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q-BETA INTEGRALS AND THE q-HERMITE POLYNOMIALS

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The continuous q-Hermite polynomials are used to give a new proof of a q-beta integral which is an extension of the Askey-Wilson integral. Multilinear generating functions, some due to Carlitz, are also established.

1. Introduction. Let $q \in (-1, 1)$ and define the q-shifted factorials by

$$(a)_0 = (a;q)_0 = 1,$$
$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n = 1, 2, \dots,$$
$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Basic hypergeometric series are defined by

$$_{r+1}\phi_r(a_1, a_2, \dots, a_{r+1}; b_1, b_2, \dots, b_r; z) \equiv {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} \middle| \, z\right]$$
$$= \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_{r+1})_n}{(q)_n(b_1)_n\cdots(b_r)_n} z^n.$$

The continuous q-Hermite polynomials $\{H_n(x|q)\}$ are given by

$$H_n(\cos\theta|q) = \sum_{k=0}^{n} \frac{(q)_n}{(q)_k(q)_{n-k}} e^{i(n-2k)\theta} \tag{1.1}$$

(see [2]). Their orthogonality [2, 3] is

$$\int_0^\pi w(\theta)H_m(\cos\theta|q)H_n(\cos\theta|q)d\theta = (q;q)_n\delta_{nm} \tag{1.2}$$

where

$$w(\theta) = \frac{(q)_\infty}{2\pi}(e^{2i\theta})_\infty(e^{-2i\theta})_\infty. \tag{1.3}$$

Rogers also introduced the continuous q-ultraspherical polynomials $\{C_n(x;\beta|q)\}$ generated by

$$\sum_{n=0}^{\infty} C_n(\cos\theta;\beta|q)t^n = \frac{(\beta te^{i\theta})_\infty(\beta te^{-i\theta})_\infty}{(te^{i\theta})_\infty(te^{-i\theta})_\infty} \tag{1.4}$$

whose weight function was found recently [8, 9]. It is easy to see that

$$C_n(x;0|q) = H_n(x|q)/(q)_n. \tag{1.5}$$

Rogers solved the connection coefficient problem of expressing $C_n(x;\beta|q)$ in terms of $C_n(x;\gamma|q)$ a consequence of which we get

$$C_n(x;\beta|q) = \sum_{k=0}^{[n/2]} \frac{(-\beta)^k q^{k(k-1)/2}(\beta)_{n-k}}{(q)_k(q)_{n-2k}} H_{n-2k}(x|q). \tag{1.6}$$

Rogers evaluated explicitly the coefficients in the linearization of products of two q-Hermite polynomials. He proved

$$H_m(x|q)H_n(x|q) = \sum_{k=0}^{\min(n,m)} \frac{(q)_m(q)_n}{(q)_k(q)_{n-k}(q)_{m-k}} H_{m+n-2k}(x|q), \tag{1.7}$$

which can be iterated to obtain the sum

$$\begin{aligned} &H_k(x|q)H_m(x|q)H_n(x|q) \\ &\quad = \sum_{r,s} \frac{(q)_k(q)_m(q)_n(q)_{m+n-2r}}{(q)_{m-r}(q)_{n-r}(q)_r(q)_{k-s}(q)_{m+n-2r-s}(q)_s} \\ &\qquad \times H_{k+m+n-2r-2s}(x|q). \end{aligned} \tag{1.8}$$

We shall also need the formula

$$\begin{aligned} &\frac{H_m(x|q)}{(q)_m} C_n(x;\beta|q) \\ &\quad = \sum_{k,j} \frac{(-\beta)^k q^{k(k-1)/2}(\beta)_{n-k}}{(q)_k(q)_{m-j}(q)_j(q)_{n-2k-j}} H_{m+n-2k-2j}(x|q), \end{aligned} \tag{1.9}$$

which follows from (1.6) and (1.7).

We shall also use the polynomials

$$h_n(x|q) = \sum_{k=0}^{n} \frac{(q)_n}{(q)_k(q)_{n-k}} x^k,$$

so that

$$H_n(\cos\theta|q) = e^{in\theta} h_n(e^{-2i\theta}|q). \tag{1.10}$$

It was shown in [1] and [14] that $\{h_n(a|q)\}$ are moments of a discrete distribution $d\psi_a(x)$, viz.,

$$h_n(a|q) = \int_{-\infty}^{\infty} x^n \, d\psi_a(x), \qquad n = 0, 1, 2 \ldots, \tag{1.11}$$

where $d\psi_a(x)$ is a step function with jumps at the points $x = q^k$ and $x = aq^k$ for $k = 0, 1, 2, \dots$ given by

$$(1.12)\quad d\psi_a(q^k) = \frac{q^k}{(a)_\infty (q)_k (q/a)_k}, \quad d\psi_a(aq^k) = \frac{q^k}{(1/a)_\infty (q)_k (aq)_k}$$

where $a < 0,\ 0 < q < 1$.

