $sy^{k_i} \in R$, then $sy^{k_i}x^{k_i} - s = s(yx-1)(y^{k_i-1}x^{k_i-1} + \cdots + 1) \in I$. Thus $s = -(sy^{k_i}x^{k_i} - s) + sy^{k_i}x^{k_i} \in I + x^{k_i}R[x]$. It follows that $s \in \mathrm{cl}(I)\backslash I$.

The elements ty and t in Arnold's Example satisfy the hypothesis of Theorem 2.1 since $ty/t = y$ is spotty over R. For linear $f(x)$ it is natural to conjecture that the spotty condition is necessary to construct a bad example. However, if $(cx-d)k[x] \cap R[x] = I$ is not almost principal, it does not follow that c/d is spotty, as the following example illustrates.

Example 2.3. Let $R = F[\{x_i\}_{i=1}^{\infty}, \{x_i y^i\}_{i=1}^{\infty}]$, where F is any field and $R \subseteq F[\{x_i\}, y]$ with $y, x_1, x_2, \ldots$ independent indeterminates over F. Set $c = x_1 y$ and $d = x_1$ with $I = (cx-d)k[x] \cap R[x]$. It is easily verified that $y = c/d$ is not spotty. We show that I is not almost principal. Let

$$h_n(x) = x_n(y^{n-1}x^{n-1} + \cdots + yx + 1).$$

Then $(cx-d)h_n(x)$ is in $R[x]$ so it suffices to show that no nonzero element s of R multiplies all $h_n(x)$ into $R[x]$. First note that a monomial $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}y^j$ in $F[y, x_1, x_2, \cdots]$ is in R if and only if $j = i_1' + 2i_2' + \cdots + ni_n'$, where $0 \le i_r' \le i_r$ for $r = 1, \ldots, n$. Suppose $sI \subseteq (cx-d)k[x] \cap R[x] = d(yx-1)R[x]$. Then since

$$x_n y^n x^n - x_n = (yx-1)(x_n y^{n-1} x^{n-1} + \cdots + x_n) \in I,$$

we have $s(x_n y^{n-1}x^{n-1} + \cdots + x_n) \in R[x]$, so $sx_n y^i \in R$ for each n with $i < n$. In particular, $sx_{2m}y^m \in R$ for all m. We may assume that s is a monomial in $F[y, x_1, x_2 \cdots]$, and write $s = x_1^{i_1} \cdots x_n^{i_n} y^j$. Since $s \in R$, we have $j = i_1' + 2i_2' + \cdots + ni_n'$ as above. Choose $m > i_1 + 2i_2 + \cdots + ni_n$. Then $sx_{2m}y^m \in R$ implies $m = i_1'' + 2i_2'' + \cdots + ni_n'' + 2ml$, where l is zero or one and $0 \leq i_r'' \leq i_r$. $l = 1$ is clearly impossible, and $l = 0$ is impossible by the choice of m. Thus I is not almost principal.

Thus we fail to establish a necessary and sufficient condition on c/d for $(cx - d)k[x] \cap R[x]$ to be almost principal. We can show, however, that the not almost integral aspect of a spotty c/d is necessary to create a bad example.

THEOREM 2.4. *If $c/d \in k$ is almost integral over R, then $I = (cx - d)k[x] \cap R[x]$ is almost principal.*

Proof. Let

$$I_1 = (cx - d)k[x] \cap R[c/d][x].$$

Then $(1/d)(cx - d)R[c/d][x]$ is principal in $R[c/d][x]$. Let t be a nonzero element of R with $t(c/d)^n \in R$ for all n, then $tdI \subseteq (cx - d)R[x]$.

