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UNIQUENESS PROBLEM WITHOUT MULTIPLICITIES IN VALUE DISTRIBUTION THEORY

SHANYU JI

Let $H_1, \ldots, H_k$ be hyperplanes in general position in $\mathbf{P}^m$ with $m \geq 2$. Let $A_1, \ldots, A_k$ be pure $(n-1)$-dimensional analytic subsets of $\mathbf{C}^n$ with $\operatorname{codim} A_i \cap A_j \geq 2$ whenever $i \neq j$. Then any linearly non-degenerate meromorphic maps $f, g, h: \mathbf{C}^n \to \mathbf{P}^m$ with $f|A_j = g|A_j = h|A_j$ and with $f^{-1}(H_j) = g^{-1}(H_j) = h^{-1}(H_j) = A_j$ for $j = 1, \ldots, k$ satisfy Property (P) if $k = 3m + 1$. Consequently such f, g, h are algebraically dependent. If even $n \geq \operatorname{rank} f = \operatorname{rank} g = \operatorname{rank} h = m$, then $k = m + 3$ suffices.

1. Introduction. Since Pólya's work **[P1]** in 1929, the uniqueness problem in value distribution theory has been studied by Nevanlinna **[N1]**, Cartan **[C1, C2]**, Fujimoto **[F1, F2]**, Schmid **[S1]**, Smiley **[S5, S6]**, Carlson **[D2]**, Drouilhet **[D1, D2]** and Stoll **[S11]**. One of the main results, given by Fujimoto in 1979 **[F2]**, is that if H_j are hyperplanes in $\mathbf{P}^m$ in general position and v_j are divisors on $\mathbf{C}^n$ whose supports have no common irreducible components for $j = 1, \ldots, m+2$ and if $\mathscr{W}$ is the set of meromorphic maps $f: \mathbf{C}^n \to \mathbf{P}^m$ with $f^*(H_j) = v_j$ for $j = 1, \ldots, m+2$, ($f^*(H_j)$ is the pull-back of the divisor of H_j on $\mathbf{P}^m$ by f), then $\mathscr{W}$ cannot contain more than $m+1$ algebraically independent maps. This theorem is in fact a generalization of the Cartan-Nevanlinna theorem (i.e., take $n = m = 1$ and replace "algebraically dependent maps" by "maps" in the above theorem) in 1928.

In this paper, we shall give some analogous results which are without multiplicities. For this kind of problem, Cartan declared **[C2]** that there are at most two meromorphic functions f, g on $\mathbf{C}$ such that $f^{-1}(a_j) = g^{-1}(a_j)$ for four distinct $a_j \in \mathbf{P}^1$. Cartan's proof appears to have a gap. But some of his original ideas are used in this paper. We also need to use some of Shiffman's and Drouilhet's results **[S3]**, **[D2]**.

Let $H_1, \ldots, H_k$ be hyperplanes in general position in $\mathbf{P}^m$ given by

$$a_0^{(j)} w_0 + \cdots + a_m^{(j)} w_m = 0 \tag{1.1}$$

for $j = 1, \ldots, k$. Let $A_1, \ldots, A_k$ be pure $(n-1)$-dimensional analytic subsets of $\mathbf{C}^n$ with $\operatorname{codim} A_i \cap A_j \geq 2$ whenever $i \neq j$. Put

$A := A_1 \cup \cdots \cup A_k$. Let $F: \mathbf{C}^n \to \mathbf{P}^m$ be a meromorphic map. Let

$$\begin{aligned}(1.2)\quad &\mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, k)\\ &:= \{f: \mathbf{C}^n \to \mathbf{P}^m \mid f \text{ is a linearly non-degenerate}\\ &\qquad\qquad \text{meromorphic map, } f|A = F|A\\ &\qquad\qquad \text{and } f^{-1}(H_j) = A_j \text{ for } j = 1, \ldots, k\}\end{aligned}$$

(for the definition of linearly non-degenerate, see §2) and

$$(1.3)\qquad \mathscr{F}_A(\mathbf{C}^n, \mathbf{P}^m, k) := \{f \in \mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, k) \mid \operatorname{rank} f = m\}.$$

We say that $f, g, h \in \mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, k)$ *satisfy Property* (P) if for each fixed $p = (p_0, \ldots, p_m) \in \mathbf{C}_*^{m+1} := \mathbf{C}^{m+1} - \{0\}$, for all but at most four values of i in the set $\{1, \ldots, k\}$, there exist $(\lambda_i, \mu_i, \nu_i) \in \mathbf{C}_*^3 = \mathbf{C}^3 - \{0\}$, such that

$$\begin{aligned}(1.4)\quad &\frac{\lambda_i(p_0 f_0 + \cdots + p_m f_m)}{a_0^{(i)} f_0 + \cdots + a_m^{(i)} f_m} + \frac{\mu_i(p_0 g_0 + \cdots + p_m g_m)}{a_0^{(i)} g_0 + \cdots + a_m^{(i)} g_m}\\ &\qquad + \frac{\nu_i(p_0 h_0 + \cdots + p_m h_m)}{a_0^{(i)} h_0 + \cdots + a_m^{(i)} h_m} = 0\end{aligned}$$

where $(f_0, \ldots, f_m)$, $(g_0, \ldots, g_m)$, and $(h_0, \ldots, h_m)$ are reduced representations of f, g, h, respectively, and $a_0^{(i)}, \ldots, a_m^{(i)}$ are given by (1.1). (The set of omitted values of i depends on p.)

Our main results are

THEOREM A. *If $f, g, h \in \mathscr{F}_A(\mathbf{C}^n, \mathbf{P}^m, m+3)$ with $m \geq 2$, then f, g, h satisfy Property* (P). *Consequently f, g, h are algebraically dependent.*

THEOREM B. *If $f, g, h \in \mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, 3m+1)$ with $m \geq 2$, then f, g, h satisfy Property* (P). *Consequently f, g, h are algebraically dependent.*

REMARK. By **[D1]**, if $k > m + 3$, $\mathscr{F}_A(\mathbf{C}^n, \mathbf{P}^m, k)$ contains at most one map. By **[S5, S6]**, if $k > 3m + 1$, $\mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, k)$ contains at most one map.

As generalization of Theorem A, we have Theorem C as follows.

Let $V \subset \mathbf{P}^M$ be a connected complex submanifold with $\dim V = m \geq 2$. Assume that V is not contained in any hyperplane of $\mathbf{P}^M$. Let $H_1, \ldots, H_k$ be hyperplanes of $\mathbf{P}^M$ such that $D_j := H_j \cap V$ are pure $(m-1)$-dimensional smooth analytic subsets of V. Let $D_1, \ldots, D_k$ have normal crossings in V. Let $A_1, \ldots, A_k$ be pure $(n-1)$-dimensional analytic subsets of $\mathbf{C}^n$ with $\operatorname{codim} A_i \cap A_j \geq 2$ whenever $i \neq j$. Put

$A := A_1 \cup \cdots \cup A_k$. Suppose k is an integer such that $K_V \otimes (H|V)^{k-2} \geq 0$ and $k \geq 5$, where K_V is the canonical bundle of V and H is the hyperplane section bundle on $\mathbf{P}^M$. Let $F: \mathbf{C}^n \to V$ be a meromorphic map. Let

$$\begin{aligned}(1.5)\qquad &\mathscr{F}_C(\mathbf{C}^n, V, k)\\ &:= \{f: \mathbf{C}^n \to V \text{ is meromorphic} | \operatorname{rank} f = m,\\ &\qquad f|A = F|A \text{ and } f^{-1}(D_j) = A_j \text{ for } j = 1, \ldots, k\}.\end{aligned}$$

Since $f: \mathbf{C}^n \to V$ can be regarded as $f: \mathbf{C}^n \to \mathbf{P}^M$ by composing with the inclusion map: $V \hookrightarrow \mathbf{P}^M$. So, $\mathscr{F}_C(\mathbf{C}^n, V, k) = \mathscr{F}_C(\mathbf{C}^n, V, k) \cap \mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^M, k)$. We say $f, g, h \in \mathscr{F}_C(\mathbf{C}^n, V, k)$ *satisfy Property* (P) if f, g, h as maps in $\mathscr{F}_B(\mathbf{C}^n, P^M, k)$ satisfy Property (P).

THEOREM C. *If $f, g, h \in \mathscr{F}_C(\mathbf{C}^n, V, k)$, then f, g, h satisfy Property* (P). *Consequently f, g, h are algebraically dependent.*

REMARK. (i) When $V = \mathbf{P}^M$, Theorem C is Theorem A.

(ii) By [**D2**], if $K_V \otimes (H|V)^{K-2} > 0$, then $\mathscr{F}_C(\mathbf{C}^n, V, k)$ contains at most one map.

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2. Preliminaries. We give or review some notions and definitions (cf. [**S7**, **S8**]).

DEFINITION 2.1. (For meromorphic maps.)

(a) Let X and Y be connected complex manifolds. Let S be a proper analytic subset of X. Then $X - S$ is dense open in X. Let $f: X - S \to Y$ be a holomorphic map. The *closed graph* of f is the closure $\Gamma(f) := \overline{\{(x, f(x)) | x \in X - S\}}$ of the graph of f over $X - S$ in $X \times Y$. Let $\pi: \Gamma(f) \to X$ and $\hat{f}: \Gamma(f) \to Y$ be the natural projections. The map f is said to be *meromorphic* on X if $\Gamma(f)$ is analytic in $X \times Y$ and π is proper. Let $\#P$ be the cardinality of P. The *indeterminacy* $I_f := \{x \in X | \#\hat{f}(\pi^{-1}(x)) > 1\}$ is analytic and contained in S. Codim $I_f \geq 2$. We can assume $S = I_f$. In this paper, we always assume that Y is algebraic and compact.

