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THE KREIN-MILMAN PROPERTY AND A MARTINGALE COORDINATIZATION OF CERTAIN NON-DENTABLE CONVEX SETS

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The concepts of (strong) martingale representations and coordinatizations are defined, and the notion of a well-separated bush is crystallized. It is proved that if $\mathscr{B}$ is a well-separated uniformly bounded bush such that $\mathscr{B}$ is a strong martingale representation for its closed convex hull W, then W contains no extreme points. It is moreover proved that if K is a closed bounded convex subset of a Banach space with an unconditional skipped-blocking decomposition, then K contains such a bush provided K fails the point of continuity property. This yields the earlier result, due to the authors (unpublished) and to W. Schachermayer, that for closed bounded convex subsets of a Banach space with an unconditional basis, the Krein-Milman property implies the point of continuity property.

1. Let X be a Banach space and C a closed convex subset of X. C is said to have the *Krein-Milman property* (the KMP) if every closed bounded convex subset K of C is the norm-closed convex hull of its extreme points; C is said to have the *point of continuity property* (the PCP) provided every non-empty closed bounded subset K of C has a weak-to-norm point of continuity (a PC) relative to K; C satisfies the *Radon-Nikodým property* (the RNP) if and only if all closed bounded convex subsets K of C are dentable. For $\varepsilon > 0$, say that K is *ε-dentable* if K has a slice of diameter less than ε. A *slice* S of K is a subset of K of the form

$$S = S(f, \alpha, K) = \{x \in K | f(x) \geq \sup f(K) - \alpha\}$$

for some $f \in X^*$, $f \neq 0$ and $\alpha > 0$. K is *dentable* if it is ε-dentable for all $\varepsilon > 0$.

It is well-known that the RNP implies the PCP as well as the KMP (cf. [**BR**], [**DU**] and [**Bo**]). The converse to the first implication is known to be false (cf. [**BR**]); the validity of the converse to the second remains as a fundamental open question.

We introduce here the notions of (strong) martingale representations and coordinatizations in order to investigate the structure of sets which fail the PCP. We obtain the following result:

THEOREM 1.1. *Let X be a Banach space with an unconditional skipped-blocking decomposition and K a closed bounded convex subset of X so that K fails the* PCP. *Then there exists a K-valued well-separated δ-bush which is a strong martingale representation for its closed convex hull.*

The conclusion of this theorem is then shown to imply the existence of a closed bounded convex non-empty set without extreme points.

THEOREM 1.2. *Let W be the closed convex hull of a δ-bush satisfying the conclusion of Theorem* 1.1. *Then W fails to have extreme points.*

Since an unconditional basis is automatically an unconditional skipped-blocking decomposition, it follows from these two theorems that for subsets of a Banach space with an unconditional basis, the KMP implies the PCP. We had originally obtained this result in the summer of 1985. Our result was motivated by an earlier unpublished manuscript of W. Schachermayer, circulated in the fall of 1984, which established that this implication holds for *subspaces* of a space with an unconditional basis. In fact, the subspace implication is easily seen directly, using the results of **[BR]**. Indeed, suppose X is a Banach space failing the PCP such that X has an unconditional skipped-blocking decomposition (G_j). It follows by the results of **[BR]** that there exists a skipped-blocking (H_j) of (G_j) which is not boundedly complete. Since (H_j) is an unconditional FDD, c_0 embeds in the closed linear span of the H_j's; so X fails the KMP because c_0 is isomorphic to a subspace of X. (See §2 for the definition of the notion of skipped-blockings and skipped-blocking decompositions; these concepts were originally introduced and developed in **[BR]** (cf. also **[R3]**).)

Schachermayer did not use this reasoning, but instead used a construction due to J. Bourgain **[B]**. This construction was employed by Bourgain in **[B]** to show that every Banach space failing the CPCP contains a subspace with an FDD which fails the RNP. (As noted in **[BR]** (see also **[R3]**), the construction shows that in fact a Banach space failing the PCP contains a subspace with an FDD which fails the PCP.) Schachermayer observed that one can exploit this construction in the unconditional setting to directly obtain a convex set without extreme points. Our earlier proof as well as the arguments of the present work follow this line of attack.

Early in 1986, Schachermayer established that the PCP and KMP jointly imply the RNP in general (in fact, he showed that the SCSP and KMP imply the RNP, cf. [S] and also [**R1**]). He also refined the argument in his earlier unpublished work to obtain the result stated above and hence deduced that the KMP and RNP are equivalent for subsets of a space with an unconditional basis. In fact it thus follows (by the work discussed in this article and the above-mentioned deep result in [S]) that the KMP and RNP are equivalent for closed bounded convex subsets of a Banach space X with an unconditional skipped-blocking decomposition. We are indebted to the referee of an earlier version of this article for pointing out that there exist Banach spaces with this property, which do not embed in a space with an unconditional basis. For example, the referee pointed out that the space J of R. C. James with $\dim J^{**}/J = 1$ has the l^2-skipped-blocking property, and the $\mathscr{L}_\infty$-Schur space B of J. Bourgain and F. Delbaen has the l^1-skipped-blocking property. (Standard known results show that neither J nor B embed in a space with an unconditional basis.)

Evidently Schachermayer's discovery that the PCP and KMP jointly imply the RNP reduces the KMP/RNP equivalence problem to the study of sets failing the PCP. It is our hope that the techniques and formulations developed here will be of use in this study.

In the rest of this first section we review the terminology used throughout and develop the machinery needed to prove Theorem 1.2. The proof of Theorem 1.1 is presented in the following section. We recall briefly the bush terminology we shall be using. This is described in greater detail in [**R1**].

$\mathscr{T}^\infty$ denotes the set of all finite sequences of integers under the following natural partial order: given $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_m)$ in $\mathscr{T}^\infty$, $\alpha \leq \beta$ if $k \leq m$ and $\alpha_i = \beta_i$ for all $1 \leq i \leq k$. We refer to the empty sequence $\varnothing$ as the top-node of $\mathscr{T}^\infty$ and alternatively denote this as "0" or α_0; if $\alpha = (\alpha_1, \dots, \alpha_k)$, the "level" of α, denoted $|\alpha|$, is defined as k. (Abstractly, $|\alpha| = \#\{\beta : \beta < \alpha\}$.) A non-empty subset G of $\mathscr{T}^\infty$ is called a sub-tree of $\mathscr{T}^\infty$ if for all $g \in G$ and $\alpha \in \mathscr{T}^\infty$ with $\alpha \leq g$, $\alpha \in G$. For such a subset G of $\mathscr{T}^\infty$ and $\alpha \in G$, we set $\mathscr{S}_\alpha = \{\beta \in G : \alpha < \beta \text{ and } |\beta| = |\alpha| + 1\}$. G is called *finitely-branching* if $2 \leq \#\mathscr{S}_\alpha < \infty$ for all $\alpha \in G$. Finally, a finitely-branching tree $\mathscr{T}$ can be defined as a partially ordered set which is order isomorphic to a finitely-branching subtree of $\mathscr{T}^\infty$. Given a finitely branching tree $\mathscr{T}$, a function $\sigma : \mathscr{T} \to \mathbf{R}^+$ is called a bush function if $\sigma(0) = 1$ and $\sum_{\beta \in \mathscr{S}_\alpha} \sigma(\beta) = 1$ for all $\alpha \in \mathscr{T}$. A sub-family $(w_\alpha)_{\alpha \in \mathscr{T}}$ of X is called a bush if for all α, $w_\alpha \in \text{co}\{w_\beta \colon \beta \in \mathscr{S}_\alpha\}$. It follows that given a bush $(w_\alpha)_{\alpha\in\mathscr{T}}$, there exists a bush function σ so that $w_\alpha = \sum_{\beta\in\mathscr{S}_\alpha} \sigma(\beta) w_\beta$ for all α; when this relation holds, we say that $(w_\alpha)_{\alpha\in\mathscr{T}}$ is a bush with function σ. Given $\delta > 0$, a bush $(w_\alpha)_{\alpha\in\mathscr{T}}$ is called a δ-bush if $\|w_\alpha - w_\beta\| > \delta$ for all $\alpha \in \mathscr{T}$ and $\beta \in \mathscr{S}_\alpha$. The *wedges* of a bush $(w_\alpha)_{\alpha\in\mathscr{T}}$ are defined as $W_\alpha = \overline{\text{co}}\{w_\beta \colon \beta \geq \alpha\}$ for $\alpha \in \mathscr{T}$; we set $W = W_0$. The *bush differences* $(d_\alpha)_{\alpha\in\mathscr{T}}$ are given by $d_0 = w_0$ and $d_\beta = w_\beta - w_\alpha$ for $\alpha \in \mathscr{T}$ and $\beta \in \mathscr{S}_\alpha$. A *branch* β of $\mathscr{T}$ is a maximal well-ordered subset of $\mathscr{T}$.

Next we formulate our notions of (strong) martingale representation and coordinatization. We first define the notion of convergence and strong convergence for a family of elements of a Banach space indexed by a finitely branching tree.

Let $(x_\alpha)_{\alpha\in\mathscr{T}}$ be such a family. We say that $\sum_{\alpha\in\mathscr{T}} x_\alpha$ *converges* if $\sum_{n=0}^\infty z_n$ converges where $z_n = \sum_{|\alpha|=n} x_\alpha$. We say that $\sum_{\alpha\in\mathscr{T}} x_\alpha$ *converges strongly* if for all $\beta \in \mathscr{T}$, $\sum_{\alpha\in\mathscr{T}_\beta} x_\alpha$ converges where $\mathscr{T}_\beta = \{\alpha \in \mathscr{T} \colon \alpha \geq \beta\}$.