It can be shown that the ideal I in Example 2.3 is not divisorial, but does satisfy condition (3) of Theorem 1.19. It is, however, easier and more interesting to show that (3) does not imply (2) of Theorem 1.19 by taking advantage of the fact that the question of which $f(x)$ produce almost principal or divisorial ideals I is symmetric in the coefficients. That is, if $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, then $f'(x) = a_0 x^n + \cdots + a_n$ behaves like f with respect to the two properties of divisorial and almost principal. To see this, let $S = \{l(x) \in k[x] \mid l(0) \neq 0\}$. The map $f \to f' = x^{\deg f} f(1/x)$ is a monoid automorphism of S which extends to a group automorphism of $S_1 = \{f/g \in k(x) \mid f(0) \neq 0, g(0) \neq 0\}$ under multiplication. Using this idea the following lemma and proposition are straightforward.

LEMMA 2.5. *Let $f \in R[x]$ with $f(0) \neq 0$ and $I = fk[x] \cap R[x]$ as usual. For $I' = f'k[x] \cap R[x]$ where $f' = x^{\deg f} f(1/x)$, then*

(1) $I = \{x^r g \mid r \geq 0, g \in I \cap S\}$.
(2) $I^{-1} = \{x^r u \mid r \geq 0, u \in I^{-1} \cap S_1\}$.
(3) $I_v = \{x^r g \mid r \geq 0, g \in I_v \cap S\}$.
(4) $(I \cap S)' = I' \cap S$.
(5) $(I^{-1} \cap S_1)' = (I')^{-1} \cap S_1$.
(6) $(I_v \cap S)' = (I')_v \cap S$.

PROPOSITION 2.6. *Let $f \in R[x]$ with $f(0) \neq 0$ and $I = fk[x] \cap R[x]$ as usual. For $I' = f'k[x] \cap R[x]$ where $f' = x^{\deg f} f(1/x)$, then*
(a) *I is almost principal if and only if I' is almost principal.*
(b) *$I^{-1} = R[x]$ if and only $(I')^{-1} = R[x]$.*
(c) *I is divisorial if and only if I' is divisorial.*

Now, the symmetry of the divisorial property provides an easy proof that (3) of Theorem 1.19 does not imply (2) because property (3) is not symmetric. If $R = F[t, \{ty^{2^n}\}]$, then y is spotty and $I = (yx-1)k[x] \cap R[x]$ is not divisorial by Theorem 2.1. By the above comments, $I' = (x-y)k[x] \cap R[x]$ is not divisorial. However, $R[y] = F[t, y]$ is a UFD and $\bigcap y^n R[y] = 0$ so I' satisfies $I' = \mathrm{cl}(I')$.

The almost principal question for a linear $f(x)$ is, of course, more accessible than a general $f(x)$, so one might hope that $I = f(x)k[x] \cap R[x]$ almost principal for all linear $f(x)$ might imply that all such I are almost principal for any $f(x)$. The authors have an inconclusive quadratic example which suggests that this is not the case. However, it is easy to handle the case where k is algebraically closed.

THEOREM 2.7. *If k is algebraically closed and if all $I = (cx-d)k[x] \cap R[x]$ are almost principal, then $R[x]$ is an almost principal ideal domain.*

Proof. This follows easily from Lemma 1.5, since every $f(x)$ factors into linear factors. Recall that it is sufficient to show all the contracted ideals or uppers to zero are almost principal.

Noetherian domains are almost principal because the finite number of generators of I in $R[x]$ allow a common denominator s to yield $sI \subseteq (f(x))$. From the first section, if a set of bounded degree generates I, I was shown to be almost principal. Our final example merely shows that neither condition is required for I to be almost principal.

EXAMPLE 2.8. Let $S = R[z, \{zy^j\}_{j=1}^{\infty}]$, where R is the ring in Arnold's example, and I is the ideal contracted from $(tyx - t)k(z)[x]$ to $S[x]$. The element z will be the element $s \neq 0$ that makes I almost principal.