(b) Let V be a complex vector space of dimension $m + 1$. Put $V_* = V - \{0\}$. Then $\mathbf{C}_* = \mathbf{C} - \{0\}$ acts by multiplication on V_*. The quotient space $\mathbf{P}(V) := V_*/\mathbf{C}_*$ is the projective space associated

to V. Let $\mathbf{P}: V_* \to P(V)$ be the residual map. If $A \subseteq V$, define $\mathbf{P}(A) := \{\mathbf{P}(a) | 0 \neq a \in A\}$. Let S and X be as in (a). Let $f: X - S \to \mathbf{P}(V)$ be a holomorphic map. Let $x \in X$ and U be an open connected neighborhood of x. A holomorphic map $\phi_f: U \to V$ is called a *representation of f at x* if $\phi_f \not\equiv 0$ and if $f(z) = \mathbf{P}(\phi_f(z))$ for all $z \in U - S$ with $\phi_f(z) \neq 0$. The representation is said to be *reduced* if $\operatorname{Codim} \phi_f^{-1}(0) \geq 2$. The map f is meromorphic if and only if there is a representation at every point of X. The representation is *global* if $U = X$. If $X = \mathbf{C}^n$, there is a global reduced representation. So, maps in $\mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, k)$ have global representations.

(c) We have assumed that Y is an algebraic compact manifold. A meromorphic map $f: X \to Y$ is called *algebraically non-degenerate* if the image of f is not contained in any proper analytic subset of Y. If $Y = \mathbf{P}^m$, then f is called *linearly non-degenerate* if the image of f is not contained in any hyperplane of $\mathbf{P}^m$.

(d) Meromorphic maps $f^1, \ldots, f^k$ of $\mathbf{C}^n$ into any algebraic manifold Y are called *algebraically dependent* if the image of the meromorphic map $f^1 \times \cdots \times f^k: \mathbf{C}^n \to Y \times \cdots \times Y$ (k times) is contained in some proper analytic subset of $Y \times \cdots \times Y$ (k times), where $(f^1 \times \cdots \times f^k)(z) := (f^1(z), \ldots, f^k(z))$ for generic points z in $\mathbf{C}^n$.

(e) Let X be as in (a). Let $\mathscr{O}$ be the sheaf of germs of holomorphic functions on X. If $x \in X$, let $\mathscr{O}_x$ be the stalk of $\mathscr{O}$ over x, and m_x be the maximal ideal in $\mathscr{O}_x$. Let m_x^p be its pth power with $m_x^0 := \mathscr{O}_x$. Take $0 \neq f \in \mathscr{O}_x$, one and only one integer $\mu(f) \geq 0$ exists such that $f \in m_x^{\mu(f)} - m_x^{\mu(f)+1}$. Here $\mu(f)$ is called the *zero-multiplicity of f*. Let $U \subset X$ be open connected, $f \not\equiv 0$ be holomorphic function on U. Take any $x \in U$ and let $f_x \in \mathscr{O}_x$ be the germ defined by f in $\mathscr{O}_x$. Then $\mu_f^0(x) := \mu(f_x) \geq 0$ is called the *zero-multiplicity of f at x*. The function $\mu_f^0: U \to \mathbf{Z}$ is called the *zero divisor of f*.

A *divisor* v on X is an integral value function $v: X \to \mathbf{Z}$ such that for every $x \in X$ there exists an open connected neighborhood U of x and holomorphic functions $g \not\equiv 0 \not\equiv h$ on U with

$$\operatorname{Codim}(g^{-1}(0) \cap h^{-1}(0)) \geq 2,$$

such that $v|U = \mu_g^0 - \mu_h^0$. $v \equiv 0$ is called the *null divisor*. If v is not the null divisor, $S := \operatorname{supp} v$ is a pure $(n-1)$-dimensional analytic subset of X where $\dim X = n$. Let $\operatorname{Reg}(S)$ be the set of regular points of S. Then $v|\operatorname{Reg}(S)$ is locally constant. Let L be the set of irreducible components of S. For each $B \in L$, $v|\operatorname{Reg}(S) \cap B \equiv p(v, B) \not\equiv 0$ is a constant integer. We say that $v \geq 0$ if $p(v, B) \geq 0$ for all $B \in L$.

For any pure $(n-1)$-dimensional analytic subset A of X, there exists a unique divisor v_A such that $v_A|\operatorname{Reg}(A) \equiv 1$. Obviously $v_A \geq 0$. The locally finite sum

$$v = \sum_{B \in L} p(v, B) v_B \tag{2.1}$$

is called the *analytic chain representation of* v. If $0 \leq n \in \mathbf{Z}$, the divisor

$$v^{(n)} := \sum_{B \in L} \min(n, p(v, B)) v_B \tag{2.2}$$

is called the *truncation of* v *at level* n.

Let $f: X \to Y$ be meromorphic. Let $v: Y \to \mathbf{Z}$ be a divisor on Y with $S = \operatorname{supp} v$. Assume that $f(X) \not\subseteq S$. Since $\operatorname{Codim} I_f \geq 2$, there exists a unique divisor $f^*(v)$ on X, called the *pull-back divisor* of v by f, such that for any pair of open, connected, non-empty subsets U of $X - I_f$ and W of Y with $f(U) \subseteq W$, there are holomorphic functions $g \not\equiv 0 \not\equiv h$ on W with $\operatorname{Codim}(g^{-1}(0) \cap h^{-1}(0)) \geq 2$ and $v|W = \mu_g^0 - \mu_h^0$, then $g \circ f|U \not\equiv 0 \not\equiv h \circ f|U$, and $f^*(v)|U = (\mu_{g \circ f}^0 - \mu_{h \circ f}^0)|U$.

DEFINITION 2.2. (For Nevanlinna theory.)

(a) For $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, let $|z|^2 := |z_1|^2 + \cdots + |z_n|^2$. $S(r) := \{z \in \mathbf{C}^n \,|\, |z| = r\}$. $B(r) := \{z \in \mathbf{C}^n \,|\, |z| < r\}$. $d^c := (\partial - \bar{\partial})/4\pi\sqrt{-1}$. Let $\omega_0 := dd^c \log |z|^2$ be the homogeneous metric form on $\mathbf{C}^n$, and $\varphi_0 := dd^c |z|^2$ be the Euclidean metric form on $\mathbf{C}^n$. Let $\sigma := d^c \log |z| \wedge \omega_0^{n-1}$ be the Poincaré form on $S(r)$. Let ω be the Fubini-Study metric form on $\mathbf{P}^{n-1}$ given by $\mathbf{P}^* \omega = \omega_0$. Denote $\mathbf{C}_*^n = \mathbf{C}^n - \{0\}$.

(b) Let v be a divisor on $\mathbf{C}^n$ with $S = \operatorname{supp} v$. For $t > 0$, the *counting function* n_v is defined by

$$n_v(t) := \begin{cases} \displaystyle\sum_{Z \in S \cap B(t)} v(z), & \text{if } n = 1, \\ \displaystyle\frac{1}{t^{2n-1}} \int_{S \cap B(t)} v \varphi_0^{n-1}, & \text{if } n > 1. \end{cases} \tag{2.3}$$

If $n > 1$, also $n_v(t) = \int_{S \cap B(t)} v \omega_0^{n-1} + v(0)$. For $0 < s < r$, the *integrate counting function* is defined by

$$N(v; r, s) := \int_s^r n_v(t) \frac{dt}{t}. \tag{2.4}$$

If $f: \mathbf{C}^n \to Y$ is a meromorphic map, and v is a divisor on Y with $f(\mathbf{C}^n) \not\subseteq \operatorname{supp} v$, and $f^* v$ is the pull-back divisor of v by f, we abbreviate $N(f^* v; r, s) = N_f(v; r, s)$. If A is a pure 1-codimensional analytic

subset of Y, abbreviate $N(f^*(v_A);r,s) = N_f(A;r,s)$ and $N(v_A;r,s) = N(A;r,s)$.

(c) Let $f: \mathbf{C}^n \to Y$ be meromorphic. Let L be a hermitian holomorphic line bundle over Y with a metric h. The *characteristic function of* f is defined by

$$A_f(L;t) = A_f(L,h;t) := \frac{1}{t^{2n-2}} \int_{B(t)} f^*(c_h) \wedge \varphi_0^{n-1}, \quad \text{for } t > 0,$$

$$(2.5) \quad T_f(L;r,s) = T_f(L,h;r,s) := \int_s^r A_f(L;t) \frac{dt}{t}, \quad \text{for } 0 < s < r.$$

Let $A_f(L;0) = \lim_{t\to 0} A_f(L;t)$. We also have

$$A_f(L;t) = \int_{B(t)} f^*(c_h) \wedge \omega_0^{n-1} + A_f(L;0), \quad \text{for } t \geq 0,$$

where c_h is the Chern form of the metric h (Note: $A_f(L;0) = 0$ if f is holomorphic at 0).

If $f: \mathbf{C}^n \to \mathbf{P}^m$ is meromorphic, and L is the hyperplane section bundle H over $\mathbf{P}^m$ with the metric $h := \{h_\alpha := |z_\alpha|^2/(|z_0|^2+\cdots+|z_m|^2)$ on $U_\alpha\}$, where $U_\alpha := \{[Z_0 : \cdots : Z_m] | Z_\alpha \neq 0\}$ for $\alpha = 0, \ldots, m$, then $c_h = \omega$. We abbreviate $T_f(H,h;r,s) = T_f(r,s)$.