Let $(c_\alpha)_{\alpha\in\mathscr{T}}$ be a family of real numbers indexed by a finitely branching tree $\mathscr{T}$. We say that $(c_\alpha)_{\alpha\in\mathscr{T}}$ is *conditionally determined* (c.d.) if $c_\alpha = \sum_{\beta\in\mathscr{S}_\alpha} c_\beta$ for all $\alpha \in \mathscr{T}$. If we also have that $c_0 = 1$ and $c_\alpha \geq 0$ for all $\alpha \in \mathscr{T}$, we say that $(c_\alpha)_{\alpha\in\mathscr{T}}$ is *normalized conditionally determined* (n.c.d.). We note that if $(c_\alpha)_{\alpha\in\mathscr{T}}$ is n.c.d. and $\beta_0 \in \mathscr{T}$, the condition that $c_{\beta_0} = 1$ ensures that $c_\alpha = 0$ unless $\alpha \geq \beta_0$ or $0 \leq \alpha \leq \beta_0$; of course then $c_\beta = 1$ if $0 \leq \beta \leq \beta_0$.

Some "intrinsic" motivation for these notions is provided by the following lemma.

LEMMA 1.3. *Let $(w_\alpha)_{\alpha\in\mathscr{T}}$ be a bush with bush differences $(d_\alpha)_{\alpha\in\mathscr{T}}$.*

(a) *If $(c_\alpha)_{\alpha\in\mathscr{T}}$ is n.c.d. and $\sum_{\alpha\in\mathscr{T}} y_\alpha$ converges where $y_\alpha = \sum_{\beta\in\mathscr{S}_\alpha} c_\beta d_\beta$, then $w_0 + \sum_{\alpha\in\mathscr{T}} y_\alpha \in W$.*

(b) *Let $(c_\alpha)_{\alpha\in\mathscr{T}}$ and y_α be as above. If $(y_\alpha)_{\alpha\in\mathscr{T}}$ converges strongly, then for all $\gamma \in \mathscr{T}$ with $c_\gamma \neq 0$, $w_\gamma + (1/c_\gamma)\sum_{\alpha\in\mathscr{T}_\gamma} y_\alpha \in W_\gamma$.*

Proof. We first note that (b) follows directly from (a). Indeed, fix $\gamma \in \mathscr{T}$ with $c_\gamma \neq 0$. By assumption $(y_\alpha)_{\alpha\in\mathscr{T}}$ converges strongly; thus $\sum_{\alpha\in\mathscr{T}_\gamma} y_\alpha$ converges. Note that $\mathscr{T}_\gamma$ is a finitely branching tree in its own right and the desired conclusion is merely a restatement of (a) for the bush $(w_\alpha)_{\alpha\in\mathscr{T}_\gamma}$ and corresponding n.c.d. $(c'_\alpha)_{\alpha\in\mathscr{T}_\gamma}$ given by $c'_\alpha = c_\alpha/c_\gamma$ for $\alpha \in \mathscr{T}_\gamma$.

To prove (a), let $x = w_0 + \sum_{\alpha\in\mathcal{T}} y_\alpha$ and

$$x_n = w_0 + \sum_{|\alpha|=1}^{n} y_\alpha = w_0 + \sum_{|\alpha|=1}^{n} \sum_{\beta\in\mathcal{S}_\alpha} c_\beta d_\beta.$$

Using the fact that $(c_\alpha)_{\alpha\in\mathcal{T}}$ is n.c.d. and that $d_\beta = w_\beta - w_\alpha$ for $\beta \in \mathcal{S}_\alpha$, a straightforward calculation yields that $x_n = \sum_{|\beta|=n+1} c_\beta w_\beta$, a *convex* combination of elements of the bush. Since $x_n \to x$, it follows that $x \in \overline{\mathrm{co}}\{w_\alpha : \alpha \in \mathcal{T}\} = W$. □

This lemma also motivates the following concepts. We say that a δ-bush $(w_\alpha)_{\alpha\in\mathcal{T}}$ is a (*strong*) *martingale representation* for its closed convex hull W, if for every $\beta_0 \in \mathcal{T}$ and $x \in W_{\beta_0}$, there exists $(c_\alpha)_{\alpha\in\mathcal{T}}$ n.c.d. with $c_{\beta_0} = 1$ representing x in the following sense:

$$x = w_0 + \sum_{\alpha\in\mathcal{T}} y_\alpha$$

the sum converging (strongly), where $y_\alpha = \sum_{\beta\in\mathcal{S}_\alpha} c_\beta d_\beta$ for all $\alpha \in \mathcal{T}$, and $(d_\alpha)_{\alpha\in\mathcal{T}}$ are the differences of the bush $(w_\alpha)_{\alpha\in\mathcal{T}}$. We say that $(w_\alpha)_{\alpha\in\mathcal{T}}$ is a (*strong*) *martingale coordinatization* if the above representation is unique.

As a convenient notational device, we extend a given finitely branching tree to include a "dummy" node "-1", which we interpret to be the predecessor to the top node 0. Thus $w_0 = d_0 = \sum_{\beta\in\mathcal{S}_{-1}} c_\beta d_\beta$, and the above equation reduces to

$$x = \sum_{\alpha\in\mathcal{T}^\#} y_\alpha$$

where $\mathcal{T}^\#$ is the extension of $\mathcal{T}$ and $y_\alpha = \sum_{\beta\in\mathcal{S}_\alpha} c_\beta d_\beta$ for all $\alpha \in \mathcal{T}^\#$.

The following martingale interpretation should clarify our choice of terminology. As is well-known, an arbitrary bush $(w_\alpha)_{\alpha\in\mathcal{T}}$ can be associated with a vector valued martingale $(\vec{w}_n)$ in a canonical fashion, as follows. Let Γ be the set of branches of $\mathcal{T}$. We then define a probability space $(\Gamma, \mathcal{A}, P)$ by setting for each $\alpha \in \mathcal{T}$, $U_\alpha = \{\gamma \in \Gamma : \alpha \in \gamma\}$, $\mathcal{A}_n = \sigma(\{U_\alpha : |\alpha| = n\})$ and $\mathcal{A} = \sigma(\mathcal{A}_n)$. The probability measure P on $(\Gamma, \mathcal{A})$ is determined by its values on the increasing finite algebras $\mathcal{A}_n$. We define these values inductively: $P(U_0) = P(\Gamma) = 1$. If P is defined on $\mathcal{A}_n$ and $\beta \in \mathcal{S}_\alpha$ with $|\alpha| = n$, then $P(U_\beta) = P(U_\alpha)\tau(\beta)$, where τ is the bush function associated with $(w_\alpha)_{\alpha\in\mathcal{T}}$.

We now let $\vec{w}_n = \sum_{|\alpha|=n} w_\alpha I_{U_\alpha}$, where I_U is the indicator function of the set U. $(\vec{w}_n)$ is easily seen to be a martingale with respect to $(\mathcal{A}_n)$. This is our *fixed vector martingale*.

Now suppose that $(w_\alpha)_{\alpha\in\mathscr{T}}$ is a martingale representation for its closed convex hull. Let $x \in W$ and let $(c_\alpha)_{\alpha\in\mathscr{T}}$ n.c.d. represent x in the sense defined above. We define

$$f_n = \sum_{|\alpha|=n} \frac{c_\alpha}{p(U_\alpha)} I_{U_\alpha}.$$

Since $(c_\alpha)_{\alpha\in\mathscr{T}}$ is c.d., (f_n) can easily be shown to be a (scalar) martingale with respect to $(\mathscr{A}_n)$; since $(c_\alpha)_{\alpha\in\mathscr{T}}$ is normalized, (f_n) is non-negative and $Ef_0 = 1$. (For a P-integrable function f on Ω, Ef, the expectation of f, is defined as usual by $Ef = \int f\,dP$.) Taking the expectation of $f_n \cdot \vec{w}_n$ we obtain for all positive n

$$E(f_n \cdot \vec{w}_n) = \sum_{|\alpha|=n} c_\alpha w_\alpha = \sum_{|\alpha|\leq n} c_\alpha d_\alpha.$$

Thus we have that $x = \lim_{n\to\infty} E(f_n \cdot \vec{w}_n)$.

Conversely, given any martingale (f_n) with respect to $(\mathscr{A}_n)$, setting c_α equal to $P(U_\alpha)$ times the fixed value of f_n on U_α for $|\alpha| = n$, we find that $(c_\alpha)_{\alpha\in\mathscr{T}}$ is normalized conditionally determined. It follows from Lemma 1.3 that if $(E(f_n \cdot \vec{w}_n))$ converges to x, say, then $x \in W$.

We also have that $(w_\alpha)_{\alpha\in\mathscr{T}}$ is a *strong martingale coordinatization* if and only if it is a martingale coordinatization so that for all martingales (f_n) with respect to $(\mathscr{A}_n)$, whenever $(E(f_n \cdot \vec{w}_n))$ converges, $(E(f_n \cdot \vec{w}_n \cdot I_A))$ converges for all sets A in the *algebra* generated by $(\mathscr{A}_n)$.

The above observations show that $(w_\alpha)_{\alpha\in\mathscr{T}}$ is a martingale coordinatization for its closed convex hull W if and only if for each $x \in W$, there is a unique martingale (f_n) with respect to $(\mathscr{A}_n)$ so that $x = \lim_{n\to\infty} E(f_n \cdot \vec{w}_n)$. We then think of the martingale (f_n) as *coordinatizing* the point x.

We need one more definition, that of a well-separated bush, before passing to the proof of Theorem 1.2. For K and L non-empty subsets of some Banach space, $\mathrm{md}(K, L)$, the *minimum distance between K and L*, is defined as $\inf\{\|k - l\| : k \in K,\ l \in L\}$. A bush $(w_\alpha)_{\alpha\in\mathscr{T}}$ is *well-separated* provided there exists a $\kappa > 0$ so that

(1) $\mathrm{md}(K_n, W_\beta) > \kappa$ for all positive integers n and $\beta \in \mathscr{T}$ with $|\beta| = n + 1$

where $K_n = \mathrm{co}\{w_\alpha : |\alpha| = n\}$. If $(w_\alpha)_{\alpha\in\mathscr{T}}$ satisfies (1) we say that $(w_\alpha)_{\alpha\in\mathscr{T}}$ is *κ-well-separated.*

Our objective is to produce a δ-bush $(w_\alpha)_{\alpha\in\mathscr{T}}$ so that $W = \overline{\mathrm{co}}\{w_\alpha : \alpha \in \mathscr{T}\}$ fails to have extreme points. The following "locates" the extreme points of W.