Askey and Wilson [9] proved

$$(1.13)\qquad \frac{(q)_\infty}{2\pi}\int_0^\pi \frac{(e^{2i\theta})_\infty (e^{-2i\theta})_\infty}{\prod\limits_{1\le j\le 4} (a_j e^{i\theta})_\infty (a_j e^{-i\theta})_\infty} d\theta = \frac{(a_1a_2a_3a_4)_\infty}{\prod\limits_{1\le r<s\le 4} (a_r a_s)_\infty},$$

where $|a_r| < 1$ for $r = 1, 2, 3, 4$. They used this integral to prove the orthogonality of what is now known as the Askey-Wilson polynomials.

Ismail and Stanton [15] observed that the left hand side of (1.13) is a generating function of the integral of the product of four q-Hermite polynomials times the weight function $w(\theta)$. They used this observation, combined with (1.8) and (1.3), to give a new proof of (1.13). Other analytic proofs of (1.13) can be found in [6] and [18]. Furthermore a combinatorial derivation of (1.13) is given in [16].

Nasrallah and Rahman [17] proved the following generalization of (1.13).

THEOREM (NASRALLAH AND RAHMAN). *If* $|a_j| < 1,\ j = 1, 2, 3, 4, 5$ *and* $|q| < 1$ *then*

$$(1.14)\quad \int_0^\pi w(\theta) \frac{(Aa_5e^{i\theta})_\infty (Aa_5e^{-i\theta})_\infty}{\prod\limits_{1\le k\le 5} (a_ke^{i\theta})_\infty (a_ke^{-i\theta})_\infty} d\theta$$
$$= \frac{(a_1a_3a_4a_5)_\infty (a_1a_2a_3a_4)_\infty (Aa_3a_5)_\infty (Aa_4a_5)_\infty (Aa_1a_5)_\infty (a_2a_5)_\infty}{(Aa_1a_3a_4a_5)_\infty \prod\limits_{1\le i<j\le 5} (a_ia_j)_\infty}$$
$$\times\ {}_8W_7(Aa_1a_3a_4a_5q^{-1}; Aa_5/a_2, A, a_1a_3, a_1a_4, a_3a_4 | a_2a_5),$$

where

$${}_8W_7(a; b, c, d, e, f | z) = {}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}\ ,\ b\ ,\ c\ ,\ d\ ,\ e\ ,\ f \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, qa/f \end{matrix} \middle| z\right].$$

Rahman [19] observed that the ${}_8\phi_7$ in (1.14) can be summed when $A = a_1a_2a_3a_4$. In this case we have

$$\int_0^\pi w(\theta)\frac{(a_1a_2a_3a_4a_5e^{i\theta})_\infty(a_1a_2a_3a_4a_5e^{-i\theta})_\infty}{\prod\limits_{1\le k\le 5}(a_ke^{i\theta})_\infty(a_ke^{-i\theta})_\infty}d\theta = \frac{\prod\limits_{k=1}^{5}\left(\frac{a_1a_2a_3a_4a_5}{a_k}\right)_\infty}{\prod\limits_{1\le k<j\le 5}(a_ka_j)_\infty}. \tag{1.15}$$

Askey [7] gave an elementary proof of (1.15) by showing that the two sides of (1.15) satisfy the same functional equation.

The main purpose of this paper is to prove (1.14) and (1.15) using different techniques that are based on the orthogonality and some multilinear generating functions for the q-Hermite polynomials. This shall be done in §3. In §2 we shall start by illustrating this technique in rederiving some results of Carlitz on the q-Hermite polynomials ((2.1) and (2.5)). We shall also obtain incidentally a transformation formula for ${}_3\phi_2$ functions. In §4 we derive a new multilinear generating function for the continuous q-Hermite polynomials. In the process of deriving such a formula we prove a reduction formula for the double series of the Kempe de Fériet type.