(d) Let $f: \mathbf{C}^n \to Y$, L, h be as in (c), Y compact. The set of all holomorphic sections in L over Y is a finite dimensional complex vector space $\Gamma(Y,L)$. Since Y is compact, we can select a hermitian metric ι on $\Gamma(Y,L)$ such that $||s(y)||_h \leq ||s||_\iota$ for all $s \in \Gamma(Y,L)$ and all $y \in Y$. If $y \in Y$, and $a \in \mathbf{P}(\Gamma(Y,L))$, then $a = \mathbf{P}(\tilde{a})$ with $0 \neq \tilde{a} \in \Gamma(Y,L)$ and

$$(2.6) \qquad 0 \leq ||y,a||_{h,\iota} := \frac{||\tilde{a}(y)||_h}{||\tilde{a}||_\iota} \leq 1$$

is a well-defined function of y and a. Let $a \in \mathbf{P}(\Gamma(Y,L))$ and assume $f(\mathbf{C}^n) \not\subseteq \operatorname{supp} \mu_a$, where μ_a is the associated divisor of a. Then $||f,a||_{h,\iota} \not\equiv 0$ and the *proximity function*

$$(2.7) \quad m_f(a;r) = m_f(a,L,h,\iota;r) := \int_{S(r)} \log \frac{1}{||f,a||_{h,\iota}} \sigma > 0 \text{ exists.}$$

If $Y = \mathbf{P}^m$, and D is a hyperplane on $\mathbf{P}^m$ defined by $a_0w_0 + \cdots + a_mw_m = 0$, assume $f(\mathbf{C}^n) \not\subseteq D$, since D associates a unique element

in $\mathbf{P}(\Gamma(\mathbf{P}^m, H))$, where H is as in (c),

$$(2.8)\quad m_f(D;r) := -\int_{S(r)} \log \frac{|a_0 f_0 + \cdots + a_m f_m|}{(|f_0|^2 + \cdots + |f_m|^2)^{1/2}(|a_0|^2 + \cdots + |a_m|^2)^{1/2}}\sigma \geq 0$$

where $\phi_f = (f_0, \ldots, f_m)$ is a reduced representation of f.

(e) The notation $||A(r) \leq B(r)$ means that there is some set $E \subset \mathbf{R}^+$ with $\int_E dr < \infty$, such that $A(r) \leq B(r)$ for all $r \in \mathbf{R}^+ - E$; $||\lim_{r\to\infty} A(r) = B$ means that there is some subset $E \subset R^+$ with $\int_E dr < \infty$, such that $\lim_{r\to\infty, r\in\mathbf{R}^+-E} A(r) = B$; $||A(r) \leq o(\log r) + B(r)$ means that there is some subset $E \subset \mathbf{R}^+$ with $\int_E dr < \infty$, and a function $0 \leq \varepsilon\colon \mathbf{R}^+ \to \mathbf{R}^+$ with $\varepsilon(r) \to 0$ as $r \to +\infty$, such that $A(r) \leq \varepsilon(r)\log r + B(r)$ for all $r \in \mathbf{R}^+ - E$.

(f) Sometimes we use C to mean a positive constant number which is independent of $r \in \mathbf{R}^+$.

3. Lemma of the logarithmic derivative. In 1925, Nevanlinna proved the "lemma of the logarithmic derivative" **[N2]**, **[H1]** in the form

$$(3.1)\quad ||m_{f'/f}(r) \leq 10\log^+ T_f(r) + 10\log r + 4\log^+\log^+(1/|f(0)|)$$

for meromorphic function f on $\mathbf{C}$. The generalization for it on $\mathbf{C}^n$ was given by Vitter in 1977 **[V1]**. Shiffman **[S4]** gave a refined lemma in which $O(\log r)$ is replaced by $o(\log r)$:

LEMMA 3.1 (**[S4]**, *Lemma* 3.11]). *Let F be a non-constant meromorphic function on $\mathbf{C}^n$. Then for any i, $1 \leq i \leq n$ and $0 < s < r$*

$$(3.2)\quad \left\| \int_{S(r)} \log^+ \left| \frac{\partial F}{\partial z_i} \Big/ F \right| \sigma \leq C\log^+(T_f(r,s)) + o(\log r). \right.$$

We need the following Lemma 3.2. The simplified proof of this lemma given below is due to W. Stoll.

For Lemma 3.2, some preparation is needed.

$\tilde{U} := \{(\tilde{z}_1, \ldots, \tilde{z}_n) \in \mathbf{C}^n | \tilde{z}_1 \neq 0\}$ is open in $\mathbf{C}^n$. Since $\tilde{U} = \mathbf{P}^{-1}(\mathbf{P}(\tilde{U}))$, the image $U := \mathbf{P}(\tilde{U})$ is open in $\mathbf{P}^{n-1}$ and $\mathbf{P}^{n-1} - U = \mathbf{P}(\{0\} \times \mathbf{C}^{n-1})$ is a hyperplane in $\mathbf{P}^{n-1}$. Now $E := \{1\} \times \mathbf{C}^{n-1}$ is an affine plane in $\mathbf{C}^n$ of dimension $n-1$ with $E \subset \tilde{U}$. The restriction $\hat{\mathbf{P}} := \mathbf{P}\colon E \to U$ is biholomorphic. If $z = \mathbf{P}(\tilde{z}) \in U$, then $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_n) \in \tilde{U}$ and $\hat{\mathbf{P}}^{-1}(z) = \tilde{z}/\tilde{z}_1$. Let

$$(3.3)\quad L\colon E \to \mathbf{C}^n$$

be the inclusion map. Let ω be the Fubini-Study form on $\mathbf{P}^{n-1}$. Then $L^*(\omega_0) = \hat{\mathbf{P}}^*(\omega)$. Define

$$\tilde{u}_j := (\underbrace{0, \ldots, 0}_{j-1}, 1, 0, \ldots, 0) \in \mathbf{C}^n.$$

Then $\tilde{u}_1, \ldots, \tilde{u}_n$ is an orthonormal base of $\mathbf{C}^n$. The dual base $\tilde{u}_1^*, \ldots, \tilde{u}_n^*$ of $(\mathbf{C}^n)^*$ is orthonormal. Put $u_j^* := \mathbf{P}(\tilde{u}_j^*)$. Pick $x \in U$. Then $e = (e_1, \ldots, e_n) = \hat{\mathbf{P}}^{-1}(x) \times E$ satisfies $\tilde{u}_1^*(e) = e_1 = 1$ and $|e| \geq 1$. Hence

$$\|x, u_1^*\| = \frac{|\tilde{u}_1^*(e)|}{|e|\,|\tilde{u}_1^*|} = \frac{1}{|e|}. \tag{3.4}$$

Let $F: \mathbf{C}^n \to \mathbf{P}^1$ be a non-constant meromorphic function. Then $S := \operatorname{supp} \mu_F^0 \cup \operatorname{supp} \mu_F^\infty$ is empty or an analytic subset of pure dimension $n-1$ in $\mathbf{C}^n$, where $\mu_F^\infty := \mu_{F^{-1}}^0$, $\phi_{F^{-1}} = (f_1, f_0)$ if $\phi_F = (f_0, f_1)$ is a reduced representation of F. Then $E_0 := \{e \in E | \mathbf{C}e \subseteq S\}$ has measure zero in E. Take any $e \in E - E_0$, a meromorphic function $F_e \not\equiv 0$ is defined by $F_e(w) := F(we)$ for all $w \in \mathbf{C}$. Obviously, if $e = (e_1, \ldots, e_n) \in E - E_0$ then $e_1 = 1$ and

$$F_e'(w) = \sum_{i=1}^n e_i \frac{\partial F}{\partial z_i}(we), \qquad \text{for all } w \in \mathbf{C}. \tag{3.5}$$

The function $g_e := F_e'/F_e$ is meromorphic on $\mathbf{C}$ for each $e \in E - E_0$.

LEMMA 3.2. *For each $0 < s < r \in \mathbf{R}^+$, the integral*

$$\int_{e \in E - E_0} m_{g_e}\left(\infty; \frac{r}{|e|}\right) L^*(\omega_0^{n-1})(e)$$

exists with

$$\int_{e \in E - E_0} m_{g_e}\left(\infty; \frac{r}{|e|}\right) L^*(\omega_0^{n-1})(e) \leq C(\log^+ T_F(r, s)) + o(\log r). \tag{3.6}$$

Proof. Take $e \in E - E_0$ and $w \in \mathbf{C}$ with $0 \neq F_e(w) \neq \infty$. Then $e = (e_1, \ldots, e_n)$ and

$$|F_e'(w)|^2 = \left|\sum_{i=1}^n e_i \frac{\partial F}{\partial z_i}(we)\right|^2 \leq |e|^2 \sum_{i=1}^n \left|\frac{\partial F}{\partial z_i}(we)\right|^2$$

or

$$\log^+|g_e(w)| = \log^+\left|\frac{F_e'(w)}{F_e(w)}\right|$$
$$\leq \log|e| + \frac{1}{2}\log^+\sum_{i=1}^{n}\left|\frac{\partial F}{\partial z_i}(we)\right|^2 |F(we)|^{-2}$$
$$\leq \log|e| + \sum_{i=1}^{n}\log^+\left|\frac{\partial F}{\partial z_i}(we)\Big/F(we)\right| + \frac{1}{2}\log n,$$

which implies

$$m_{g_e}\left(\infty;\ \frac{r}{|e|}\right)$$
$$\leq \log|e| + \sum_{i=1}^{n}\frac{1}{2\pi}$$
$$\times \int_0^{2\pi}\log^+\left|\frac{\partial F}{\partial z_i}\left(r\exp(\sqrt{-1}\theta)\frac{e}{|e|}\right)\Big/F\left(r\exp(\sqrt{-1}\theta)\frac{e}{|e|}\right)\right| d\theta$$
$$+ \frac{1}{2}\log n.$$

By [**W1**, p. 130, (2.12)] we have

$$\int_{e\in E-E_0}\log|e|L^*(\omega_0^{n-1})(e) \tag{3.7}$$
$$= \int_{e\in E-E_0}\log\frac{1}{||P(e), u_1^*||}\hat{\mathbf{P}}^*(\omega^{n-1})(e)$$
$$= \int_{x\in\mathbf{P}^{n-1}}\log\frac{1}{||x, u_1^*||}\omega^{n-1}(x) = \frac{1}{2}\sum_{\lambda=1}^{n-1}\frac{1}{\lambda}.$$