PROPOSITION 1.4. *Let a δ-bush $(w_\alpha)_{\alpha\in\mathscr{T}}$ be given satisfying:*

$$(*) \qquad \text{For all } \alpha \in \mathscr{T},\ W_\alpha = \mathrm{co}\{W_\beta : \beta \in \mathscr{S}_\alpha\}.$$

Then for every extreme point x of W, there exists a branch γ of $\mathscr{T}$ such that x is an element of the wedge intersection $\bigcap_{\alpha\in\gamma} W_\alpha$.

Proof. We sketch a quick inductive proof. Fix an extreme point x of W. By assumption $x \in W = W_0$. Assume that $\gamma_0 = 0 < \gamma_1 < \cdots < \gamma_n \in \mathscr{T}$ have been found with $|\gamma_j| = j$ and $x \in W_{\gamma_j}$ for all $j = 0, \dots, n$. Applying $(*)$ we have

$$x \in W_{\gamma_n} = \mathrm{co}\{W_\beta : \beta \in \mathscr{S}_{\gamma_n}\}.$$

Since x is an extreme point it follows that $x \in W_\beta$ for some $\beta \in \mathscr{S}_{\gamma_n}$. Let $\gamma_{n+1} = \beta$. Continuing in this manner, we obtain a branch $\gamma = \{\gamma_0, \gamma_1, \gamma_2, \dots\}$ so that $x \in \bigcap_{\alpha\in\gamma} W_\alpha$. □

As a consequence of this proposition, given a δ-bush $(w_\alpha)_{\alpha\in\mathscr{T}}$ which satisfies $(*)$ and has *empty wedge intersections* (i.e., $\bigcap_{\alpha\in\gamma} W_\alpha$ is empty for all branches γ of $\mathscr{T}$), $W = \overline{\mathrm{co}}\{w_\alpha : \alpha \in \mathscr{T}\}$ will fail to have extreme points. As we demonstrate below, a well-separated δ-bush has empty wedge intersections (EWI); a bush which is a strong martingale representation for its closed convex hull satisfies $(*)$.

REMARK. The concept of a bush with EWI was introduced by R. C. James who established in **[J]** that a closed bounded convex subset K of a Banach space X such that K fails the RNP contains a δ-bush with EWI (see also **[R2]**). Condition $(*)$ is related to, but more comprehensive than, the notion of a *complemented* bush, introduced by A. Ho in **[Ho]**.

We prove the first of the assertions made above.

PROPOSITION 1.5. *Let $\kappa > 0$. If $(w_\alpha)_{\alpha\in\mathscr{T}}$ is a κ-well-separated δ-bush, then*

(a) $\mathrm{md}(E, W) \geq \kappa$ *where* $E = \bigcap_{n=0}^{\infty} \bigcup_{|\alpha|=n} \tilde{W}_\alpha$.

(b) $(w_\alpha)_{\alpha\in\mathscr{T}}$ *has empty wedge intersections (EWI).*

(c) $\mathrm{md}(X, E) \geq \kappa/2$.

[Recall, we regard X as a subset of X^{**}; $\tilde{W}_\alpha$ equals the weak-closure of W_α under this identification.]*

Proof. (a) Fix $e \in E$ and $w \in W$. We shall show that $\|e - w\| \geq \kappa$. Fix $\eta > 0$ and choose $w' \in \mathrm{co}\{w_\alpha : \alpha \in \mathcal{T}\}$ so that $\|w' - w\| < \eta$. Since $(w_\alpha)_{\alpha \in \mathcal{T}}$ is a bush, there exists a level n so that $w' \in K_n = \mathrm{co}\{w_\alpha : |\alpha| = n\}$. Since $e \in E$ we may choose $\beta \in \mathcal{T}$ with $|\beta| = n + 1$ and $e \in \tilde{W}_\beta$. By assumption $\mathrm{md}(K_n, W_\beta) > \kappa$; thus $d(w', W_\beta) > \kappa$ and by the Hahn-Banach theorem $d(w', \tilde{W}_\beta) > \kappa$. It follows that $\|e - w'\| > \kappa$; thus $\|e - w\| > \kappa - \eta$. Since η is arbitrary, the desired result follows.

(b) This follows immediately from (a) since E contains the set of all possible wedge intersections.

(c) This follows from (a) and the following elementary result, established in Lemma 2.6 of **[R2]**.

PROPOSITION 1.6. *Let W be a closed convex non-empty subset of a Banach space X and E a subset of $\tilde{W}$. Then*

$$\mathrm{md}(E, X) \geq \tfrac{1}{2}\,\mathrm{md}(E, W). \qquad \square$$

We are now prepared for the proof of Theorem 1.2. Let $(w_\alpha)_{\alpha \in \mathcal{T}}$ be a well-separated δ-bush which is a strong martingale representation for its closed convex hull W. To prove that W has no extreme points, by Proposition 1.4 it suffices to show that $(w_\alpha)_{\alpha \in \mathcal{T}}$ satisfies $(*)$ and has EWI. Since our δ-bush is well-separated, it follows from Proposition 1.5 that it has EWI. It remains to show that it satisfies $(*)$. This follows from the strong martingale representation and Lemma 1.3. Fix $\alpha_1 \in \mathcal{T}$. We must show that $W_{\alpha_1} = \mathrm{co}\{W_\beta : \beta \in \mathcal{S}_{\alpha_1}\}$. Fix $x \in W_{\alpha_1}$. By assumption there exists $(c_\alpha)_{\alpha \in \mathcal{T}}$ n.c.d. with $c_{\alpha_1} = 1$ so that $x = \sum_{\alpha \in \mathcal{T}^*} y_\alpha$ with the series converging strongly, where $y_\alpha = \sum_{\beta \in \mathcal{S}_\alpha} c_\beta d_\beta$ for all α. Thus $x = w_{\alpha_1} + \sum_{\alpha \in \mathcal{T}_{\alpha_1}} y_\alpha$. If we let $k_\gamma = w_\gamma + (1/c_\gamma) \sum_{\alpha \in \mathcal{T}_\gamma} y_\alpha$ for all $\gamma \in \mathcal{S}_{\alpha_1}$ with $c_\gamma \neq 0$, then k_γ is thus well defined and $k_\gamma \in W_\gamma$ by Lemma 1.3(b). Let $\mathcal{S}_1 = \{\gamma \in \mathcal{S}_{\alpha_1} : c_\gamma \neq 0\}$. Since $c_{\alpha_1} = 1$ and $(c_\alpha)_{\alpha \in \mathcal{T}}$ is c.d., we have $\sum_{\gamma \in \mathcal{S}_1} c_\gamma = 1$. It then follows that

$$\begin{aligned} x &= \sum_{\alpha \in \mathcal{T}^*} y_\alpha = w_{\alpha_1} + \sum_{\alpha \in \mathcal{T}_{\alpha_1}} y_\alpha \\ &= \sum_{\gamma \in \mathcal{S}_{\alpha_1}} c_\gamma w_\gamma + \sum_{\gamma \in \mathcal{S}_{\alpha_1}} \sum_{\alpha \in \mathcal{T}_\gamma} y_\alpha = \sum_{\gamma \in \mathcal{S}_{\alpha_1}} c_\gamma k_\gamma. \end{aligned}$$

Thus $x \in \mathrm{co}\{W_\gamma : \gamma \in \mathcal{S}_{\alpha_1}\}$. $\square$

2. This section is devoted to the proof of Theorem 1.1. Our construction actually exhibits a δ-bush which is a strong martingale *coordinatization* for its closed convex hull (in the proof of Theorem 2.4 use the δ-bush $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ instead of $(w_\alpha)_{\alpha\in\mathscr{T}}$); however, this δ-bush lies in a "bubble" of the original set K (i.e., in a set of the form $K+\alpha B_X$ where α is an arbitrary positive number and B_X is the closed unit ball of X) and may no longer lie in K.

We first assemble various structural results. Following **[BR]**, **[R3]**, we say that $(G_j)_{j=1}^\infty$ is a *decomposition* for a Banach space X provided X is the closed linear span of the G_j's and for all positive integers j, G_j is finite-dimensional and there exist continuous linear projections P_j from X onto G_j satisfying $P_jP_k = 0$ for all $j \neq k$. $(G_j)_{j=1}^\infty$ is called a *finite-dimensional decomposition* (an FDD) if it is a decomposition so that in addition for all $x \in X$, $x = \sum_{j=1}^\infty P_j(x)$, the sum converging in norm. If this sum converges *unconditionally* for all $x \in X$ we say that $(G_j)_{j=1}^\infty$ is an *unconditional* FDD.

Let $(G_j)_{j=1}^\infty$ be a sequence of subspaces of a Banach space X. $(H_j)_{j=1}^\infty$ is a *skipped-blocking* of $(G_j)_{j=1}^\infty$ if there exist sequences of positive integers (m_k) and (n_k) so that $m_k < n_k + 1 < m_{k+1}$, and H_k equals the closed linear subspace spanned by $\{G_i\colon i = m_k, \dots, n_k\}$ (we write $H_k = \mathrm{sp}\{G_i\}_{i=m_k}^{n_k}$). A decomposition $(G_j)_{j=1}^\infty$ is a *skipped*-FDD (resp. *unconditional skipped-blocking decomposition*) if every skipped-blocking $(H_j)_{j=1}^\infty$ of $(G_j)_{j=1}^\infty$ is an FDD (resp. an unconditional FDD) for its closed linear span $\overline{\mathrm{sp}}\{H_j\}_{j=1}^\infty$.

We define the *bi-FDD constant* for an FDD $(G_j)_{j=1}^\infty$ to be $\lambda = \sup_{k\leq j} \|\sum_{i=k}^j P_i\|$. It is easily seen that if (G_j) is a skipped-FDD, then there is a $\lambda < \infty$ so that the bi-FDD constant of (H_j) is less than or equal to λ for every skipped-blocking (H_j) of (G_j). We call the infimum of these possible λ's the skipped-bi-FDD constant of (G_j). It is evident that if (G_j) is an FDD, then the skipped-bi-FDD constant of (G_j) equals its bi-FDD constant. (As noted in **[BR]**, **[R3]**, any separable infinite-dimensional Banach space admits a skipped-FDD with constant at most $1 + \varepsilon$, for a given $\varepsilon > 0$. Also, it follows from the results in **[BR]** that if X has an unconditional skipped-blocking decomposition, then so does Y for every infinite-dimensional closed linear subspace Y of X.)