2. Generating functions. To illustrate our technique we begin by deriving Carlitz [11] extension of Mehler formula

$$\begin{aligned} S &= \sum_{n=0}^{\infty} h_n(a|q)h_{n+k}(b|q)\frac{z^n}{(q)_n} \\ &= \frac{(abz^2)_\infty}{(z)_\infty(bz)_\infty(az)_\infty(abz)_\infty}\sum_{r=0}^{k}\frac{(q)_k(bz)_r(abz)_r}{(q)_r(q)_{k-r}(abz^2)_r}b^{k-r}. \end{aligned} \tag{2.1}$$

We begin by the generating function

$$\sum_{n=0}^{\infty} h_n(a|q)\frac{z^n}{(q)_n} = \frac{1}{(z)_\infty(az)_\infty}. \tag{2.2}$$

Multiply by z^k, then replace z by xz and use (1.11). We get

$$\sum_{n=0}^{\infty} h_n(a|q)h_{n+k}(b|q)\frac{z^n}{(q)_n} = \frac{1}{(b)_\infty(z)_\infty(az)_\infty}{}_2\phi_1\left[\begin{matrix} z, az \\ q/b \end{matrix}\middle| q^{k+1}\right] + \frac{b^k}{(1/b)_\infty(bz)_\infty(abz)_\infty}{}_2\phi_1\left[\begin{matrix} bz, abz \\ bq \end{matrix}\middle| q^{k+1}\right].$$

Now using a transformation formula of Sears **[21]** (see also **[12]**)

$$(2.3)\quad {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}\middle| z\right] + \frac{(b)_\infty(q/c)_\infty(c/a)_\infty(az/q)_\infty(q^2/az)_\infty}{(c/q)_\infty(bq/c)_\infty(q/a)_\infty(az/c)_\infty(qc/az)_\infty} \times {}_2\phi_1\left[\begin{matrix} qa/c, qb/c \\ q^2/c \end{matrix}\middle| z\right] = \frac{(abz/c)_\infty(q/c)_\infty}{(q/a)_\infty(az/c)_\infty}{}_2\phi_1\left[\begin{matrix} c/a, cq/abz \\ cq/az \end{matrix}\middle| bq/c\right]$$

we get that the left hand side of (2.1) is

$$S = \frac{(abz^2q^k)_\infty}{(z)_\infty(az)_\infty(bzq^k)_\infty(q/z)_\infty}{}_2\phi_1\left[\begin{matrix} q/bz, q^{1-k}/abz^2 \\ q^{1-k}/bz \end{matrix}\middle| abz\right].$$

By Heine's transformation formula

$$(2.4)\quad {}_2\phi_1\left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}\middle| z\right] = \frac{(\beta)_\infty(\alpha z)_\infty}{(\gamma)_\infty(z)_\infty}{}_2\phi_1\left[\begin{matrix} \gamma/\beta, z \\ \alpha z \end{matrix}\middle| \beta\right],$$

we get that

$$S = \frac{(abz^2q^k)_\infty(z)_k b^k}{(z)_\infty(az)_\infty(bz)_\infty(abz)_\infty}{}_2\phi_1\left[\begin{matrix} q^{-k}, abz \\ q^{1-k}/z \end{matrix}\middle| \frac{q}{bz}\right].$$

Now by a transformation formula **[12**; ex 1.14(ii)]

$${}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}\middle| z\right] = \frac{b^n(c/b)_n}{(c)_n}\sum_{j=0}^{n}\frac{(q^{-n})_j(b)_j(q/z)_j(-1)^j q^{-j(j-1)/2}}{(q)_j(bq^{1-n}/c)_j}(z/c)^j$$

we get the right hand side of (2.1).

We next consider the sum

$$(2.5)\ G = G(a,b,x,y,z) = \sum_{m,n,k} \frac{x^m y^n z^k}{(q)_m (q)_n (q)_k} h_{m+k}(a|q) h_{n+k}(b|q)$$

$$= \sum_{m,n,k} \frac{x^m y^n z^k}{(q)_m (q)_n (q)_k} h_{n+k}(b|q) \int_{-\infty}^{\infty} u^{m+k} d\psi_a(u)$$

$$= \int_{-\infty}^{\infty} \frac{1}{(xu)_\infty} \sum_{k,n} \frac{y^n (zu)^k}{(q)_n (q)_k} d\psi_a(u)$$

$$= \int_{-\infty}^{\infty} \frac{1}{(xu)_\infty} \sum_{r=0}^{\infty} \frac{y^r}{(q)_r} h_r(b|q) h_r(zu/y|q) d\psi_a(a).$$

Using the q-Mehler formula (formula (2.1) with $k = 0$)

$$G(a,b,x,y,z) = \frac{1}{(y)_\infty (by)_\infty} \int_{-\infty}^{\infty} \frac{(byzu)_\infty}{(xu)_\infty (zu)_\infty (bzu)_\infty} d\psi_a(u).$$