A diffeomorphism Φ: $E\times\mathbf{R}(0,2\pi)\to\tilde{E}$ onto a subset $\tilde{E}$ of $S(r)$ is defined by $\Phi(e,\theta) = r\exp(\sqrt{-1}\theta)e/|e|$ such that $\tilde{E}$ is open in the topology of $S(r)$ and such that $S(r)-\tilde{E}$ has measure zero. Then

$$\int_{e\in E-E_0}\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{\partial F}{\partial z_i}(\Phi(e,\theta))\Big/F(\Phi(e,\theta))\right| d\theta\hat{\mathbf{P}}^*(\omega)^{n-1}(e)$$
$$= \int_{S(r)}\log^+\left|\frac{\partial F}{\partial Z_i}\Big/F\right|\sigma.$$

Finally,

$$\int_{e\in E-E_0} m_{g_e}\left(\infty;\frac{r}{|e|}\right) L^*(\omega_0^{n-1})(e)$$
$$\leq \sum_{i=1}^{n}\int_{S(r)} \log^+\left|\frac{\partial F}{\partial Z_i}\Big/ F\right| \sigma + \frac{1}{2}\log n + \sum_{\lambda=1}^{n-1}\frac{1}{\lambda},$$

then (3.6) is proved by applying Lemma 3.1. □

4. Lemmas from Nevanlinna Main Theorems.

LEMMA 4.1. *Let* $f \in \mathscr{F}_C(\mathbf{C}^n, V, k)$, $K_V \otimes (H|V)^{k-2} \geq 0$. *Then for* $0 < s < r \in \mathbf{R}^+$,

$$\|2T_f(H|V;r,s) \leq \sum_{j=1}^{k} N(A_j;r,s) + C(\log^+ T_f(r,s)) + o(\log r). \tag{4.1}$$

In particular, if $V = \mathbf{P}^m$, *i.e.,* $f \in \mathscr{F}_A(\mathbf{C}^n, \mathbf{P}^m, m+3)$,

$$\|2T_f(r,s) \leq \sum_{j=1}^{m+3} N(A;r,s) + C(\log^+ T_f(r,s)) + o(\log r). \tag{4.2}$$

Proof. From [**S4**, Theorem 3.1] we have

$$\|2T_f(H|V;r,s) \leq \sum_{j=1}^{k} N_f(H_j;r,s) - N(R_f;r,s) + C(\log^+ T_f(r,s)) + o(\log r), \tag{4.3}$$

where R_f is the ramification divisor of f which is defined in [**S4**, p. 73], or [**D2**]. By [**D2**, Lemma 3.2],

$$\sum_{j=1}^{k} N_f(H_j;r,s) - N(R_f;r,s) \leq \sum_{j=1}^{k} \overline{N}_f(H_j;r,s) + C \tag{4.4}$$

where $\overline{N}_f(v;r,s) := N((f^*v)^{(1)};r,s)$ (cf. (2.2)). Note that $\overline{N}_f(H_j;r,s) = N(A_j;r,s)$; then (4.1) is proved by (4.3) and (4.4). □

LEMMA 4.2. *Let* $f \in \mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, 3m+1)$. *Then for* $0 < s < r \in \mathbf{R}^+$,

$$\|2T_f(r,s) \leq \sum_{j=1}^{3m+1} N(A_j;r,s) + C(\log^+ T_f(r,s)) + o(\log r). \tag{4.5}$$

Proof. From [**V1**, (5.5), (5.6), p. 103],

$$[(3m+1)-m-1]T_f(r,s) = 2mT_f(r,s) \leq \log(|g_1 \cdots g_{3m+1}|/J) + \log D + C, \tag{4.6}$$

where

(1) $g_i := f_0 a_0^{(i)} + \cdots + f_m a_m^{(i)}$, and $a_0^{(i)}, \ldots, a_m^{(i)}$ are given by (1.1), for $i = 1, \ldots, 3m+1$, and $(f_0, \ldots, f_m)$ is a reduced representation of f.

(2) $J := \|f \wedge D^{\alpha_1} f \wedge \cdots \wedge D^{\alpha_m} f\| = \det(D^{\alpha_i} f)_{0 \leq i \leq m, \alpha_0 = 0}$, where $(0, \alpha_1, \ldots, \alpha_m)$ are nonnegative integers such that there exists a dense open subset U of $\mathbf{C}^n$, for any $z \in U$, $f(z), D^{\alpha_1} f(z), \ldots, D^{\alpha_m} f(z)$ are linearly independent.

(3)

$$D := \det \begin{pmatrix} 1 & \cdots & 1 \\ D^{\alpha_1} f_0/f_0 & \cdots & D^{\alpha_1} f_m/f_m \\ \cdots & \cdots & \cdots \\ D^{\alpha_m} f_0/f_0 & \cdots & D^{\alpha_m} f_m/f_m \end{pmatrix}.$$

From [**F3**, p. 255], we have

$$\int_{S(r)} \log(|g_1 \cdots g_{3m+1}|/J)\sigma \leq \sum_{j=1}^{3m+1} N_f^{H_j}(r,s)^{[k_0]} \leq m \sum_{j=1}^{3m+1} \overline{N}_f(H_j; r, s), \tag{4.7}$$

where

(1) By [**F3**, Proposition 4.5], k_0 always can be m for nondegenerate meromorphic map from $\mathbf{C}^n$ into $\mathbf{P}^m$.

(2) By [**F3**, p. 250], the second inequality holds because

$$N_f^{H_j}(r,s)^{[m]} := N((f^* H_j)^{(m)}; r, s) \quad \text{(cf. (2.2))} \leq m\overline{N}_f(H_j; r, s). \tag{4.8}$$

With (4.6), (4.7) and (4.8),

$$2mT_f(r,s) \leq \sum_{j=1}^{3m+1} m\overline{N}_f(H_j; r, s) + \int_{S(r)} \log D \cdot \sigma + C.$$

Apply Lemma 3.1 repeatedly to $\int_{S(r)} \log D \cdot \sigma$, then (4.5) follows. □

LEMMA 4.3. *For any $n, m \in \mathbf{Z}^+$, let $f, g: \mathbf{C}^n \to \mathbf{P}^m$ be linearly nondegenerate meromorphic maps with $f \not\equiv g$. Let D be any hypersurface on $\mathbf{P}^m$. If $f^{-1}(D) = g^{-1}(D)$ and $f|f^{-1}(D) = g|g^{-1}(D)$, then*

$$\overline{N}_f(D; r, s) \leq T_f(r,s) + T_g(r,s) + C. \tag{4.9}$$

Proof. See [D2, Lemma 3.4]. (Note: Lemma 3.4 in [D2] needs the condition $n > m$, but this is seen to be unnecessary by checking the proof.) □

LEMMA 4.4. *Let $f, g \in \mathcal{F}_B(\mathbf{C}^n, \mathbf{P}^m, k)$ with $f \not\equiv g$ and with*

$$\text{(4.10)} \quad \begin{cases} 2T_f(r,s) \le \sum_{j=1}^{k} N(A_j; r, s) + C(\log^+ T_f(r,s)) + o(\log r), \\ 2T_g(r,s) \le \sum_{j=1}^{k} N(A_j; r, s) + C(\log^+ T_g(r,s)) + o(\log r). \end{cases}$$

Then

$$\text{(4.11)} \quad \left\| \lim_{r\to\infty} \sum_{j=1}^{k} N(A_j; r, s) \Big/ T_f(r,s) = 2 \quad \textit{and} \right.$$

$$\text{(4.12)} \quad \left\| \lim_{r\to\infty} T_f(r,s) \Big/ T_g(r,s) = 1. \right.$$

Proof. From (4.10) and Lemma 4.3,

$$\begin{aligned} \left\| T_f(r,s) + T_g(r,s) \right. &\le \sum_{j=1}^{k} N(A_j; r, s) \\ &\quad + C(\log^+ T_f(r,s) + \log T_g^+(r,s)) + o(\log r) \\ &\le T_f(r,s) + T_g(r,s) + C(\log^+ T_f(r,s) + \log^+ T_g(r,s)) \\ &\quad + o(\log r). \end{aligned}$$

So,

$$\text{(4.13)} \quad \left\| 1 \le \frac{[\sum_{j=1}^{k} N(A_j; r, s) + C(\log^+ T_f(r,s) + \log^+ T_g(r,s)) + o(\log r)]}{[T_f(r,s) + T_g(r,s)]} \right.$$
$$\le 1 + \frac{[C(\log^+ T_f(r,s) + \log^+ T_g(r,s)) + o(\log r)]}{[T_f(r,s) + T_g(r,s)]}.$$

By (4.10),

$$\text{(4.14)} \quad \left\| \overline{\lim_{r\to\infty}} T_f(r,s) \Big/ \sum_{j=1}^{k} N(A_j; r, s) \le 1/2 \quad \text{and} \right.$$

$$\text{(4.15)} \quad \left\| \overline{\lim_{r\to\infty}} T_g(r,s) \Big/ \sum_{j=1}^{k} N(A_j; r, s) \le 1/2. \right.$$

Apply (4.15) to (4.13), we obtain (4.11). Then apply (4.11) to (4.13), we obtain (4.12). □

COROLLARY 4.5. *Let* $f, g \in \mathscr{F}_A(\mathbf{C}^n, \mathbf{P}^m, m+3)$ *(respectively* $\mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, 3m+1)$, *or* $\mathscr{F}_C(\mathbf{C}^n, V, k)$*) with* $f \not\equiv g$. *Then*

$$\left\| \lim_{r\to\infty} \sum_{j=1}^{m+3} N(A_j; r, s) \Big/ T_f(r, s) = 2, \right. \tag{4.16}$$

(resp.

$$\left\| \lim_{r\to\infty} \sum_{j=1}^{3m+1} N(A_j; r, s) \Big/ T_f(r, s) = 2, \quad \text{or} \right. \tag{4.17}$$

$$\lim_{r\to\infty} \sum_{j=1}^{k} N(A_j; r, s) \Big/ T_f(r, s) = 2) \tag{4.18}$$

$$\left\| \lim_{r\to\infty} T_f(r, s) \Big/ T_g(r, s) = 1. \right. \tag{4.19}$$

5. Proof for holomorphic curves. In this section, we prove Theorem B when $n = 1$. Although this case is only for holomorphic curves, the method of proof will be essentially used for the remaining cases of Theorems A, B and C.