Let $\mathscr{T}^\#$ be an extended finitely branching tree. We say that a mapping $\tau\colon \mathscr{T}^\# \to \mathbf{N}$ is *strongly order preserving* if $|\alpha| < |\beta|$ implies $\tau(\alpha) < \tau(\beta)$. The next theorem gives the main constructive element of our proof.

THEOREM 2.1. *Let $(G_j)_{j=1}^{\infty}$ be a skipped-FDD for a Banach space X and K a closed bounded convex subset of X such that K fails the PCP. Then there exists a $\delta > 0$ so that for all sequences of positive numbers $(\varepsilon_j)_{j=-1}^{\infty}$ there exist a finitely branching tree $\mathscr{T}$, δ-bushes $(w_\alpha)_{\alpha\in\mathscr{T}}$ and $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ sharing a common bush function σ with $(w_\alpha)_{\alpha\in\mathscr{T}} \subset K$, a skipped-blocking $(H_j)_{j=1}^{\infty}$ of $(G_j)_{j=1}^{\infty}$ and a strongly order preserving bijection $\tau: \mathscr{T}^{\#} \to \mathbf{N}$ so that the following hold for all $\alpha \in \mathscr{T}^{\#}$ and $\beta \in \mathscr{S}_\alpha$ (as usual $(d_\alpha)_{\alpha\in\mathscr{T}}$ and $(\bar{d}_\alpha)_{\alpha\in\mathscr{T}}$ denote the differences of $(w_\alpha)_{\alpha\in\mathscr{T}}$ and $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ respectively):*

(i) $\|d_\beta - \bar{d}_\beta\| < \varepsilon_{|\alpha|}$,

(ii) $\bar{d}_\beta \in H_{\tau(\alpha)}$,

(iii) $\{\bar{d}_\beta : \beta \in \mathscr{S}_\alpha\}$ *is affinely independent.*

We shall soon see that if (G_j) is an unconditional skipped-blocking decomposition and if $\varepsilon = \sum_{j=-1}^{\infty} \varepsilon_j$ is sufficiently small, then the δ-bush $(w_\alpha)_{\alpha\in\mathscr{T}}$ constructed above will satisfy the conclusion of Theorem 1.1. To this end, some preliminary results are useful. We delay the proof of Theorem 2.1.

We first concentrate on the conditions on $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ obtained above.

LEMMA 2.2. *Let $(H_j)_{j=1}^{\infty}$ be an FDD for a Banach space Y with corresponding projections $P_j: Y \to H_j$ and $\lambda = \sup_j \|P_j\|$. Suppose $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ is a δ-bush in Y with differences $(\bar{d}_\alpha)_{\alpha\in\mathscr{T}}$ and $\tau: \mathscr{T}^{\#} \to \mathbf{N}$ is a strongly order preserving bijection so that for all $\alpha \in \mathscr{T}^{\#}$:*

(i) $\bar{d}_\beta \in H_{\tau(\alpha)}$ *for all* $\beta \in \mathscr{S}_\alpha$, *and*

(ii) $\{\bar{d}_\beta : \beta \in \mathscr{S}_\alpha\}$ *is affinely independent.*

Then $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ is δ/λ-well-separated and a martingale coordinatization for its closed convex hull. Moreover the levels of $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ are affinely independent.

Proof. We first deal with the algebraic assertion at the end. This is trivial for the 0th level. Proceeding by induction, we fix $n \geq 0$ and assume that for all real numbers (λ_α) indexed by $\mathscr{T}_n$, the nth level of $\mathscr{T}$, $\sum_{\alpha\in\mathscr{T}_n} \lambda_\alpha = 0$ and $\sum_{\alpha\in\mathscr{T}_n} \lambda_\alpha w_\alpha = 0$ implies that $\lambda_\alpha = 0$ for all $\alpha \in \mathscr{T}_n$. We now fix real numbers (λ_β) indexed by $\mathscr{T}_{n+1} = \{\alpha \in \mathscr{T} : |\alpha| = n+1\}$ with $\sum_{\beta\in\mathscr{T}_{n+1}} \lambda_\beta = 0 = \sum_{\beta\in\mathscr{T}_{n+1}} \lambda_\beta w_\beta$ and show that $\lambda_\beta = 0$ for all

$\beta \in \mathscr{T}_{n+1}$. Let $\lambda_\alpha = \sum_{\beta \in \mathscr{S}_\alpha} \lambda_\beta$ for all $\alpha \in \mathscr{T}_n$. We then have

$$0 = \sum_{\beta \in \mathscr{T}_{n+1}} \lambda_\beta \bar{w}_\beta = \sum_{\alpha \in \mathscr{T}_n} \sum_{\beta \in \mathscr{S}_\alpha} \lambda_\beta (\bar{w}_\alpha + \bar{d}_\beta)$$
$$= \sum_{\alpha \in \mathscr{T}_n} \lambda_\alpha \bar{w}_\alpha + \sum_{\alpha \in \mathscr{T}_n} \sum_{\beta \in \mathscr{S}_\alpha} \lambda_\beta \bar{d}_\beta.$$

It follows that $\sum_{\alpha \in \mathscr{T}_n} \lambda_\alpha \bar{w}_\alpha = 0$ and $\sum_{\beta \in \mathscr{S}_\alpha} \lambda_\beta \bar{d}_\beta = 0$ for all $\alpha \in \mathscr{T}_n$ since each of these terms lie in disjoint blocks of our FDD. Since also $\sum_{\alpha \in \mathscr{T}_n} \lambda_\alpha = \sum_{\beta \in \mathscr{T}_{n+1}} \lambda_\beta = 0$, by our induction hypothesis we have $\lambda_\alpha = 0$ for all $\alpha \in \mathscr{T}_n$. We now have for all $\alpha \in \mathscr{T}_n$, $\sum_{\beta \in \mathscr{S}_\alpha} \lambda_\beta = \lambda_\alpha = 0 = \sum_{\beta \in \mathscr{S}_\alpha} \lambda_\beta \bar{d}_\beta$ and $\{\bar{d}_\beta : \beta \in \mathscr{S}_\alpha\}$ is affinely independent, hence $\lambda_\beta = 0$ for all $\alpha \in \mathscr{S}_\alpha$. Since this holds for all $\alpha \in \mathscr{T}_n$, $\lambda_\beta = 0$ for all $\beta \in \mathscr{T}_{n+1}$ and we are done.

We now show that $(\bar{w}_\alpha)_{\alpha \in \mathscr{T}}$ is a martingale coordinatization for its closed convex hull. Fix $\beta_0 \in \mathscr{T}$. We must show that for each $x \in \bar{W}_{\beta_0} = \overline{\text{co}}\{\bar{w}_\alpha : \alpha \geq \beta_0\}$

(**) there exists a unique n.c.d $(c_\alpha)_{\alpha \in \mathscr{T}}$ with $c_{\beta_0} = 1$ and $x = \sum_{\alpha \in \mathscr{T}^\#} \bar{y}_\alpha$ where $\bar{y}_\alpha = \sum_{\beta \in \mathscr{S}_\alpha} c_\beta \bar{d}_\beta$ for all $\alpha \in \mathscr{T}^\#$.

Let L equal the set of all $x \in \bar{W}_{\beta_0}$ for which (**) holds (without the uniqueness assertion). We easily see that $\bar{w}_\beta \in L$ for all $\beta \geq \beta_0$. Indeed, fix $\beta \geq \beta_0$. We define $(c_\alpha)_{\alpha \in \mathscr{T}}$ inductively as follows:

For $0 \leq \alpha \leq \beta$ let $c_\alpha = 1$. If c_α has been defined for all $\alpha \geq \beta$ with $|\alpha| = |\beta| + n$ for a non-negative integer n, and $\gamma \in \mathscr{S}_\alpha$, α as above, let $c_\gamma = c_\alpha \sigma(\gamma)$ where σ is the bush function associated with $(\bar{w}_\alpha)_{\alpha \in \mathscr{T}}$. This defines c_α inductively for all $\alpha \geq \beta$. If α is incomparable with β (i.e., neither $\alpha \leq \beta$ nor $\alpha \geq \beta$), let $c_\alpha = 0$.

Since σ is the bush function for $(\bar{w}_\alpha)_{\alpha \in \mathscr{T}}$, this choice of $(c_\alpha)_{\alpha \in \mathscr{T}}$ yields

$$y_\alpha = \sum_{\gamma \in \mathscr{S}_\alpha} c_\gamma \bar{d}_\gamma = \sum_{\gamma \in \mathscr{S}_\alpha} c_\alpha \sigma(\gamma) \bar{d}_\gamma = c_\alpha \sum_{\gamma \in \mathscr{S}_\alpha} \sigma(\gamma) \bar{d}_\gamma = 0 \quad \text{for all } \alpha > \beta.$$

$(c_\alpha)_{\alpha \in \mathscr{T}}$ is obviously n.c.d. and $\sum_{\alpha \in \mathscr{T}^\#} y_\alpha = \sum_{0 \leq \alpha \leq \beta} \bar{d}_\alpha = \bar{w}_\beta$.

Next we observe that L is *closed*. Indeed, let $x_n \to x$ with $x_n \in L$ for all n. Letting $(c_\alpha^n)_{\alpha \in \mathscr{T}}$ represent x_n and passing to a subsequence, we may assume that for all $\alpha \in \mathscr{T}$, $c_\alpha^n \to c_\alpha$ as $n \to \infty$. Note that $(c_\alpha)_{\alpha \in \mathscr{T}}$ is n.c.d. and $c_{\beta_0} = 1$. By hypothesis $x_n = \sum_{\alpha \in \mathscr{T}^\#} \bar{y}_\alpha^n$ where $\bar{y}_\alpha^n = \sum_{\beta \in \mathscr{S}_\alpha} c_\beta^n \bar{d}_\beta$ and by (i), $y_\alpha^n \in H_{\tau(\alpha)}$ for all $\alpha \in \mathscr{T}^\#$.