From (1.11) we get

$$(2.6)\quad G(a,b,x,y,z) = \frac{(byz)_\infty}{(y)_\infty (by)_\infty (a)_\infty (x)_\infty (z)_\infty (bz)_\infty} {}_3\phi_2\left[\begin{matrix} x, z, bz \\ q/a, bzy \end{matrix}\middle| q\right] + \frac{(abzy)_\infty}{(y)_\infty (by)_\infty (1/a)_\infty (ax)_\infty (az)_\infty (abz)_\infty} \times {}_3\phi_2\left[\begin{matrix} ax, az, abz \\ aq, abzy \end{matrix}\middle| q\right]$$

But Carlitz **[11]** showed that

$$(2.7)\quad G(a,b,x,y,z) = \frac{(axz)_\infty (byz)_\infty}{(x)_\infty (ax)_\infty (y)_\infty (by)_\infty (z)_\infty (az)_\infty (bz)_\infty} {}_3\phi_2\left[\begin{matrix} x, y, z \\ axz, byz \end{matrix}\middle| abz\right].$$

Although (2.6) and (2.7) are the same we shall nevertheless need to use (2.6) for the representation of $G(a,b,x,y,z)$. Equating G in (2.6) and (2.7) we get the transformation formula

$$(2.8)\ {}_3\phi_2\left[\begin{matrix} x, y, z \\ axz, byz \end{matrix}\middle| abz\right] = \frac{(ax)_\infty (az)_\infty}{(a)_\infty (axz)_\infty} {}_3\phi_2\left[\begin{matrix} x, z, bz \\ q/a, bzy \end{matrix}\middle| q\right] + \frac{(abzy)_\infty (x)_\infty (z)_\infty (bz)_\infty}{(bzy)_\infty (1/a)_\infty (abz)_\infty (axz)_\infty} {}_3\phi_2\left[\begin{matrix} ax, az, abz \\ aq, abzy \end{matrix}\middle| q\right].$$

An interesting special case of (2.8) is $x = q^{-n}$ for $n = 0, 1, 2, \ldots$. We get

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, y, z \\ azq^{-n}, byz \end{matrix}\middle| abz\right] = \frac{(q/a)_n}{(q/az)_n z^n}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, z, bz \\ q/a, bzy \end{matrix}\middle| q\right] \tag{2.9} $$

which is due to Sears [20]. Formula (2.9) in turn implies Jackson's Theorem for the summation of a terminating balanced (Saalschültzian) ${}_3\phi_2$ with argument q, viz.,

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, a, b \\ c, abq^{1-n}/c \end{matrix}\middle| q\right] = \frac{(c/a)_n (c/b)_n}{(c)_n (c/ab)_n}. \tag{2.10} $$

Formula (2.8) can also be obtained as a limiting case of Bailey's transformation [12; (3.3.1)]

3. The q-beta integral. . We consider in this section the Nasrallah-Rahman formula (1.14). We first consider the integral

$$ J = \int_0^\pi w(\theta) \frac{(Aa_5 e^{i\theta})_\infty (Aa_5 e^{-i\theta})_\infty}{\prod\limits_{1 \le k \le 5} (a_k e^{i\theta})_\infty (a_k e^{-i\theta})_\infty} d\theta. \tag{3.1} $$

We recall from (1.4) and (1.5) that

$$ \sum_{n=0}^{\infty} H_n(\cos\theta|q) \frac{t^n}{(q)_n} = \frac{1}{(te^{i\theta})_\infty (te^{-i\theta})_\infty}, \tag{3.2} $$

so that

$$ \begin{aligned} J = \sum_{n_l} \int_0^\pi w(\theta) &\left\{ \prod_1^3 H_{n_l}(\cos\theta|q) a_l^{n_l} \right\} \\ &\times \left\{ \frac{H_{n_4}(\cos\theta|q)}{(q)_{n_4}} C_{n_5}(\cos\theta; a_1 a_2 a_3 a_4|q) a_4^{n_4} a_5^{n_5} \right\} d\theta. \end{aligned} $$

We now linearize the quantities in braces using (1.7) and (1.9) respectively. We get

$$ \begin{aligned} J = \sum_{n_l, k, j, r, s} &\frac{a_1^{n_1} a_2^{n_2} a_3^{n_3} a_4^{n_4} a_5^{n_5} (-A)^r q^{r(r-1)/2} (q)_{n_2+n_3-2k} (A)_{n_5-r}}{(q)_j (q)_r (q)_s (q)_k (q)_{n_2-k} (q)_{n_3-k} (q)_{n_1-j} (q)_{n_2+n_3-2k-j} (q)_{n_4-s} (q)_{n_5-2r-s}} \\ &\times \int_0^\pi H_{n_4+n_5-2r-2s}(\cos\theta|q) H_{n_1+n_2+n_3-2j-2k}(\cos\theta|q) w(\theta) d\theta. \end{aligned} $$

We apply the orthogonality relation (1.2) and then shift the summation indices so that $n_1 \to n_1 + j$, $n_2 \to n_2 + k$, $n_3 \to n_3 + k$, $n_4 \to n_4 + s$, $n_5 \to n_5 + 2r + s$.