LEMMA 5.1. *Let* $n, m, k \in \mathbf{Z}^+$. *Let* $\mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, k) \neq \emptyset$. *Then there exists a dense subset* $\mathscr{P} \subset \mathbf{C}_*^{m+1}$, *such that for any* $p = (p_0, \ldots, p_m) \in \mathscr{P}$, *the hyperplane* H_p *defined by* $p_0 w_0 + \cdots + p_m w_m = 0$ *satisfies*

$$\text{Codim} \left(\bigcup_{j=1}^{k} A_j \cap f^{-1}(H_p) \right) \geq 2 \quad \textit{for all } f \in \mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, k). \tag{5.1}$$

Proof. For any irreducible pure $(n-1)$-dimensional component σ of $\bigcup_{j=1}^{k} A_j$, set $K_\sigma := \{(x_0, \ldots, x_m) \in \mathbf{C}^{m+1} | x_0 f_0(z) + \cdots + x_m f_m(z) = 0$ for all $z \in \sigma\}$, where $\phi_f = (f_0, \ldots, f_m)$ is a reduced representation of $f \in \mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, k)$. K_σ is independent of the choice of f for $f|A = F|A$. K_σ is a complex vector subspace of $\mathbf{C}^{m+1}$ with $\dim K_\sigma \leq m$.

Let $K := \bigcup_\sigma K_\sigma$. Then K is a union of at most a countable number of m-dimensional complex vector subspaces in $\mathbf{C}^{m+1}$. Let $\mathscr{P} := \mathbf{C}^{m+1} - K$. Then $\mathscr{P}$ meets the requirement of the lemma. □

Now for any $n, m, k \in \mathbf{Z}^+$ and for $f, g, h \in \mathscr{F}_B(\mathbf{C}, \mathbf{P}^m, k)$, let $\mathscr{P}$ be associated to $\mathscr{F}_B(\mathbf{C}, \mathbf{P}^m, k)$ as in Lemma 5.1, and take $p = (p_0, \ldots, p_m) \in \mathscr{P}$. Take $i \in \{1, \ldots, k\}$. We define a meromorphic function φ on $\mathbf{C}$ by

$$\begin{aligned}(5.2)\qquad \varphi = \varphi(f, g, h, p, i) &:= \det \begin{pmatrix} F & G & H \\ F'/F & G'/G & H'/H \\ 1 & 1 & 1 \end{pmatrix} \\ &= \left(\frac{G'}{G} - \frac{H'}{H}\right) F + \left(\frac{H'}{H} - \frac{F'}{F}\right) G + \left(\frac{F'}{F} - \frac{G'}{G}\right) H,\end{aligned}$$

where

$$(5.3)\qquad \begin{cases} F = F(f, p, i) := (a_0^{(i)} f_0 + \cdots + a_m^{(i)} f_m)/(p_0 f_0 + \cdots + p_m f_m), \\ G = G(g, p, i) := (a_0^{(i)} g_0 + \cdots + a_m^{(i)} g_m)/(p_0 g_0 + \cdots + p_m g_m), \\ H = H(h, p, i) := (a_0^{(i)} h_0 + \cdots + a_m^{(i)} h_m)/(p_0 h_0 + \cdots + p_m h_m), \end{cases}$$

where $\phi_f = (f_0, \ldots, f_m)$, $\phi_g = (g_0, \ldots, g_m)$, $\phi_h = (h_1, \ldots, h_m)$ are reduced representations of f, g, h, respectively, and $(a_0^{(i)}, \ldots, a_m^{(i)})$ is defined in (1.1).

Sometimes we define

$$(5.4)\qquad \begin{aligned} P(f) &= p_0 f_0 + \cdots + p_m f_m, \\ H_i(f) &= a_0^{(i)} f_0 + \cdots + a_m^{(i)} f_m, \text{ etc.} \end{aligned}$$

LEMMA 5.2. *Let $f, g, h \in \mathscr{F}_B(\mathbf{C}, \mathbf{P}^m, 3m+1)$. Assume*

$$\varphi = \varphi(f, g, h, p, i) \not\equiv 0,$$

for some $p \in \mathscr{P}$ and some $i \in \{1, \ldots, 3m+1\}$. Then for $0 < s < r$,

$$(5.5)\qquad \left\| \varliminf_{r \to \infty} \frac{N(A_i, r, s)}{T_f(r, s)} \geq \frac{1}{2}. \right.$$

Proof. First we prove

$$(5.6)\qquad \bigcup_{j=1, j \neq i}^{3m+1} A_j \subset \varphi^{-1}(0) \cap (\varphi')^{-1}(0).$$

For any $a \in A_j$ with $j \neq i$, $j \in \{1, \ldots, 3m+1\}$, because the A_k are disjoint in $\mathbf{C}$ for $k = 1, \ldots, 3m+1$ and (5.1), we know that

$F(a) = G(a) = H(a) \in \mathbf{C}_*$. Then from (5.2), we have

(5.7) $\varphi(a) = 0$ and

$$(5.8)\quad \varphi'(a) = \left(\frac{G'(a)}{G(a)} - \frac{H'(a)}{H(a)}\right) F'(a) + \left(\frac{H'(a)}{H(a)} - \frac{F'(a)}{F(a)}\right) G'(a) + \left(\frac{F'(a)}{F(a)} - \frac{G'(a)}{G(a)}\right) H'(a) + \left[\left(\frac{G'}{G}\right)'(a) - \left(\frac{H'}{H}\right)'(a)\right] F(a) + \left[\left(\frac{H'}{H}\right)'(a) - \left(\frac{F'}{F}\right)'(a)\right] G(a) + \left[\left(\frac{F'}{F}\right)'(a) - \left(\frac{G'}{G}\right)'(a)\right] H(a) = 0.$$

Next we show that if $b \in A_i$, then

$$(5.9)\qquad \varphi(b) \neq \infty.$$

When w is near b in $\mathbf{C}$, $F(w) = (w-b)^{k_1} F^*(w)$, $F^*(b) \neq 0$; $G(w) = (w-b)^{k_2} G^*(w)$, $G^*(b) \neq 0$; $H(w) = (w-b)^{k_3} H^*(w)$, $H^*(b) \neq 0$, where integers $k_1, k_2, k_3 \geq 1$. Then $\varphi(b) \neq \infty$ follows from (5.2).

Since $\varphi \not\equiv 0$ and (5.6) and the First Main Theorem,

$$(5.10)\quad 2 \sum_{j=1, j\neq i}^{3m+1} N(A_j; r, s) \leq T_\varphi(r, s) + O(1) = N_\varphi(\infty; r, s) + m_\varphi(\infty; r) - m_\varphi(\infty; s) + O(1).$$

By (5.2), (5.3) and (5.9), all the poles of φ must be the ones of F, G, H. Hence

$$(5.11)\qquad N_\varphi(\infty; r, s) \leq N_F(\infty; r, s) + N_G(\infty; r, s) + N_H(\infty; r, s).$$

By Lemma 3.1,

$$(5.12)\quad \| m_\varphi(\infty; r) \leq m_F(\infty; r) + m_G(\infty; r) + m_H(\infty; r) + C(\log^+ T_F(r, s) + \log^+ T_G(r, s) + \log^+ T_H(r, s)) + o(\log r).$$

Note $T_F(r, s) \leq T_f(r, s) + C$, etc. Note $f \not\equiv g$ because $\varphi \not\equiv 0$, so we can apply the Corollary 4.5. By (5.10), (5.11), (5.12) and (4.19),

$$\left\| 2 \sum_{j=1, j\neq i}^{3m+1} N(A_j; r, s) \leq 3T_f(r, s) + o(\log r) + C(\log^+ T_f(r, s)). \right.$$

Then by (4.6) we prove (5.5). □

LEMMA 5.3. *Let* $\mathscr{F}_B(\mathbf{C}, \mathbf{P}^m, k)$, $\mathscr{P}$, $p \in \mathscr{P}$, $i \in \{1, \ldots, k\}$, $f, g, h \in \mathscr{F}_B(\mathbf{C}, \mathbf{P}^m, k)$, $\varphi = \varphi(f, g, h, p, i)$, $F = F(f, p, i)$, $G = G(g, p, i)$, $H = H(h, p, i)$ *be as in* (5.2), (5.3). *Assume* $k \geq 2$. *Then*

(5.13) $\varphi \equiv 0 \Leftrightarrow$ *There is constant* $(\lambda, \mu, \nu) \in \mathbf{C}_*^3$ *such that*

$$\frac{\lambda}{F} + \frac{\mu}{G} + \frac{\nu}{H} \equiv 0. \tag{5.14}$$

Proof. Set $\alpha := 1/F$, $\beta := 1/G$, $\gamma := 1/H$. Since

$$\varphi = \det \begin{pmatrix} 1/\alpha & 1/\beta & 1/\gamma \\ -\alpha'/\alpha & -\beta'/\beta & -\gamma'/\gamma \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{\alpha\beta\gamma} \det \begin{pmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix}$$

and $\alpha \not\equiv \infty$, $\beta \not\equiv \infty$, $\gamma \not\equiv \infty$ for f, g, h are linearly non-degenerate, put

$$\Delta := \det \begin{pmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix}.$$

Thus (5.13) is equivalent to

$$\Delta \equiv 0. \tag{5.15}$$

By Wronsky determinant, (5.14) is equivalent to

$$\det \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{pmatrix} = 0. \tag{5.16}$$

It suffices to prove (5.15)$\Leftrightarrow$(5.16).