Since $(H_j)_{j=1}^\infty$ is an FDD, x has a unique representation of the form $x = \sum_{\alpha \in \mathscr{T}^\#} u_\alpha$ where $u_\alpha \in H_{\tau(\alpha)}$ for all $\alpha \in \mathscr{T}^\#$, the series converging in the order induced by $\tau\colon \mathscr{T}^{\#} \to \mathbf{N}$. The sequence (x_n) converges to x, hence by continuity of the projection onto each $H_{\tau(\alpha)}$ we must have

$$\sum_{\beta\in\mathscr{S}_\alpha} c_\beta^n \bar{d}_\beta \to u_\alpha \quad \text{for all } \alpha\in\mathscr{T}^{\#}.$$

We trivially have that $\sum_{\beta\in\mathscr{S}_\alpha} c_\beta^n \bar{d}_\beta \to \sum_{\beta\in\mathscr{S}_\alpha} c_\beta \bar{d}_\beta$ which must therefore equal u_α. We thus have $x = \sum_{\alpha\in\mathscr{T}^{\#}} u_\alpha$ where $u_\alpha = \sum_{\beta\in\mathscr{S}_\alpha} c_\beta \bar{d}_\beta$ for all $\alpha\in\mathscr{T}^{\#}$, the series converging in the order induced by τ. Since τ is *strongly order preserving*, τ enumerates fully each level of $\mathscr{T}^{\#}$ before passing to the next. It follows that $\sum_{\alpha\in\mathscr{T}^{\#}} u_\alpha$ converges to x in the usual sense. This proves that L is closed. Since L is obviously *convex*, it follows that $L = \bar{W}_{\beta_0}$.

To show that each $x \in \bar{W}_{\beta_0}$ has a *unique* n.c.d. representation $(c_\alpha)_{\alpha\in\mathscr{T}}$, it suffices to show that if $(c_\alpha)_{\alpha\in\mathscr{T}}$ is c.d., $\sum_{\alpha\in\mathscr{T}^{\#}} \bar{y}_\alpha = 0$ where $\bar{y}_\alpha = \sum_{\beta\in\mathscr{S}_\alpha} c_\beta \bar{d}_\beta$ for all $\alpha\in\mathscr{T}^{\#}$ and $\sum_{\beta\in\mathscr{S}_\alpha} c_\beta = 0$ for some $\alpha\in\mathscr{T}^{\#}$, then $c_\beta = 0$ for $\beta\in\mathscr{S}_\alpha$. This is easy to show. Since we have an FDD, $\bar{y}_\alpha = \sum_{\beta\in\mathscr{S}_\alpha} c_\beta \bar{d}_\beta = 0$ for all α. Since $\sum_{\beta\in\mathscr{S}_\alpha} c_\beta = 0$, the affine independence of $\{d_\beta\colon \beta\in\mathscr{S}_\alpha\}$ ensures that $c_\beta = 0$ for all $\beta\in\mathscr{S}_\alpha$.

Next, we show that $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ is δ/λ-well-separated. Fix $x \in \bar{K}_n = \operatorname{co}\{\bar{w}_\alpha\colon |\alpha| = n\}$ and $y \in \bar{W}_\beta = \overline{\operatorname{co}}\{\bar{w}_\gamma\colon \gamma \geq \beta\}$ where $\beta \in \mathscr{S}_{\alpha_1}$ for some α_1 with $|\alpha_1| = n$. Using (i) and the fact that τ is strongly order preserving, we can find integers $j < k$ so that $\operatorname{co}\{\bar{w}_\alpha\colon |\alpha| = n\} \subset H = \operatorname{sp}\{H_0, \dots, H_j\}$ and $\bar{d}_\beta \in H_k$. We next note that $y = \bar{w}_{\alpha_1} + \bar{d}_\beta + r$ where $r \in \overline{\operatorname{sp}}\{H_i\colon i > k\}$. We now obtain

$$||x-y|| \geq \frac{1}{\lambda}||P_k(x-y)|| = \frac{1}{\lambda}||\bar{d}_\beta||$$

where P_k is the projection associated with H_k and $\lambda = \sup_j ||P_j||$. Since $||\bar{d}_\beta|| \geq \delta$ we obtain the desired result. □

We now draw some further consequences from the results of Theorem 2.1.

THEOREM 2.3. *Let* $(w_\alpha)_{\alpha\in\mathscr{T}}$ *and* $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ *be as in Theorem* 2.1, $W = W_0 = \overline{\operatorname{co}}\{w_\alpha\colon \alpha \geq 0\}$, $\bar{W} = \bar{W}_0 = \overline{\operatorname{co}}\{\bar{w}_\alpha\colon \alpha\geq 0\}$. *Then*

(a) $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ *is* δ/λ*-well-separated and a martingale coordinatization for* $\bar{W}$, *where* λ *is the skipped bi-FDD constant for* $(G_j)_{j=1}^\infty$.

(b) *Let* $(c_\alpha)_{\alpha\in\mathscr{T}}$ *be n.c.d. and for all* $\alpha\in\mathscr{T}^{\#}$ *let* $y_\alpha = \sum_{\beta\in\mathscr{S}_\alpha} c_\beta d_\beta$ *and* $\bar{y}_\alpha = \sum_{\beta\in\mathscr{S}_\alpha} c_\beta \bar{d}_\beta$. *Then* $\sum_{\alpha\in\mathscr{T}^{\#}} y_\alpha$ *converges (respectively converges unconditionally) if and only if* $\sum_{\alpha\in\mathscr{T}^{\#}} \bar{y}_\alpha$ *converges (respectively* *converges unconditionally). Here the mode of convergence is taken to be that induced by* $\tau\colon \mathcal{T}^{\#} \to \mathbf{N}$.

(c) *There is a uniformly continuous closed surjective affine map* $\varphi\colon \bar{W} \to W$ *with* $\varphi(\bar{w}_\alpha) = w_\alpha$ *and* $\varphi(\bar{W}_\alpha) = W_\alpha$ *for all* $\alpha \in \mathcal{T}$ *and* $\|\varphi(\bar{x}) - \bar{x}\| \le \varepsilon = \sum_{j=-1}^{\infty} \varepsilon_j$ *for all* $\bar{x} \in \bar{W}$.

(d) $(w_\alpha)_{\alpha\in\mathcal{T}}$ *is* $(\delta/\lambda - 2\varepsilon)$*-well-separated provided* $\varepsilon = \sum_{j=-1}^{\infty} \varepsilon_j < \delta/(2\lambda)$.

Theorem 1.1 now follows immediately. In fact, we obtain the following result.

THEOREM 2.4. *Let X be a Banach space with an unconditional skipped-blocking decomposition and let K be a closed bounded convex subset of X so that K fails the PCP. Then there exists a well-separated K-valued* δ*-bush* $(w_\alpha)_{\alpha\in\mathcal{T}}$ *which is a strong martingale representation for its closed convex hull.*

Proof of Theorem 2.4. Let K and X be as above and $(G_j)_{j=1}^{\infty}$ be an unconditional skipped-blocking decomposition for X. We first apply Theorem 2.1 with $\sum_{j=-1}^{\infty} \varepsilon_j < \delta/(2\lambda)$, where λ is the skipped-bi-FDD constant for $(G_j)_{j=1}^{\infty}$, to obtain a skipped-blocking $(H_j)_{j=1}^{\infty}$ of $(G_j)_{j=1}^{\infty}$, δ-bushes $(w_\alpha)_{\alpha\in\mathcal{T}}$ and $(\bar{w}_\alpha)_{\alpha\in\mathcal{T}}$, and a strongly order preserving bijection $\tau\colon \mathcal{T}^{\#} \to \mathbf{N}$ satisfying (i)–(iii) of Theorem 2.1.

It now follows from Theorem 2.3(a) that $(\bar{w}_\alpha)_{\alpha\in\mathcal{T}}$ is δ/λ-well-separated and a martingale coordinatization for $\bar{W}$. Thus for all $\beta_0 \in \mathcal{T}$ and $\bar{x} \in \bar{W}_{\beta_0} = \overline{\text{co}}\{\bar{w}_\beta\colon \beta \ge \beta_0\}$, there exists $(c_\alpha)_{\alpha\in\mathcal{T}}$ n.c.d. with $c_{\beta_0} = 1$ so that $\bar{x} = \sum_{\alpha\in\mathcal{T}^{\#}} \bar{y}_\alpha$ where $\bar{y}_\alpha = \sum_{\beta\in\mathcal{S}_\alpha} c_\beta \bar{d}_\beta$ for all $\alpha \in \mathcal{T}^{\#}$. Since $(H_j)_{j=1}^{\infty}$ is an unconditional FDD and $y_\alpha \in H_\tau(\alpha)$ for all $\alpha \in \mathcal{T}$ (condition (ii) of Theorem 2.1), $\bar{x} = \sum_{\alpha\in\mathcal{T}^{\#}} \bar{y}_\alpha$, the sum converging *unconditionally* in the order induced by τ. Since τ is a strongly order preserving bijection, it follows that this sum converges *strongly*. Thus $(\bar{w}_\alpha)_{\alpha\in\mathcal{T}}$ is a *strong* martingale coordinatization for its closed convex hull.

We now show that $(w_\alpha)_{\alpha\in\mathcal{T}}$ is a martingale representation for its closed convex hull. It then follows immediately from Theorem 2.3(b) and the argument above that it is a *strong* martingale representation and Theorem 2.3(d) tells us it is $(\delta/\lambda - 2\varepsilon)$-well-separated,where $\varepsilon = \sum_{j=-1}^{\infty} \varepsilon_j$.