We get

$$J = \sum_{n_i,k,j,r,s} \frac{a_1^{n_1+j} a_2^{n_2} a_3^{n_3} a_4^{n_4} a_5^{n_5} (a_2a_3)^k (a_4a_5)^s (-Aa_5^2)^r q^{r(r-1)/2}}{(q)_j (q)_r (q)_s (q)_k (q)_{n_1} (q)_{n_2} (q)_{n_3} (q)_{n_4} (q)_{n_5}} \times \frac{(q)_{n_4+n_5} (q)_{n_2+n_3} (A)_{n_5+r+s}}{(q)_{n_2+n_3-j}} \delta_{n_1+n_2+n_3-j,n_4+n_5},$$

so that $j = n_1 + n_2 + n_3 - n_4 - n_5$.

Evaluating the sums over s and k

$$J = \sum_{n_i,r} \frac{a_1^{2n_1} (a_1a_2)^{n_2} (a_1a_3)^{n_3} (a_4/a_1)^{n_4} (a_5/a_1)^{n_5} (-Aa_5^2)^r q^{r(r-1)/2}}{(q)_j (q)_r (q)_{n_1} (q)_{n_2} (q)_{n_3} (q)_{n_4} (q)_{n_5}} \times \frac{(q)_{n_4+n_5} (q)_{n_2+n_3} (A)_{n_5+r} (Aa_4a_5q^{r+n_5})_\infty}{(q)_{n_4+n_5-n_1} (a_4a_5)_\infty (a_2a_3)_\infty}.$$

The sum over r is

$$\sum_r \frac{(Aq^{n_5})_r q^{r(r-1)/2} (-Aa_5^2)^r}{(q)_r (Aa_4a_5q^{n_5})_r} = \lim_{\lambda\to\infty} {}_2\phi_1 \left[\begin{matrix} \lambda, Aq^{n_5} \\ Aa_4a_5q^{n_5} \end{matrix} \middle| \frac{A}{\lambda} a_5^2 \right].$$

Apply Heine transformation (2.4) to the ${}_2\phi_1$ in the above limit to identify the r-sum as

$$\frac{(Aq^{n_5})_\infty (Aa_5^2)_\infty}{(Aa_4a_5q^{n_5})_\infty} {}_2\phi_1 \left[\begin{matrix} a_4a_5, 0 \\ Aa_5^2 \end{matrix} \middle| Aq^{n_5} \right].$$

We therefore get

$$\frac{(a_4a_5)_\infty (a_2a_3)_\infty}{(A)_\infty (Aa_5^2)_\infty} J = \sum_{n_i,r} \frac{a_1^{2n_1} (a_1a_2)^{n_2} (a_1a_3)^{n_3} (a_4/a_1)^{n_4} (a_5/a_1)^{n_5} (Aq^{n_5})^r}{(q)_{n_1+n_2+n_3-n_4-n_5} (q)_r (q)_{n_1} (q)_{n_2} (q)_{n_3} (q)_{n_4} (q)_{n_5}} \times \frac{(q)_{n_4+n_5} (q)_{n_2+n_3} (a_4a_5)_r}{(q)_{n_4+n_5-n_1} (Aa_5^2)_r}.$$