For (5.16)$\Rightarrow$(5.15): By the definition of $\mathscr{F}_B(\mathbf{C}, \mathbf{P}^m, k)$ and $k \geq 2$, $H|A = G|A = H|A \not\equiv 0$, so $\lambda+\mu+\nu = 0$ from (5.14). So (5.16)$\Leftrightarrow$(5.14) $\Rightarrow$ (5.15) is obvious.

Assume (5.15). If $(\alpha, \beta, \gamma) = \zeta(1, 1, 1)$, where ζ is a meromorphic function, (5.16) is obvious; if not, then $(\alpha', \beta', \gamma') = \rho(1, 1, 1) + \eta(\alpha, \beta, \gamma)$, where ρ, η are meromorphic functions. Differentiating, we conclude that $(\alpha'', \beta'', \gamma'')$ is a linear combination of (α, β, γ) and $(\alpha', \beta', \gamma')$. □

The proof of Theorem B *when* $n = 1$.

(1) Let $f, g, h \in \mathscr{F}_B(\mathbf{C}, \mathbf{P}^m, 3m + 1)$. Let $\mathscr{P}$ be associated to $\mathscr{F}_B(\mathbf{C}, \mathbf{P}^m, 3m + 1)$ as in Lemma 5.1.

It suffices to show that for each $p \in \mathscr{P}$, for all but at most four $i \in \{1, \ldots, 3m+1\}$ (here the condition $m \geq 2$ is used),

$$\frac{\lambda_i}{F(f,p,i)} + \frac{\mu_i}{G(g,p,i)} + \frac{\nu_i}{H(h,p,i)} \equiv 0 \tag{5.17}$$

holds for some constant $(\lambda_i, \mu_i, \nu_i) \in \mathbf{C}_*^3$, where $F(f,p,i)$, $G(g,p,i)$, $H(h,p,i)$ are defined as in (5.3).

In fact, assume that the statement above has been proved, we prove that f, g, h satisfy Property (P) as follows: for any $p = (p_0, \ldots, p_m) \in \mathbf{C}_*^{m+1}$, since $\mathscr{P}$ is dense in $\mathbf{C}_*^{m+1}$ by Lemma 5.1, we choose a sequence of $p_t \in \mathscr{P}$ with $p_t \to p$ as $t \to \infty$. For each p_t, by the assumption, for all but at most four $i \in \{1, \ldots, 3m+1\}$ (the exceptional set depends on t),

$$\frac{\lambda_{t_i}}{F(f,p_t,i)} + \frac{\mu_{t_i}}{G(g,p_t,i)} + \frac{\nu_{t_i}}{H(h,p_t,i)} \equiv 0 \tag{5.18}$$

for some constant $(\lambda_{t_i}, \mu_{t_i}, \nu_{t_i}) \in \mathbf{C}_*^3$. We assume

$$\operatorname{Max}(|\lambda_{t_i}|, |\mu_{t_i}|, |\nu_{t_i}|) = 1 \quad \text{for all } t \text{ and } i. \tag{5.19}$$

By taking a convergent subsequence of $\{p_t\}$ if necessary, we can assume that

(i) There is an exceptional set $\Delta_p \subset \{1, \ldots, 3m+1\}$ which consists of at most four numbers and is independent of t, such that for any p_t, and any $i \in \{1, \ldots, 3m+1\} - \Delta_p$, (5.18) holds for some constant $(\lambda_{t_i}, \mu_{t_i}, \nu_{t_i}) \in \mathbf{C}_*^3$.

(ii) $\lambda_{t_i} \to \lambda_i$, $\mu_{t_i} \to \mu_i$, $\nu_{t_i} \to \nu_i$ for each $i \in \{1, \ldots, 3m+1\} - \Delta_p$ as $t \to \infty$.

Because (5.19), $(\lambda_i, \mu_i, \nu_i) \in \mathbf{C}_*^3$. Just let $t \to \infty$ in (5.18), we have proved (1.4), i.e., f, g, h satisfy Property (P).

(2) Now suppose that the statement (5.17) is not true. Then there exist $p \in \mathscr{P}$ and $i_1, \ldots, i_5 \in \{1, \ldots, 3m+1\}$, such that there is no constant $(\lambda, \mu, \nu) \in \mathbf{C}_*^3$ so that

$$\frac{\lambda}{F(f,p,i_s)} + \frac{\mu}{G(g,p,i_s)} + \frac{\nu}{H(h,p,i_s)} \equiv 0$$

for $s = 1, 2, 3, 4, 5$. By Lemma 5.3, $\varphi(f,g,h,p,i_s) \not\equiv 0$ for $s = 1, 2, 3, 4, 5$. Then by Lemma 5.2,

$$\left\| \varliminf_{r \to \infty} N(A_{i_s}; r, s)/T_f(r,s) \geq \frac{1}{2}, \quad \text{for } s = 1, 2, 3, 4, 5. \right.$$

So,

$$\left\| \lim_{r\to\infty} \sum_{j=1}^{3m+1} N(A_j; r, s)/T_f(r, s) \right\| \geq \sum_{j=1}^{3m+1} \varliminf_{r\to\infty} N(A_j, r, s)/T_f(r, s) \geq \frac{5}{2} > 2,$$

but this is contrary to (4.17). □

6. Lemmas for φ_{ae}. For proving Theorems A, B and C, we make some preparation in this section. Since §5, we assume $n \geq 2$ through §6, §7.

Let n, m, $k \in \mathbf{Z}^+$ with $k \geq 2$, and let $f, g, h \in \mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, k)$. Let $\mathscr{P}$ be associated to $\mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, k)$ as in Lemma 5.1. Take $p \in \mathscr{P}$ and $i \in \{1, \ldots, k\}$. Suppose that there is no constant $(\lambda, \mu, \nu) \in \mathbf{C}_*^3$, such that

$$\frac{\lambda}{F} + \frac{\mu}{G} + \frac{\nu}{H} \equiv 0, \tag{6.1}$$

where $F = F(f, p, i)$, $G = G(g, p, i)$, $H = H(h, p, i)$ are defined by (5.3).

Let

$$K := K(f, g, h, p, i) := \{F = 0\} \cup \{1/F = 0\} \cup \{G = 0\} \cup \{1/G = 0\} \cup \{H = 0\} \cup \{1/H = 0\} \cup \{F = G\} \cup \{G = H\} \cup \{H = F\}.$$

Since f, g, h are linearly non-degenerate and the assumption in (6.1), K is a proper analytic subset of $\mathbf{C}^n$.

Take $a = (a_1, \ldots, a_n) \in \mathbf{C}^n - K$. Since $n \geq 2$, let $e = (e_1, \ldots, e_n) \in E := \{1\} \times \mathbf{C}^{n-1}$ (see §3). Then define a meromorphic function $\varphi_{ae} = \varphi_{ae}(f, g, h, p, i)$ on $\mathbf{C}$ by

$$\varphi_{ae} := \det \begin{pmatrix} F_{ae} & G_{ae} & H_{ae} \\ F'_{ae}/F_{ae} & G'_{ae}/G_{ae} & H'_{ae}/H_{ae} \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{where} \tag{6.2}$$

$$F_{ae}(w) := F(a + we), \text{ etc.} \tag{6.3}$$

LEMMA 6.1.

$$\varphi_{ae}(0) = A_1(a) + A_2(a)e_2 + \cdots + A_n(a)e_n, \tag{6.4}$$

where

$$A_j(a) = \left(\frac{D^jG(a)}{G(a)} - \frac{D^jH(a)}{H(a)}\right)F(a) + \left(\frac{D^jH(a)}{H(a)} - \frac{D^jF(a)}{F(a)}\right)G(a) + \left(\frac{D^jF(a)}{F(a)} - \frac{D^jG(a)}{G(a)}\right)H(a) \neq \infty$$

for $j = 1, \ldots, n$ *and* $D^j = \dfrac{\partial}{\partial z^j}$.

Proof. By (6.3), $F_{ae}(0) = F(a)$, and

$$F'_{ae}(w) = D^1F(a + we) + e_2D^2F(a + we) + \cdots + e_nD^nF(a + we),$$

so

$$F'_{ae}(0) = D^1F_{ae}(0) + e_2D^2F_{ae}(0) + \cdots + e_nD^nF_{ae}(0). \tag{6.5}$$

Then by (6.2) and (6.5), we obtain (6.4).

Since $a \notin K$, $F(a) \neq 0, \infty$, $G(a) \neq 0, \infty$, $H(a) \neq 0, \infty$, so $A_j(a) \neq \infty$ for $j = 1, \ldots, n$. □

LEMMA 6.2. *There is no open subset* $U \subset \mathbf{C}^n - K$ *such that*

$$\varphi_{ae}(0) = 0 \quad \textit{for all } a \in U \textit{ and all } e \in E. \tag{6.6}$$

Proof. (1) Suppose that there is some open subset $U \subset \mathbf{C}^n - K$ such that $\varphi_{ae}(0) = 0$ for all $a \in U$ and all $e \in E$. From Lemma 6.1, it implies that $A_j(a) = 0$ for all $a \in U$ and for $j = 1, \ldots, n$. Since A_j are meromorphic, we know

$$A_j \equiv 0 \quad \text{on } \mathbf{C}^n \quad \text{for } j = 1, \ldots, n. \tag{6.7}$$

We shall find a contradiction with (6.7).