We now fix $x \in W_{\beta_0}$. By Theorem 2.3(c) there exists an $\bar{x} \in \bar{W}_{\beta_0}$ with $\varphi(\bar{x}) = x$. Since $(\bar{w}_\alpha)_{\alpha\in\mathcal{T}}$ is a martingale coordinatization, we can represent $\bar{x}$ (using the alternate notation) $\bar{x} = \lim_{n\to\infty} \sum_{|\alpha|=n} c_\alpha \bar{w}_\alpha$

where $(c_\alpha)_{\alpha\in\mathcal{T}}$ is n.c.d. and $c_{\beta_0} = 1$. Since φ is affine, continuous and maps each $\bar{w}_\alpha$ to w_α, it follows that $x = \lim_{n\to\infty}\sum_{|\alpha|=n} c_\alpha w_\alpha$. Thus $(w_\alpha)_{\alpha\in\mathcal{T}}$ is indeed a (strong) martingale representation for its closed convex hull. □

Proof of Theorem 2.3. (a) follows immediately from Lemma 2.2.

(b) is immediate from the fact that $\sum_{\alpha\in\mathcal{T}^\#} \|y_\alpha - \bar{y}_\alpha\| < \infty$. The latter is a consequence of the fact that each d_α is a suitably small perturbation of $\bar{d}_\alpha$.

$$\sum_{\alpha\in\mathcal{T}^\#} \|y_\alpha - \bar{y}_\alpha\| \le \sum_{\alpha\in\mathcal{T}^\#}\sum_{\beta\in\mathcal{S}_\alpha} c_\beta \|d_\beta - \bar{d}_\beta\|$$
$$\le \sum_{n=-1}^{\infty}\sum_{|\alpha|=n}\sum_{\beta\in\mathcal{S}_\alpha} c_\beta\varepsilon_n = \sum_{n=-1}^{\infty}\sum_{|\alpha|=n} c_\alpha\varepsilon_n = \sum_{n=-1}^{\infty}\varepsilon_n = \varepsilon < \infty.$$

(c) We define φ as follows: Given $\bar{x} = \sum_{\alpha\in\mathcal{T}^\#}\sum_{\beta\in\mathcal{S}_\alpha} c_\beta \bar{d}_\beta$, we let $\varphi(\bar{x}) = \sum_{\alpha\in\mathcal{T}^\#}\sum_{\beta\in\mathcal{S}_\alpha} c_\beta d_\beta$. (Equivalently if $\bar{x} = \lim_{n\to\infty}\sum_{|\alpha|=n} c_\alpha \bar{w}_\alpha$, let $\varphi(\bar{x}) = \lim_{n\to\infty}\sum_{|\alpha|=n} c_\alpha w_\alpha$, from whence it follows that φ is affine and $\varphi(\bar{w}_\alpha) = w_\alpha$ for all $\alpha\in\mathcal{T}$.) Note that the estimate $\|\varphi(\bar{x}) - \bar{x}\| \le \sum_{j=-1}^{\infty}\varepsilon_j$ for all $\bar{x}\in\bar{W}$ follows from the proof of (b). It follows from (a) and (b) that φ is a well-defined map.

We next show that φ is *uniformly continuous*. Let $\bar{x}_n - \bar{y}_n \to 0$ in $\bar{W}$, and let $(c_\alpha^n)_{\alpha\in\mathcal{T}}$ and $(e_\alpha^n)_{\alpha\in\mathcal{T}}$ represent $\bar{x}_n$ and $\bar{y}_n$ respectively. We shall show that $\varphi(\bar{x}_n) - \varphi(\bar{y}_n) \to 0$. Since (H_j) is an FDD it is immediate that

$$\sum_{\beta\in\mathcal{S}_\alpha} c_\beta^n \bar{d}_\beta - \sum_{\beta\in\mathcal{S}_\alpha} e_\beta^n \bar{d}_\beta \to 0 \quad \text{for all } \alpha\in\mathcal{T}^\#. \tag{2}$$

We claim that

$$c_\alpha^n - e_\alpha^n \to 0 \quad \text{for all } \alpha\in\mathcal{T}. \tag{3}$$

This follows from a simple inductive argument once the following fact is established. Fix $d_1,\dots,d_k$ affinely independent elements of a Banach space X, and sequences of scalars $(c_1^n),\dots,(c_k^n)$ and $(e_1^n),\dots,(e_k^n)$. If

$$\sum_{i=1}^{k} c_i^n d_i - \sum_{i=1}^{k} e_i^n d_i \to 0 \quad \text{and} \quad \sum_{i=1}^{k} c_i^n - \sum_{i=1}^{k} e_i^n \to 0,$$

then $c_i^n - e_i^n \to 0$ for all $i = 1,\dots,k$. This is easily established. We work in the product space $X\times\mathbf{R}$ and choose u so that $d_1+u,\dots,d_k+u$

are *linearly* independent (e.g., $u = 0 \oplus 1$). We then have

$$\sum_{i=1}^{k}(c_i^n - e_i^n)(d_i + u) \to 0;$$

by the linear independence of these terms, the desired result follows.

We now present the inductive argument that $c_\alpha^n - e_\alpha^n \to 0$ for all $\alpha \in \mathcal{T}$. This is trivially true for $\alpha = 0$. Fix $n \geq 0$ and assume that $c_\alpha^n - e_\alpha^n \to 0$ for $|\alpha| = n$. Fix such an $\alpha \in \mathcal{T}$. It follows from the argument just given that $c_\beta^n - e_\beta^n \to 0$ for all $\beta \in \mathcal{S}_\alpha$. Indeed, we know that $\sum_{\beta\in\mathcal{S}_\alpha} c_\beta^n \bar{d}_\beta - \sum_{\beta\in\mathcal{S}_\alpha} e_\beta^n \bar{d}_\beta \to 0$, $\{\bar{d}_\beta \colon \beta \in \mathcal{S}_\alpha\}$ is affinely independent, and by the induction hypothesis and the fact that for all n, $(c_\alpha^n)_{\alpha\in\mathcal{T}}$ and $(e_\alpha^n)_{\alpha\in\mathcal{T}}$ are c.d., we have

$$\sum_{\beta\in\mathcal{S}_\alpha} c_\beta^n - \sum_{\beta\in\mathcal{S}_\alpha} e_\beta^n = c_\alpha^n - e_\alpha^n \to 0.$$

We now conclude the proof that φ is uniformly continuous. Fix $\eta > 0$. First choose N so that

$$\sum_{j\geq N} \varepsilon_j < \eta. \tag{4}$$

Recall that $\bar{x}_n - \bar{y}_n \to 0$. Since (H_j) is an FDD, it follows that

$$\left\| \sum_{|\alpha|\geq N} \sum_{\beta\in\mathcal{S}_\alpha} (c_\beta^n - e_\beta^n)\bar{d}_\beta \right\| \to 0.$$

Choose an integer n_1 so that

$$\left\| \sum_{|\alpha|\geq N} \sum_{\beta\in\mathcal{S}_\alpha} (c_\beta^n - e_\beta^n)\bar{d}_\beta \right\| < \eta \quad \text{for all } n \geq n_1. \tag{5}$$

We can now estimate

$$\begin{aligned} \|\varphi(\bar{x}_n) - \varphi(\bar{y}_n)\| &= \left\| \sum_{\alpha\in\mathcal{T}^\#} \sum_{\beta\in\mathcal{S}_\alpha} (c_\beta^n - e_\beta^n) d_\beta \right\| \\ &\leq \left\| \sum_{\substack{\alpha\in\mathcal{T}^\# \\ |\alpha|<N}} \sum_{\beta\in\mathcal{S}_\alpha} (c_\beta^n - e_\beta^n) d_\beta \right\| + \left\| \sum_{\substack{\alpha\in\mathcal{T}^\# \\ |\alpha|\geq N}} \sum_{\beta\in\mathcal{S}_\alpha} (c_\beta^n - e_\beta^n) d_\beta \right\|. \end{aligned}$$

Since the first expression contains only finitely many terms and $c_\beta^n - e_\beta^n \to 0$ for each $\beta \in \mathcal{T}$, we can find an integer $n_2 \geq n_1$ so that this expression is less than η for $n \geq n_2$. The second expression satisfies

$$\begin{aligned}
&\left\| \sum_{\substack{\alpha\in\mathcal{T}^\# \\ |\alpha|\geq N}} \sum_{\beta\in\mathcal{S}_\alpha} (c_\beta^n - e_\beta^n) d_\beta \right\| \\
&\quad \leq \left\| \sum_{\substack{\alpha\in\mathcal{T}^\# \\ |\alpha|\geq N}} (c_\beta^n - e_\beta^n)(d_\beta - \bar{d}_\beta) \right\| + \left\| \sum_{\substack{\alpha\in\mathcal{T}^\# \\ |\alpha|\geq N}} (c_\beta^n - e_\beta^n)\bar{d}_\beta \right\| \\
&\quad \leq \sum_{n=N}^{\infty} \sum_{\substack{\alpha\in\mathcal{T}^\# \\ |\alpha|=n}} \sum_{\beta\in\mathcal{S}_\alpha} (c_\beta^n + e_\beta^n)\|d_\beta - \bar{d}_\beta\| + \eta \quad \text{for } n \geq n_1 \qquad \text{by (5)} \\
&\quad \leq \sum_{n=N}^{\infty} \sum_{\substack{\alpha\in\mathcal{T}^\# \\ |\alpha|=n}} \sum_{\beta\in\mathcal{S}_\alpha} (c_\beta^n + e_\beta^n)\varepsilon_n + \eta \qquad \text{by (i) of Theorem 2.1} \\
&\quad \leq 2\sum_{n=N}^{\infty} \varepsilon_n + \eta \quad \text{since } (c_\beta^n)_{\beta\in\mathcal{T}} \text{ and } (e_\beta^n)_{\beta\in\mathcal{T}} \text{ are n.c.d.} \\
&\quad < 3\eta \qquad \text{by (4).}
\end{aligned}$$

It follows that $\|\varphi(\bar{x}_n) - \varphi(\bar{y}_n)\| < 4\eta$ for all $n \geq n_2$.