Now set $m = n_4 + n_5$, $n = n_2 + n_3$, $k = n_4 + n_5 - n_1$. Thus

$$(3.3)\quad \frac{(a_4a_5)_\infty(a_2a_3)_\infty}{(A)_\infty(Aa_5^2)_\infty}J$$
$$= \sum \frac{a_1^{n+m-2k}a_2^{n_2}a_3^{n-n_2}a_4^{m-n_5}a_5^{n_5}(q)_m(q)_n(a_4a_5)_r(Aq^{n_5})^r}{(q)_{n_2}(q)_{m-k}(q)_{n-n_2}(q)_{m-n_5}(q)_{n_5}(q)_k(q)_{n-k}(q)_r(Aa_5^2)_r}$$
$$= \sum \frac{a_3^n a_4^m a_1^{m+n-2k}(a_4a_5)_rA^r}{(q)_r(q)_k(q)_{m-k}(q)_{n-k}(Aa_5^2)_r}h_m\left(\frac{a_5}{a_4}q^r|q\right)h_n\left(\frac{a_2}{a_3}|q\right)$$
$$= \sum \frac{a_1^{m+n}a_3^n a_4^m A^r(a_3a_4)^k(a_4a_5)_r}{(q)_r(q)_m(q)_n(Aa_5^2)_r(q)_k} \times h_{n+k}(a_2/a_3|q)h_{m+k}(a_5q^r/a_4|q)$$
$$= \sum_r \frac{A^r(a_4a_5)_r}{(q)_r(Aa_5^2)_r}G\left(\frac{a_5}{a_4}q^r, \frac{a_2}{a_3}, a_1a_4, a_1a_3, a_3a_4\right).$$

Using (2.6) for the value of G we get, after some simplifications,

$$\frac{\prod_{1\le j<k\le 5}(a_ja_k)_\infty}{(a_1a_2a_3a_4)_\infty(a_1a_2a_3a_4)_\infty(A)_\infty(Aa_5^2)_\infty}J$$
$$= \frac{(a_1a_5)_\infty(a_2a_5)_\infty(a_3a_5)_\infty}{(a_1a_2a_3a_5)_\infty(a_5/a_4)_\infty}$$
$$\times \sum_j \frac{(a_1a_4)_j(a_2a_4)_j(a_3a_4)_jq^j}{(q)_j(a_1a_2a_3a_4)_j(qa_4/a_5)_j}{}_2\phi_1\left[\begin{matrix}a_4a_5, a_5q^{-j}/a_4\\ Aa_5^2\end{matrix}\middle| Aq^j\right]$$
$$+ \frac{(a_1a_4)_\infty(a_2a_4)_\infty(a_3a_4)_\infty}{(a_1a_2a_3a_4)_\infty(a_4/a_5)_\infty}$$
$$\times \sum_j \frac{(a_1a_5)_j(a_3a_5)_j(a_2a_5)_jq^j}{(q)_j(a_1a_2a_3a_5)_j(qa_5/a_4)_j}{}_2\phi_1\left[\begin{matrix}q^{-j}, a_4a_5\\ Aa_5^2\end{matrix}\middle| \frac{Aa_5q^j}{a_4}\right]$$
$$= \frac{(a_1a_5)_\infty(a_2a_5)_\infty(a_3a_5)_\infty(Aa_5/a_4)_\infty(Aa_4a_5)_\infty}{(a_5/a_4)_\infty(a_1a_2a_3a_5)_\infty(Aa_5^2)_\infty(A)_\infty} \times {}_4\phi_3\left[\begin{matrix}a_1a_4, a_2a_4, a_3a_4, A\\ qa_4/a_5, a_1a_2a_3a_4, Aa_4a_5\end{matrix}\middle| q\right]$$
$$+ \frac{(a_1a_4)_\infty(a_2a_4)_\infty(a_3a_4)_\infty}{(a_4/a_5)_\infty(a_1a_2a_3a_4)_\infty}{}_4\phi_3\left[\begin{matrix}a_1a_5, a_2a_5, a_3a_5, Aa_5/a_4\\ Aa_5^2, qa_5/a_4, a_1a_2a_3a_5\end{matrix}\middle| q\right].$$

We now can use Bailey's transformation of a very well-poised ${}_8\phi_7$ series in terms of two balanced ${}_4\phi_3$ series ([**10**; p. 69] and [**12**; (2.10.10)]). In that transformation put $qa = Aa_1a_3a_4a_5$, $f = A$, $g = a_3a_4$, $h = a_1a_4$, $d = Aa_5/a_2$, $e = a_1a_3$ we get the Nasrallah-Rahman formula (1.14).

However if we choose

$$f = a_1a_4, \quad g = a_2a_4, \quad h = a_3a_4, \quad qa = a_1a_2a_3a_4^2a_5,$$
$$d = a_1a_2a_3a_4/A, \quad e = a_4a_5$$

we get

$$(3.4)\ J = \frac{(Aa_4a_5)_\infty(aq/a_4a_1)_\infty(aq/a_2a_4)_\infty(aq/a_3a_4)_\infty(a_1a_2a_3a_5)_\infty(Aa_5/a_4)_\infty}{(qa)_\infty\prod_{1\le j<k\le 5}(a_ja_k)_\infty} \times {}_8W_7\left(a; \frac{qa}{a_4a_5A}, a_4a_5, a_2a_4, a_1a_4, a_3a_4 \middle| A\frac{a_5}{a_4}\right).$$

Formula (3.4) seems to be the more useful form of (1.14). In fact if we put $A = a_1a_2a_3a_4$ it then follows that $qa = a_4a_5A$ and in this case the ${}_8W_7$ in (3.4) becomes 1 and (1.15) follows immediately. In contrast Askey [7] used the summation of a very well-poised ${}_6\phi_5$ to show that in that case (1.14) reduces to (1.15).