3. Generators for $I = f(x)k[x] \cap R[x]$ with $f(x) = a_0 + a_1x + \cdots + a_nx^n$. In his 1973 paper **[R]**, Ratliff developed equivalent conditions to $I = f(x)k[x] \cap R[x]$ being linearly generated for $f(x) = a_0 + a_1x$ ($n = 1$). Observe that if I is linearly generated then I is almost principal. We extend many of the results to when $f(x)$ is of arbitrary degree n.

THEOREM 3.1. *For $f(x) = a_0+a_1x+\cdots+a_nx^n$, elements of degree n generate $I = f(x)k[x]\cap R[x]$ if and only if for each $g \in I$, the constant term of $g(x)$ is an element of $\bigcap_{i=1}^{n}[(a_0)\colon (a_i)]_R$.*

Proof. We assume that elements of degree n generate I. Let $g(x) \in I$ with $\deg g(x) = m > n$. If we write

$$g(x) = b_mx^m + \cdots + b_0,$$

then since nth degree terms of the form $t_if(x)$ generate I,

$$g(x) = \sum_{i=1}^{s} h_i(x)t_if(x) \quad \text{for } h_i(x) \in R[x],\ t_i \in k,$$

and

$$b_0 = \sum_{i=1}^{s} t_ia_0c_{0_i} = \sum_{i=1}^{s}(t_ic_{0_i})a_0 \in R,$$

for c_{0_i} the constant term in $h_i(x)$. Observe that $t_ia_j \in R$ for each i and j since $t_if(x) \in R[x]$. Thus

$$b_0a_j = \left(\sum_{i=1}^{s}(a_jt_ic_{0_i})a_0\right) \in (a_0) \quad \text{for each } j.$$

Hence $b_0 \in \bigcap[(a_0)\colon (a_i)]_R$.

For the other direction, assume that the constant terms are in the intersection. Let $g(x) \in I$ be of least degree in I so that$g(x)$ is not an $R[x]$-linear combination of nth degree terms. Since $g(x) \in I$, we may write $g(x) = h(x)f(x)$ for some $h(x)$ in $k[x]$. Let $h(x) = \sum_{j=1}^{m}(c_j/d_j)x^j$. Then $a_0(c_0/d_0)$ is a constant term in I and is in the intersection $\bigcap[(a_0)\colon (a_i)]_R$. Hence$(c_0/d_0)a_i \in R$ for each i. Thus

$$g(x) = \left(h(x) - \frac{c_0}{d_0}\right) f(x) + \left(\frac{c_0}{d_0}\right) f(x)$$

and

$$g(x) - \left(\frac{c_0}{d_0}\right) f(x) = \left(h(x) - \frac{c_0}{d_0}\right) f(x).$$

Now, $g(x) - (c_0/d_0)f(x)$ is a multiple of x and can be written in the form $xf(x)l(x)$ with $f(x)l(x) \in R[x]$ and $l(x)$ in $k[x]$. By the definition of I, $f(x)l(x) \in I$. But the degree of $f(x)l(x)$ is $m-1$, so

$f(x)l(x)$ can be written in terms of degree n polynomials. It follows that

$$g(x) = xf(x)l(x) + c_0/d_0\, f(x)$$

can be so expressed and I is generated by degree n polynomials.

COROLLARY 3.2. *The image of I under the homomorphism given by $x \to 0, r \to r$ for each $r \in R$, is $\bigcap_{i=1}^{n}[(a_0):(a_i)]_R$ if and only if degree n polynomials generate I.*

Proof. If $b \in \bigcap_{i=1}^{n}[(a_0):(a_i)]_R$, then $(b/a_0)f(x) \in I$ has constant term b so the intersection is always contained in the image. Theorem 3.1 implies I is generated by degree n polynomials if and only if the other containment holds.

Note that for $n = 1$, I has a "linear base" if and only if $[(a_0):(a_1)]_R$ equals the constant terms in I.