(2) Consider $A_n \equiv 0$. Write $z = (z', z_n) \in \mathbf{C}^{n-1} \times \mathbf{C}$. $A_n \equiv 0$ means

$$\det \begin{pmatrix} F(z', z_n) & G(z', z_n) & H(z', z_n) \\ \dfrac{D^nF(z', z_n)}{F(z', z_n)} & \dfrac{D^nG(z', z_n)}{G(z', z_n)} & \dfrac{D^nH(z', z_n)}{H(z', z_n)} \\ 1 & 1 & 1 \end{pmatrix} = 0$$

for all $(z', z_n) \in \mathbf{C}^n$.

For any $(z', z_n) \in \mathbf{C}^n - K$, fixing $z' \in \mathbf{C}^{n-1}$, by the proof of Lemma 5.3 for $F(z', \cdot) \not\equiv 0$, $G(z', \cdot) \not\equiv 0$, $H(z', \cdot) \not\equiv 0$, there is constant $(\tilde{\lambda}(z'), \tilde{\mu}(z'), \tilde{\nu}(z')) \in \mathbf{C}^3_*$ such that

$$\begin{cases} \dfrac{\tilde{\lambda}(z')}{F(z', z_n)} + \dfrac{\tilde{\mu}(z')}{G(z', z_n)} + \dfrac{\tilde{\nu}(z')}{H(z', z_n)} \equiv 0, \\ \tilde{\lambda}(z') + \tilde{\mu}(z') + \tilde{\nu}(z') \equiv 0, \quad \text{for all } z_n \in \mathbf{C}. \end{cases} \tag{6.8}$$

The second identity is because $F|A = G|A = H|A \not\equiv 0$ for $k \geq 2$.

(3) Consider the equation on $\mathbf{C}^n$,

$$\begin{cases} \dfrac{\lambda(z)}{F(z)} + \dfrac{\mu(z)}{G(z)} + \dfrac{\nu(z)}{H(z)} \equiv 0, \\ \lambda(z) + \mu(z) + \nu(z) \equiv 0. \end{cases} \tag{6.9}$$

For any $z \in \mathbf{C}^n - K$, we solve (6.9) to obtain

$$\begin{cases} \lambda(z) = \dfrac{1/H(z) - 1/G(z)}{1/F(z) - 1/G(z)}\nu(z), \\ \mu(z) = \dfrac{1/F(z) - 1/H(z)}{1/F(z) - 1/G(z)}\nu(z). \end{cases} \tag{6.10}$$

So, for any $\nu \not\equiv 0$, we obtain a solution (λ, μ, ν) for (6.9) on $\mathbf{C}^n - K$, with $\mu \not\equiv 0$ and $\lambda \not\equiv 0$. If we fix $\nu \not\equiv 0$ to be a meromorphic function, then $(\lambda(z), \mu(z), \nu(z))$ is a solution for (6.9) on all of $\mathbf{C}^n$, where $\lambda \not\equiv 0$, $\mu \not\equiv 0$ are also meromorphic functions.

Such (λ, μ, ν) is not unique, but for any $z \in \mathbf{C}^n - K$,

$$\frac{\lambda(z)}{\nu(z)} \neq 0, \infty, \quad \frac{\mu(z)}{\nu(z)} \neq 0, \infty, \quad \frac{\lambda(z)}{\mu(z)} \neq 0, \infty$$

are uniquely determined by (6.9) because of (6.10).

We want to prove that such λ/ν, μ/ν, λ/μ are constant on $\mathbf{C}^n$ from (6.7), which is contrary to (6.1). Thus the lemma is proved.

(4) For any $z = (z', z_n) \in \mathbf{C}^n - K$, by (6.8) and the uniqueness property in (3), assume $\tilde{\nu}(z') \not\equiv 0$; then we have

$$\frac{\tilde{\lambda}(z')}{\tilde{\nu}(z')} = \frac{\lambda(z', z_n)}{\nu(z', z_n)} \neq 0, \infty, \quad \frac{\tilde{\mu}(z')}{\tilde{\nu}(z')} = \frac{\mu(z', z_n)}{\nu(z', z_n)} \neq 0, \infty,$$

Thus

$$\frac{\tilde{\lambda}(z')}{\tilde{\mu}(z')} = \frac{\lambda(z', z_n)}{\mu(z', z_n)} \neq 0, \infty.$$

i.e., for any $(z', z_n) \in \mathbf{C}^n - K$, the meromorphic functions λ/μ, ν/μ, λ/ν are independent of z_n. So, they are independent of z_n on $\mathbf{C}^n$.

Repeating the same procedure for $j = 1, \ldots, n-1$, we prove that λ/μ, ν/μ, λ/ν are independent of z_j for $j = 1, \ldots, n$. □

7. Proof for Theorems A, B, C. As in §6, assume that $n \geq 2$.

LEMMA 7.1. *Let* $f, g, h \in \mathscr{F}_A(\mathbf{C}^n, \mathbf{P}^m, m+3)$ *(respectively* $\mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, 3m+1)$, *or* $\mathscr{F}_C(\mathbf{C}^n, V, k)$*). Let* $p \in \mathscr{P}$, *where* $\mathscr{P}$ *is associated to* $\mathscr{F}_A(\mathbf{C}^n, \mathbf{P}^m, m+3)$ *(resp.* $\mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, 3m+1)$, *or* $\mathscr{F}_C(\mathbf{C}^n, V, k)$*)*

and let $i \in \{1, \ldots, m+3\}$ *(resp.* $\{1, \ldots, 3m+1\}$, *or* $\{1, \ldots, k\}$*). Let* $F = F(f, p, i)$, $G = G(g, p, i)$, $H = H(h, p, i)$ *be defined by* (5.3). *Suppose that there is no constant* $(\lambda, \mu, \nu) \in \mathbf{C}_*^3$ *such that*

$$\frac{\lambda}{F} + \frac{\mu}{G} + \frac{\nu}{H} \equiv 0. \tag{7.1}$$

For each $b \in \mathbf{C}^n$, *define a bioholomorphic map* $\zeta_b : \mathbf{C}^n \to \mathbf{C}^n$ *by* $\zeta_b(z) = z + b$ *for all* $z \in \mathbf{C}^n$. *Abbreviate* $A_{jb} = \zeta_b^{-1}(A_j)$ *and* $f_b = f \circ \zeta_b$. *Then for almost every* $b \in \mathbf{C}^n$ *we have*

$$\left\| \varliminf_{r \to \infty} N(A_{jb}; r, s)/T_{f_b}(r, s) \geq \frac{1}{2}, \quad \text{for } 0 < s < r. \right. \tag{7.2}$$

Proof. We only need to consider $\mathscr{F}_A(\mathbf{C}^n, \mathbf{P}^m, m+3)$. For the cases of $\mathscr{F}_B(\mathbf{C}^n, \mathbf{P}^m, 3m+1)$, or $\mathscr{F}_C(\mathbf{C}^n, V, k)$, the proof is the same.

(1) By §6, there is a proper analytic subset $K := K(f, g, h, p, i) \in \mathbf{C}^n$ and $a \in \mathbf{C}^n - K$ and some $e \in E$ such that

$$\varphi_{ae}(0) \neq 0, \tag{7.3}$$

where $\varphi_{ae} = \varphi_{ae}(f, g, h, p, i)$ is defined by (6.2).

By taking $w = (w_1, \ldots, w_n) = a + (z_1, \ldots, z_n) = a + z$, we assume $a = 0$. From Lemma 6.1, $E_0 := \{e \in E | \varphi_{0e}(0) = 0\} \subset E$ is a proper linear variety, so

$$\text{Measure } E_0 = 0. \tag{7.4}$$

(2) For any $e \in E - E_0$ and for any $j \in \{1, \ldots, m+3\}$ with $j \neq i$, we shall prove

$$I_{ej} \subset \varphi_{0e}^{-1}(0) \cap (\varphi_{0e}')^{-1}(0), \text{ where} \tag{7.5}$$

$$I_{ej} := \left\{ w \in \mathbf{C} \,|\, we \in A_j - f^{-1}(H_p) \cup \bigcup_{\substack{1 \leq u \leq m+3 \\ u \neq j}} A_u \right\}. \tag{7.6}$$

Assume $I_{ej} \neq \varnothing$. For any $b \in I_{ej}$, since $F_{0e}(b) = F(be) = G_{0e}(b) = G(be) = H_{0e}(b) = H(be) \in \mathbf{C}_*$ by definition (7.6), by using the proof of Lemma 5.2 ((5.7), (5.8)), we can prove

$$\varphi_{0e}(b) = \varphi_{0e}'(b) = 0.$$

So, (7.5) is proved.

(3) Now we estimate $2\sum_{j=1, j\neq i}^{m+3} N(A_j; r, s)$.

Because $p \in \mathscr{P}$ and $\operatorname{Codim} A_i \cap A_j \geq 2$ for $i \neq j$, we know that

$$\operatorname{Codim} A_j \cap \left\{ f^{-1}(H_p) \cup \bigcup_{\substack{1 \leq u \leq m+3 \\ u \neq j}} A_u \right\} \geq 2, \quad \text{so}$$

$$N(A_j; r, s) = \int_s^r \frac{dt}{t^{2n-1}} \int_{A_j' \cap B(t)} \varphi_0^{n-1},$$

where

$$A_j' = A_j - f^{-1}(H_p) \cup \bigcup_{\substack{1 \leq u \leq m+3 \\ u \neq j}} A_u.$$

Therefore

$$\begin{aligned} &2 \sum_{j=1, j \neq i}^{m+3} N(A_j; r, s) \\ &\quad = 2 \int_{e \in E - E_0} \sum_{j=1, j \neq i}^{m+3} N\left(I_{ej}; \frac{r}{|e|}, \frac{s}{|e|}\right) L^*(\omega_0^{n-1})(e) \\ &\qquad\qquad \left(\begin{array}{c} \because \text{ (7.4) and } L^* \\ \text{is defined in (3.3)} \end{array} \right) \\ &\quad \leq \int_{e \in E - E_0} N_{\varphi_{0e}}\left(0; \frac{r}{|e|}, \frac{s}{|e|}\right) L^*(\omega_0^{n-1})(e). \end{aligned}$$

The last inequality is because of (7.5) and the fact that if $I_{ej} = \varnothing$, then $N(I_{ej}; r, s) = 0$.