4. Miscellaneous results. We next consider the integral

$$I = \frac{\prod_{j=1}^4(t_j)_\infty(at_j)_\infty}{(s_1)_\infty(s_2)_\infty(as_1)_\infty(as_2)_\infty}\int_{-\infty}^{\infty}\frac{(xs_1)_\infty(xs_2)_\infty}{(xt_1)_\infty(xt_2)_\infty(xt_3)_\infty(xt_4)_\infty}d\psi_a(x)$$

and integrate it in two different ways. If we evaluate I directly using (1.12) we get

$$I = \frac{(at_1)_\infty(at_2)_\infty(at_3)_\infty(at_4)_\infty}{(a)_\infty(as_1)_\infty(as_2)_\infty}{}_4\phi_3\left[\begin{matrix}t_1, t_2, t_3, t_4\\ s_1, s_2, q/a\end{matrix}\middle| q\right] + \frac{(t_1)_\infty(t_2)_\infty(t_3)_\infty(t_4)_\infty}{(q/a)_\infty(s_1)_\infty(s_2)_\infty}{}_4\phi_3\left[\begin{matrix}at_1, at_2, at_3, at_4\\ as_1, as_2, aq\end{matrix}\middle| q\right].$$

Thus if we choose $s_1 = s_2 = q/a$, $a^3t_2t_3t_4 = q^2$, and use a transformation formula of Bailey [**10**, p. 69]

$$(4.1) \qquad I = \frac{(at_3t_4)_\infty(at_2t_3)_\infty(at_2t_4)_\infty(at_1)_\infty}{(at_2t_3t_4)_\infty(as_1)_\infty(as_2)_\infty}{}_8W_7(q/a^2; q/at_1, q/at_1, t_2, t_3, t_4|at_1).$$

On the other hand if we first expand the integrand in I we get

$$\begin{aligned}
&\frac{(s_1)_\infty(s_2)_\infty(as_1)_\infty(as_2)_\infty}{\prod_{j=1}^4\{(t_j)_\infty(at_j)_\infty\}} I \\
&= \sum_{n_1,n_2,n_3,n_4} \frac{t_1^{n_1}t_2^{n_2}t_3^{n_3}t_4^{n_4}(s_1/t_1)_{n_1}(s_2/t_2)_{n_2}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}} \int_{-\infty}^{\infty} x^{n_1+n_2+n_3+n_4}\,d\psi_a(x) \\
&= \sum_{n_1,n_2,n_3,n_4} \frac{t_1^{n_1}t_2^{n_2}t_3^{n_3}t_4^{n_4}(s_1/t_1)_{n_1}(s_2/t_2)_{n_2}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}} h_{n_1+n_2+n_3+n_4}(a|q) \\
&= \sum_{m,n_1,n_2} \frac{t_1^{n_1}t_2^{n_2}t_4^{m}(s_1/t_1)_{n_1}(s_2/t_2)_{n_2}}{(q)_{n_1}(q)_{n_2}(q)_m} h_m\left(\frac{t_3}{t_4}|q\right) h_{n_1+n_2+m}(a|q) \\
&= \sum_{m,k} \frac{t_4^m t_2^k (q/at_2)_k}{(q)_m(q)_k}\, {}_2\phi_1\left[\begin{matrix} q^{-k}, q/at_1 \\ at_2q^{-k}\end{matrix}\middle| at_1\right] h_m(t_3/t_4|q) h_{m+k}(a|q).
\end{aligned}$$

Equating the two values of I we get

$$\begin{aligned}
(4.2)\quad &\sum_{m,k} \frac{t_4^m t_2^k (q/at_2)_k}{(q)_m(q)_k}\, {}_2\phi_1\left[\begin{matrix} q^{-k}, q/at_1 \\ at_2q^{-k}\end{matrix}\middle| at_1\right] h_m(t_3/t_4|q) h_{m+k}(a|q) \\
&= \frac{(q/a)_\infty(q/a)_\infty(q^2/a^2t_4)_\infty(q^2/a^2t_3)_\infty(q^2/a^2t_2)_\infty(at_1)_\infty}{(q^2/a^2)_\infty \prod_{j=1}^4\{(t_j)_\infty(at_j)_\infty\}} \\
&\quad \times {}_8W_7(q/a^2; q/at_1, q/at_1, t_2, t_3, t_4|at_1).
\end{aligned}$$