THEOREM 3.3. *Let $I = (a_n x^n + \cdots + a_1 x + a_0)k[x] \cap R[x]$ and I_n the R-module consisting of elements of I of degree n. Then*

$$\mu\left(\bigcap_{i=1}^{n}[(a_n):(a_i)]_R\right) = \mu(I_n)$$

where μ denotes the minimum number of generators.

Proof. Let $G = \{t_j\}$ be a generating set for $\bigcap_{i=1}^{n}[(a_n):(a_i)]$. We let $f_j(x) = (t_j/a_n)f(x)$. To show that $\{f_j(x)\}$ generates the polynomials in I of degree n, note first that $f_j(x) \in I$ since each $t_j \in \bigcap[(a_n):(a_i)]$ and thus $(t_j/a_n)a_i \in R$ for each i. Now let $g(x)$ be of degree n in I. Since $g(x) \in I, g(x) = sf(x)$ for some $s \in k$. For $t = sa_n$, $t \in \bigcap[(a_n):(a_i)]$ and thus $t = \sum r_j t_j$. We rewrite $g(x)$ as

$$g(x) = \frac{\sum r_j t_j}{a_n} f(x) = \sum r_j \left(\frac{a_n f_j(x)}{a_n}\right) = \sum r_j f_j(x).$$

To see the correspondence in the other direction, let $\{g_j(x)\}$ be a generating set for I_n. For each $j, g_j(x) = s_j f(x)$. In this case let $t_j = s_j a_n$. Each $t_j \in \bigcap[(a_n):(a_i)]$ since

$$g_j(x) = s_j a_n \left(x^n + \cdots + \frac{a_1}{a_n}x + \frac{a_0}{a_n}\right) \in R[x].$$

To see that the set $\{t_j\}$ generates $\bigcap[(a_n):(a_i)]$, let $t \in \bigcap[(a_n):(a_i)]$. Then $(t/a_n)f(x) \in R[x]$, hence $(t/a_n)f(x) \in I$. Thus

$$\frac{t}{a_n}f(x) = \sum r_j g_j(x)$$

$$tf(x) = \sum a_n r_j g_j(x) = \sum a_n r_j s_j f(x)$$

or

$$t = \sum r_j a_n s_j = \sum r_j t_j.$$

COROLLARY 3.4 (*to both theorems*). *If* $\Phi: R[x] \to R$ *is defined by setting* $x = 0$ *and* $\Phi(I) \subseteq \bigcap_{i=1}^{n}[(a_0):(a_i)]_R$ *then*

$$\mu_{R[x]}(I) = \mu\left(\bigcap_{i=1}^{n}[(a_n):(a_i)]_R\right).$$

Proof. The hypothesis and proof of Corollary 3.2 guarantee that degree n terms generate I. The minimal number of degree n generators of I is equal to the minimal number of generators of $\bigcap[(a_n):(a_i)]$.

We can see from this final corollary that for $n = 1$, the ideals $[a_0 : a_1]_R$ and $[a_1 : a_0]_R$ "determine" the number of generators of $I = ((a_0 + a_1 x)k[x]) \cap R[x]$.

REFERENCES

[G] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, N. Y., 1972.

[H] J. R. Hedstrom and E. G. Houston, *Some remarks on star operators*, J. Pure Appl. Algebra, **18** (1980), 37–44.

[J] J. L. Johnson, *Three topological properties from Noetherian rings*, Canad J. Math., **34** (1982), 525–534.

[R] L. J. Ratliff, *Conditions for* Ker($R[x] \to R[c/b]$) *to have a linear base*, Proc. Amer. Math. Soc., **39** (1973), 509–514.

Received April 21, 1987 and in revised form October 30, 1987.

SAN JOSE STATE UNIVERSITY
SAN JOSE, CA 95192-0103

UNIVERSITY OF NORTH CAROLINA
CHARLOTTE, NC 28223

AND

ELMHURST COLLEGE
ELMHURST, IL 60126