We now establish that φ is *closed*. It is then immediate that it is surjective (since $\mathrm{co}\{w_\alpha : \alpha \in \mathcal{T}\}$ is trivially dense in W). Let A be a closed non-empty subset of $\bar{W}$ and $(\bar{x}_n)$ be a sequence in A, $x_n = \varphi(\bar{x}_n)$ for all n, and assume that $x_n \to x$. For all n let $(c_\alpha^n)_{\alpha\in\mathcal{T}}$ represent x_n; by passing to a subsequence we may assume without loss of generality that $c_\alpha^n \to c_\alpha$ for all $\alpha \in \mathcal{T}$, where $(c_\alpha)_{\alpha\in\mathcal{T}}$ is, of course, n.c.d.. (It is *not* yet apparent that $(c_\alpha)_{\alpha\in\mathcal{T}}$ represents x!)

We claim that $(\bar{x}_n)$ is Cauchy. It then follows that $(\bar{x}_n)$ converges. By the arguments presented above, we know that $\bar{x}_n \to \bar{x}$ where $\bar{x}$ is represented by $(c_\alpha)_{\alpha\in\mathcal{T}}$. Hence $x = \varphi(\bar{x})$, and $\bar{x} \in A$ since A is closed. Let $\eta > 0$. We show that for n, m sufficiently large, $\|\bar{x}_n - \bar{x}_m\| < 4\eta$. This follows from another perturbation estimate. First, choose N so that $\sum_{j\geq N} \varepsilon_j < \eta$. As before, we need only show that

$$\left\| \sum_{\substack{\alpha\in\mathcal{T}^\# \\ |\alpha|\geq N}} \sum_{\beta\in\mathcal{S}_\alpha} (c_\beta^n - c_\beta^m)\bar{d}_\beta \right\| < 3\eta \tag{6}$$

for n, m sufficiently large, since we can then control the finite number of initial terms.

Since (x_n) converges and also $(\sum_{\alpha\in\mathcal{T}^*;|\alpha|<N}\sum_{\beta\in\mathcal{S}_\alpha} c^n_\beta d_\alpha)$ converges, we have that $(\sum_{\alpha\in\mathcal{T}^*;|\alpha|\geq N}\sum_{\beta\in\mathcal{S}_\alpha} c^n_\beta d_\alpha)$ converges and is thus Cauchy. Hence for n, m sufficiently large,

$$\left\|\sum_{\substack{\alpha\in\mathcal{T}^*\\|\alpha|\geq N}}\sum_{\beta\in\mathcal{S}_\alpha}(c^n_\beta - c^m_\beta)d_\beta\right\| < \eta. \tag{7}$$

Estimating the left-hand-side of (6) by means of (7) and the perturbation technique used above yields the desired result.

Since φ is affine and $\varphi(\bar{w}_\alpha) = w_\alpha$ for all $\alpha\in\mathcal{T}$, to prove that $\varphi(\bar{W}_\alpha) = \varphi(W_\alpha)$ for all $\alpha\in\mathcal{T}$, it suffices to regard the subtree $\mathcal{T}_\alpha = \{\beta\in\mathcal{T}: \beta\geq\alpha\}$ as a finitely branching tree in its own right and apply the proof just given that $\varphi(\bar{W}) = W$.

(d) Fix an integer n, $x\in K_n = \mathrm{co}\{w_\alpha: |\alpha| = n\}$ and $y\in W_\beta$ with $\beta\in\mathcal{T}$ and $|\beta| = n+1$. It follows from (c) that there exist $\bar{x}\in\bar{K}_n = \mathrm{co}\{\bar{w}_\alpha: |\alpha| = n\}$ and $\bar{y}\in\bar{W}_\beta$ with $\|\bar{x}-x\|\leq\varepsilon$ and $\|\bar{y}-y\|\leq\varepsilon$. Since $(\bar{w}_\alpha)_{\alpha\in\mathcal{T}}$ is δ/λ-well-separated, $\|x-y\|\geq\delta/\lambda-2\varepsilon$ and hence $(w_\alpha)_{\alpha\in\mathcal{T}}$ is $(\delta/\lambda-2\varepsilon)$-well-separated provided $(\delta/\lambda-2\varepsilon)$ is positive. □

We finally present the proof of Theorem 2.1. We construct the desired δ-bush by "averaging back" from a suitably chosen approximate bush. Let $(\varepsilon_j)_{j=0}^\infty$ be a sequence of positive numbers. An *approximate δ-bush corresponding to* $(\varepsilon_j)_{j=0}^\infty$ is a bounded family of elements of a Banach space $(w_\alpha)_{\alpha\in\mathcal{T}}$, indexed by a finitely branching tree $\mathcal{T}$, which satisfies:

$$\left\|w_\alpha - \sum_{\beta\in\mathcal{S}_\alpha}\sigma(\beta)w_\beta\right\|\leq\varepsilon_{|\alpha|}\quad\text{and}\quad\|w_\alpha - w_\beta\|\geq\delta\quad\text{for }\beta\in\mathcal{S}_\alpha,$$

where σ is a *bush function.*

The following result is a useful "trick-of-the-trade". It can be proved directly. The most elegant proofs, however, involve martingale techniques (cf. **[KR]**) and make use of the correspondence between bushes and approximate bushes on one hand, and martingales and quasi-martingales on the other (see the discussion following Definition 7.5.2 in **[Bo]**).

PROPOSITION 2.5. *Let $(w_\alpha)_{\alpha\in\mathcal{T}}$ be an approximate δ-bush with bush function σ corresponding to a summable sequence of positive numbers*

$(\eta_j)_{j=0}^{\infty}$ *which satisfy* $\delta' = \delta - 2\sum \eta_j > 0$. *Then there exists an "averaged back"* δ'*-bush* $(\tilde{w}_\alpha)_{\alpha\in\mathscr{T}}$ *such that*

$$\tilde{w}_\alpha \in \overline{\text{co}}\{w_\beta : \beta \in \mathscr{T}\}, \quad \tilde{w}_\alpha = \sum_{\beta\in\mathscr{S}_\alpha} \sigma(\beta)\tilde{w}_\beta \quad \textit{and}$$

$$\|\tilde{w}_\alpha - w_\alpha\| \le \sum_{j=|\alpha|}^{\infty} \eta_j \quad \textit{for all } \alpha \in \mathscr{T}.$$

We shall need the following two lemmas, which investigate the neighborhood structure of a closed bounded convex set failing the PCP. The first lemma is a slight modification of Lemma 2 in **[BR]** and is proved in **[R3]**.

LEMMA 2.6. *If a closed bounded convex subset* K *of a Banach space* X *fails the PCP, then there exist* $\delta > 0$ *and a nonempty closed subset* A *of* K *such that every relative weak neighborhood in* A *has diameter greater than* 4δ.

The next result is a "localized" version of Lemma 10 in **[B]** and is proved in **[W]**.

LEMMA 2.7. *Let* A *and* δ *be as in the conclusion of Lemma* 2.6. *Then for all* $x \in A$ *and relative weak neighborhoods* $N(x)$ *of* x,

$$x \in \overline{\text{co}}\{(A\backslash B(x, 2\delta)) \cap N(x)\} \tag{8}$$

where $B(x, \tau)$ *is the closed ball of radius* τ *centered at* x.

We now begin the main construction. Let X, K, and $(G_j)_{j=1}^{\infty}$ be as in the statement of Theorem 2.1. Applying Lemmas 2.6 and 2.7, we obtain a closed subset A of K and $\delta > 0$ so that (8) holds for all $x \in A$ and relative weak neighborhoods $N(x)$ of x.

Let $(\varepsilon_j)_{j=-1}^{\infty}$ be an arbitrary sequence of positive numbers. We need to produce a K-valued δ-bush $(w_\alpha)_{\alpha\in\mathscr{T}}$ with bush function σ, a δ-bush $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ sharing the same bush function, bush differences $(d_\alpha)_{\alpha\in\mathscr{T}}$ and $(\bar{d}_\alpha)_{\alpha\in\mathscr{T}}$, a strongly order preserving bijection $\tau: \mathscr{T}^{\#} \to \mathbf{N}$, and a skipped-blocking $(H_j)_{j=1}^{\infty}$ of $(G_j)_{j=1}^{\infty}$ satisfying for all $\alpha \in \mathscr{T}^{\#}$ and $\beta \in \mathscr{S}_\alpha$:

(i) $\|d_\beta - \bar{d}_\beta\| < \varepsilon_{|\alpha|}$,

(ii) $\bar{d}_\beta \in H_{\tau(\alpha)}$,

(iii) $\{\bar{d}_\beta : \beta \in \mathscr{S}_\alpha\}$ is affinely independent.

We first claim it suffices to construct a δ-bush $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ along with an A-valued *approximate* 2δ-bush $(w_\alpha)_{\alpha\in\mathscr{T}}$ with differences $(d_\alpha)_{\alpha\in\mathscr{T}}$ corresponding to a suitably chosen $(\eta_j)_{j=0}^\infty$ depending on $(\varepsilon_j)_{j=-1}^\infty$, satisfying (i) (with $\eta_{|\alpha|}$ replacing $\varepsilon_{|\alpha|}$), (ii), and (iii). Indeed, let $(w_\alpha)_{\alpha\in\mathscr{T}}$ be such an approximate bush. Proposition 2.5 then guarantees the "averaged back" bush $(\tilde{w}_\alpha)_{\alpha\in\mathscr{T}}$ is K-valued, a δ-bush so long as $2\sum\eta_j \le \delta$ and satisfies

$$||\tilde{w}_\alpha - w_\alpha|| \le \sum_{j=|\alpha|}^{\infty} \eta_j.$$

Letting $(\tilde{d}_\alpha)_{\alpha\in\mathscr{T}}$ denote the differences of $(\tilde{w}_\alpha)_{\alpha\in\mathscr{T}}$ and fixing $\alpha \in \mathscr{T}^\#$ and $\beta \in \mathscr{S}_\alpha$, we then have

$$||\tilde{d}_\beta - d_\beta|| = ||w_\beta - w_\alpha - (\tilde{w}_\beta - \tilde{w}_\alpha)|| \le \sum_{j=|\beta|}^{\infty} \eta_j + \sum_{j=|\alpha|}^{\infty} \eta_j.$$

Since by hypothesis we additionally have $||d_\beta - \bar{d}_\beta|| < \eta_{|\alpha|}$, it follows that

$$||\tilde{d}_\beta - \bar{d}_\beta|| < 2\sum_{j=|\alpha|}^{\infty} \eta_j.$$

Thus given $(\varepsilon_j)_{j=-1}^\infty$, for suitably chosen $(\eta_j)_{j=0}^\infty$ (e.g., choose $(\eta_j)_{j=0}^\infty$ so that $2\sum_{j=0}^\infty \eta_j \le \delta$ and $\eta_{j+k} \le \varepsilon_j/(4\cdot 2^k)$ for $j = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$) and corresponding approximate bush, the "averaged back" bush will have the desired properties.