Since I is symmetric in t_1 and t_2 it follows from (4.1) that

$$\begin{aligned}
(4.3)\quad &{}_8W_7(q/a^2; q/at_1, q/at_1, t_2, t_3, t_4|at_1) \\
&= \frac{(at_2)_\infty}{(at_1)_\infty}\, {}_8W_7(q/a^2; q/at_2, q/at_2, t_1, t_3, t_4|at_2).
\end{aligned}$$

Let us now reconsider the last step in the derivation in (3.3). Instead of replacing the inside sum by (2.6) we use (2.7). The result is

$$\begin{aligned}
J &= \frac{(A)_\infty(Aa_5^2)_\infty(a_1a_3a_4a_5)_\infty(a_1a_2a_3a_4)_\infty(a_2a_5)_\infty}{\prod_{1\le j<k\le 5}(a_ja_k)_\infty} \\
&\quad \times \sum_r \frac{A^r(a_4a_5)_r(a_1a_5)_r(a_3a_5)_r}{(q)_r(Aa_5^2)_r(a_1a_3a_4a_5)_r}\, {}_3\phi_2\left[\begin{matrix} a_1a_4\ ,\ a_1a_3\ ,\ a_3a_4 \\ a_1a_3a_4a_5q^r, a_1a_2a_3a_4\end{matrix}\middle| a_2a_5q^r\right] \\
&= \sum_j \frac{(a_1a_4)_j(a_1a_3)_j(a_3a_4)_j(a_2a_5)^j}{(q)_j(a_1a_3a_4a_5)_j(a_1a_2a_3a_4)_j}\, {}_3\phi_2\left[\begin{matrix} a_4a_5, a_3a_5, a_1a_5 \\ Aa_5^2, a_1a_3a_4a_5q^j\end{matrix}\middle| Aq^j\right].
\end{aligned}$$

We then transform the ${}_3\phi_2$ using Hall's formula [13]

$$ {}_3\phi_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix}\middle| ed/abc\right] = \frac{(e/c)_\infty (ed/ab)_\infty}{(e)_\infty (ed/abc)_\infty} {}_3\phi_2\left[\begin{matrix} d/a,d/b,c \\ d,de/ab \end{matrix}\middle| e/c\right] $$

we obtain, after some simplification, that

$$ J = \frac{(Aa_5/a_3)_\infty (Aa_3a_5)_\infty (a_1a_2a_3a_4)_\infty (a_1a_3a_4a_5)_\infty (a_2a_5)_\infty}{\prod_{1\le j<k\le 5}(a_ja_k)_\infty} \times \sum_{m,n} \frac{(A)_n(a_1a_4)_n(a_3a_5)_m(a_1a_3)_{m+n}(a_3a_4)_{n+m}}{(q)_n(q)_m(a_1a_2a_3a_4)_n(Aa_3a_5)_{m+n}(a_1a_3a_4a_5)_{m+n}} \times (a_2a_5)^n(Aa_5/a_3)^m. $$

If we compare this value for J with that in (3.4) we get a reduction formula of a q-analog of a Kampé de Fériet type function to a single very well-poised series. After some simple change of notation this can be stated as

$$ (4.4)\quad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(A)_m(\alpha)_m(\beta)_n(\delta)_{n+m}(\gamma)_{m+n}}{(q)_m(q)_n(\eta)_m(\alpha\beta)_{n+m}(A\beta)_{m+n}} \left(\frac{\eta\beta}{\delta\gamma}\right)^m \left(A\frac{\alpha\beta}{\delta\gamma}\right)^n = \frac{(A\alpha\beta/\delta)_\infty(\beta\eta/\delta)_\infty(\alpha\beta\eta/\delta\gamma)_\infty(\beta\eta/\gamma)_\infty(A\beta/\gamma)_\infty}{(\alpha\beta\eta/\delta)_\infty(A\alpha\beta/\delta\gamma)_\infty(A\beta)_\infty(\eta)_\infty(\beta\eta/\delta\gamma)_\infty} \times {}_8W_7\left(\frac{\alpha\beta\eta}{q\delta}; \frac{\eta}{A}, \frac{\alpha\beta}{\delta}, \frac{\eta}{\delta}, \alpha, \gamma \middle| A\beta/\gamma\right). $$

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