(4) For any $e \in E - E_0$, we estimate $N_{\varphi_{0e}}(0; r/|e|, s/|e|)$ now.

If $\varphi_{0e} \equiv$ const., since $\varphi_{0e}(0) \neq 0$, $N_{\varphi_{0e}}(0; r/|e|, s/|e|) \equiv 0$. So, we consider $\varphi_{0e} \not\equiv$ const. Then by the First Main Theorem,

$$\begin{aligned} N_{\varphi_{0e}}\left(0; \frac{r}{|e|}, \frac{s}{|e|}\right) &\leq T_{\varphi_{0e}}\left(\frac{r}{|e|}, \frac{s}{|e|}\right) + m_{\varphi_{0e}}\left(0; \frac{s}{|e|}\right) + C \\ &\leq N_{\varphi_{0e}}\left(\infty; \frac{r}{|e|}, \frac{s}{|e|}\right) + m_{\varphi_{0e}}\left(\infty; \frac{r}{|e|}\right) + m_{\varphi_{0e}}\left(0; \frac{s}{|e|}\right) + C. \end{aligned}$$

By (6.2) and as in (5.9), since $\varphi_{0e} \not\equiv$ const., all the poles of φ_{0e} must be the ones of F_{0e}, G_{0e}, and H_{0e}. Then

$$\begin{aligned} N_{\varphi_{0e}}\left(\infty; \frac{r}{|e|}, \frac{s}{|e|}\right) \leq\ & N_{F_{0e}}\left(\infty; \frac{r}{|e|}, \frac{s}{|e|}\right) \\ & + N_{G_{0e}}\left(\infty; \frac{r}{|e|}, \frac{s}{|e|}\right) + N_{H_{0e}}\left(\infty; \frac{r}{|e|}, \frac{s}{|e|}\right). \end{aligned}$$

Furthermore,

$$m_{\varphi_{0e}}\left(\infty;\frac{t}{|e|}\right)$$
$$\leq m_{F_{0e}}\left(\infty;\frac{t}{|e|}\right)+m_{G_{0e}}\left(\infty;\frac{t}{|e|}\right)+m_{H_{0e}}\left(\infty;\frac{t}{|e|}\right)$$
$$+2\left(m_{F'_{0e}/F_{0e}}\left(\infty;\frac{t}{|e|}\right)+m_{G'_{0e}/G_{0e}}\left(\infty;\frac{t}{|e|}\right)+m_{H'_{0e}/H_{0e}}\left(\infty;\frac{t}{|e|}\right)\right)$$
$$+\log 6.$$

So,

$$N_{\varphi_{0e}}\left(0;\frac{r}{|e|},\frac{s}{|e|}\right)$$
$$\leq T_{F_{0e}}\left(\frac{r}{|e|},\frac{s}{|e|}\right)+T_{G_{0e}}\left(\frac{r}{|e|},\frac{s}{|e|}\right)+T_{H_{0e}}\left(\frac{r}{|e|},\frac{s}{|e|}\right)$$
$$+2\left(m_{F'_{0e}/F_{0e}}\left(\infty;\frac{r}{|e|}\right)+m_{G'_{0e}/G_{0e}}\left(\infty;\frac{r}{|e|}\right)\right.$$
$$\left.+m_{H'_{0e}/H_{0e}}\left(\infty;\frac{r}{|e|}\right)\right)+\log 6.$$

(5) By (3), (4), and Lemma 3.2,

$$(7.7)\qquad \left\|2\sum_{j=1,j\neq i}^{m+3}N(A_j;r,s)\right.$$
$$\leq\int_{e\in E-E_0}N_{\varphi_{0e}}\left(0;\frac{r}{|e|},\frac{s}{|e|}\right)L^*(\omega_0^{n-1})(e)$$
$$\leq T_F(r,s)+T_G(r,s)+T_H(r,s)$$
$$+C(\log^+T_F(r,s)+\log^+T_G(r,s)+\log^+T_H(r,s))$$
$$+o(\log r)+C.$$

By Corollary 4.5, and (7.7), the lemma for $\mathscr{F}_A(\mathbf{C}^n,\mathbf{P}^m,m+3)$ follows as in the proof of Lemma 5.2. □

REMARK. If f, g, h, p are as in Lemma 7.1, suppose for distinct $i_t\in\{1,\ldots,m+3\}$ (resp. $\{1,\ldots,3m+1\}$, or $\{1,\ldots,k\}$), there are no constant $(\lambda_t,\mu_t,\nu_t)\in\mathbf{C}_*^3$, such that

$$\frac{\lambda_t}{F(f,p,i_t)}+\frac{\mu_t}{G(g,p,i_t)}+\frac{\nu_t}{H(h,p,i_t)}\equiv 0,\quad \text{for } t=1,\ldots,5.$$

Then there is $a \in \mathbf{C}^n$, which is independent of t, so that under the new coordinates system $w = a + z$,

$$\left\| \varliminf_{r\to\infty} N(A_{i_t}; r, s)/T_f(r, s) \geq \frac{1}{2}.$$

In fact, from §6, we first consider i_1, there is $a_1 \in \mathbf{C}^n$, and some $e_1 \in E$, $\varphi_{a_1e_1}(0) \neq 0$, where $\varphi_{a_1e_1} = \varphi_{a_1e_1}(f, g, h, p, i_1)$. By Lemma 6.1, there are neighborhoods U_1 of a_1 in $\mathbf{C}^n$ and V_1 of e_1 in E, so that for all $a \in U_1$, $e \in V_1$, $\varphi_{ae}(0) \neq 0$. Then, we consider i_2, also from §6, there is $a_2 \in U_1$ and some $e_2 \in V_1$, $\varphi_{a_2e_2}(0) \neq 0$, and then go on for $t = 3, 4, 5$. So, we finally find some $a \in \mathbf{C}^n$, and $e \in E$, such that $\varphi_{ae}(f, g, h, p, i)(0) \neq 0$, for $t = 1, 2, 3, 4, 5$.

Proof of Theorems A, B *and* C. From Lemma 7.2 and the remark above and as the proof of Theorem B when $n = 1$ in §5, Theorems A and B are proved immediately.

For Theorem C, for $f, g, h \in \mathscr{F}_C(\mathbf{C}^n, V, k)$, we can prove as above that f, g, h satisfy Property (P). It remains to show that f, g, h are algebraically dependent.

Since f, g, h satisfy Property (P) and $k \geq 5$, we can take $p = (p_0, \ldots, p_m) \in \mathbf{C}_*^{m+1}$ and $i \in \{1, \ldots, k\}$ such that

$$(7.8)\quad (a_0^{(i)}w_0 + \cdots + a_M^{(i)}w_M)/(p_0w_0 + \cdots + p_Mw_M) \not\equiv \text{const.} \quad \text{and}$$

$$(7.9)\qquad \frac{\lambda P(f)}{H_i(f)} + \frac{\mu P(g)}{H_i(g)} + \frac{\nu P(h)}{H_i(h)} \equiv 0$$

for some $(\lambda, \mu, \nu) \in \mathbf{C}_*^3$, where $P(f) = p_0f_0 + \cdots + p_Mf_M$, $H_i(f) = a_0^{(i)}f_0 + \cdots + a_M^{(i)}f_M$, etc., as in (5.4).

Let

$$\begin{aligned} Q(w, u, w) := {} & \lambda P(w)H_i(u)H_i(v) + \mu H_i(w)P(u)H_i(v) \\ & + \nu H_i(w)H_i(u)P(v) \end{aligned}$$

be a polynomial of $\mathbf{C}[w_0, \ldots, w_M; u_0, \ldots, u_M; v_0, \ldots, v_M]$. Note that

$$(7.10)\quad Q(w, u, v) = H_i(w)H_i(u)H_i(v)\left(\frac{\lambda P(w)}{H_i(w)} + \frac{\mu P(u)}{H_i(u)} + \frac{\nu P(v)}{H_i(v)}\right).$$

Assume $\lambda \neq 0$. Since V is not contained in any hyperplane in $\mathbf{P}^M$, we can choose $u', v' \in \mathbf{C}^{M+1}$ with $\mathbf{P}(u') \in V$ and $\mathbf{P}(v') \in V$, such that $H_i(u') \neq 0$, $H_i(v') \neq 0$. Hence $Q(\cdot, u', v') \not\equiv 0$ by (7.10). For the same reason as above we can choose $w' \in \mathbf{C}^{M+1}$ with $\mathbf{P}(w') \in V$ such

that $Q(w', u', w') \neq 0$. This means $\{(\mathbf{P}(w), \mathbf{P}(u), \mathbf{P}(v)) \in V \times V \times V | Q(w, u, v) = 0\}$ is a proper analytic subset of $V \times V \times V$. Then from (7.9), $Q(\phi_f, \phi_g, \phi_h) \equiv 0$, where ϕ_f is a reduced representation of f, etc., and therefore f, g, h are algebraically dependent. □

REMARK. Recently, Stoll has proved some closely related results **[S11]** which implies that if $f, g, h \in \mathcal{F}_A(\mathbf{C}^n, \mathbf{P}^m, m + 3)$, or $\mathcal{F}_B(\mathbf{C}^n, \mathbf{P}^m, 3m + 1)$, or $\mathcal{F}_C(\mathbf{C}^n, V, k)$, then f, g, h are not in general position, hence f, g, h are algebraically dependent. Stoll applies the First Main Theorem for exterior product **[S10]** to prove his theorems. The method is very interesting.

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THE JOHNS HOPKINS UNIVERSITY
BALTIMORE, MD 21218