Thus to prove Theorem 2.1, given $(\varepsilon_j)_{j=-1}^\infty$ an arbitrary sequence of positive numbers, it suffices to construct an A-valued approximate 2δ-bush $(w_\alpha)_{\alpha\in\mathscr{T}}$ and a δ-bush $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}}$ sharing the same bush function σ, with differences $(d_\alpha)_{\alpha\in\mathscr{T}}$ and $(\bar{d}_\alpha)_{\alpha\in\mathscr{T}}$ respectively, and $\tau\colon \mathscr{T}^\# \to \mathbf{N}$ and (H_j) as above, satisfying (i)–(iii) for all $\alpha \in \mathscr{T}^\#$ and $\beta \in \mathscr{S}_\alpha$ *and* (iv)

$$\left\| w_\alpha - \sum_{\beta\in\mathscr{S}_\alpha} \sigma(\beta) w_\beta \right\| = \left\| \sum_{\beta\in\mathscr{S}_\alpha} \sigma(\beta) d_\beta \right\| \le \varepsilon_{|\alpha|} \quad \text{for all } \alpha \in \mathscr{T}.$$

We carry out the construction using Lemma 2.7; the weak neighborhoods we use will be of the form:

$$N(x) = \{y \in A\colon ||S_m(y - x)|| < \varepsilon\}$$

for some $\varepsilon > 0$ and positive integer m, where $S_m(y) = \sum_{j=1}^m P_j(y)$, the natural projection of y onto the span of the first m G_j's. (Recall that for all i, P_i is the projection associated with G_i.)

We construct the bushes by induction on the lexicographic order induced by the strongly order preserving bijection τ, which we simultaneously define, beginning with the top-most level and enumerating each node of a given level before passing to the next level. Thus when all the nodes of a given level have been enumerated by τ, all the nodes of the succeeding level are defined. At a given node α, we define $\tau(\alpha)$, construct the successors $\mathcal{S}_\alpha$ of α, the differences of the bushes (d_β) and $(\bar{d}_\beta)$ for $\beta \in \mathcal{S}_\alpha$ and a skipped-blocking $H_{\tau(\alpha)}$ of $(G_j)_{j=1}^\infty$ so that (i)–(iv) hold for this α and $\beta \in \mathcal{S}_\alpha$.

We begin with the "dummy" index "-1", define $\tau(-1) = 1$ and set $\mathcal{S}_{-1} = \{\alpha_0\}$, where α_0, the empty sequence, is the top node of our finitely branching tree. (To avoid confusion, we refrain from using the notation $\alpha_0 = 0$ here.) Recall that the sole purpose of the dummy index is to write α_0 as a successor. We now let $w_{\alpha_0} = d_{\alpha_0}$ be an arbitrary element of A, set $m_1 = 1$, and choose $n_1 \geq m_1$ so that there exists

$$\bar{d}_{\alpha_0} \in H_1 \stackrel{\text{def}}{=} \text{sp}\{G_j\}_{j=m_1}^{n_1} \quad \text{with } \|\bar{d}_{\alpha_0} - d_{\alpha_0}\| \leq \varepsilon_1.$$

Set $\bar{w}_{\alpha_0} = \bar{d}_{\alpha_0}$. (i)–(iii) then trivially hold for $\alpha = -1$; (iv) does not yet apply.

Let $j \geq 1$ and assume that the construction has been carried out for α with $\tau(\alpha) \leq j$, with $m_1 < n_1 + 1 < m_2 < \cdots < m_j < n_j + 1$ and $H_i = \text{sp}\{G_k\}_{k=m_i}^{n_i}$ for $i = 1, \ldots, j$. Choose α with $\tau(\alpha) = j$ and let $k = |\alpha|$. If there exists a node α_1 on the kth level for which τ has not yet been defined, let $\tau(\alpha_1) = j + 1$. Otherwise choose an arbitrary node α_1 on the $(k + 1)$st level (these nodes all exist since by hypothesis the successors to the kth level have then been constructed) and let $\tau(\alpha_1) = j + 1$.

Let $\eta = \varepsilon_{j+1}/5$. We apply (8) with $x = w_{\alpha_1}$ and $N(w_{\alpha_1}) = \{y \in A : \|S_{n_j+1}(y - x)\| < \eta\}$.

Since $w_{\alpha_1} \in A' = \overline{\text{co}}\{(A \backslash B(w_{\alpha_1}, 2\delta)) \cap N(w_{\alpha_1})\}$ there exist an integer n, $w_1, \ldots, w_n \in A'$ and $\lambda_1, \ldots, \lambda_n$ non-negative numbers with $\sum \lambda_i = 1$ so that

$$\left\| w_{\alpha_1} - \sum_{i=1}^{n} \lambda_i w_i \right\| < \eta \quad \text{and} \quad \|w_{\alpha_1} - w_i\| \geq 2\delta \quad \text{for } i = 1, \ldots, n.$$

By our choice of $N(w_{\alpha_1})$, we additionally have

$$\|S_{n_j+1}(d_i)\| < \eta \quad \text{for } d_i = w_i - w_{\alpha_1} \text{ and } i = 1, \ldots, n.$$

Note that

$$(9)\qquad \left\|\sum_{i=1}^{n}\lambda_i d_i\right\| = \left\|\sum_{i=1}^{n}\lambda_i w_i - \sum_{i=1}^{n}\lambda_i w_{\alpha_1}\right\| < \eta.$$

We set $m_{j+1} = n_j+2$ and choose $n_{j+1} \geq m_{j+1}$ so that $||d_i - S_{n_{j+1}}d_i|| < \eta$ for $i = 1, \dots, n$. Setting $d_i' = (S_{n_{j+1}} - S_{n_j+1})d_i$, we then have

$$(10)\qquad ||d_i' - d_i|| < 2\eta \quad \text{and} \quad d_i' \in H_{j+1} \overset{\text{def}}{=} \text{sp}\{G_l\}_{l=m_{j+1}}^{n_{j+1}} \quad \text{for } i = 1, \dots, n.$$

Before we define our bush elements, we make adjustments and pare down these sets. Let $d_i'' = d_i' - \sum_{j=1}^{n}\lambda_j d_j'$ for $i = 1, \dots, n$. Then

$$\sum_{i=1}^{n}\lambda_i d_i'' = 0, d_i'' \in H_{j+1} \quad \text{for } i = 1, \dots, n, \quad \text{and}$$

$$\begin{aligned} ||d_i'' - d_i|| &\leq ||d_i' - d_i|| + \left\|\sum_{j=1}^{n}\lambda_j d_j'\right\| \\ &\leq 2\eta + \left\|\sum_{j=1}^{n}\lambda_j d_j\right\| + \sum \lambda_j ||d_j' - d_j|| \qquad \text{by (10)} \\ &\leq 2\eta + \eta + 2\eta \qquad \text{by (9) and (10)} \\ &= \varepsilon_{j+1}. \end{aligned}$$

Since $\sum_{i=1}^{n}\lambda_i d_i'' = 0$, we can choose an affinely independent subset $\{\bar{d}_1, \dots, \bar{d}_r\}$ of $\{d_1'', \dots, d_n''\}$ and non-negative numbers $\sigma_1, \dots, \sigma_r$ with $\sum_{i=1}^{r}\sigma_i = 1$ and $\sum_{i=1}^{r}\sigma_i\bar{d}_i = 0$. Relabeling if necessary, we now let $\{d_1, \dots, d_r\}$ be the corresponding subset of the original d_i's. We then still have

$$||d_i|| \geq 2\delta, \quad ||d_i - \bar{d}_i|| < \varepsilon_{j+1}, \quad \bar{d}_i \in H_{j+1} \quad \text{and}$$
$$w_{\alpha_1} + d_i \in A \quad \text{for } i = 1, \dots, r.$$

Note that

$$\left\|\sum_{i=1}^{r}\sigma_i d_i\right\| \leq \left\|\sum_{i=1}^{r}\sigma_i\bar{d}_i\right\| + \sum_{i=1}^{r}\sigma_i||d_i - \bar{d}_i|| \leq 0 + \varepsilon_{j+1}.$$

We now *define* the successors of α_1 to be $(\alpha_1, 1), \dots, (\alpha_1, r)$, where (α, i) is the $(|\alpha| + 1)$-tuple of integers for which the first $|\alpha|$ terms coincide with α and the last term equals i. Relabeling with these indices, and setting $\sigma((\alpha, i)) = \sigma_i$ for $i = 1, \dots, r$, it is clear that (i)–(iv) are satisfied. This completes the proof of Theorem 2.1. □

We wish finally to isolate out the main constructive step in the proof of Theorem 2.1, which we use to produce a well-separated δ-bush which is a strong martingale representation for its closed convex hull. We thus obtain a criterion which insures that a closed convex set fails the KMP. The criterion is essentially the same as the one formulated by W. Schachermayer in [S] and in his earlier unpublished work. For possible future applications, we formulate the criterion in terms of unconditional decompositions which are not necessarily finite-dimensional.

Let X be a Banach space and $(H_j)_{j=1}^{\infty}$ be a sequence of closed linear subspaces of X so that (H_j) is an *unconditional decomposition* for its closed linear span. It follows that there exists a $\lambda < \infty$ so that $\|\sum \varepsilon_j h_j\| \le \lambda \|\sum h_j\|$ for all choices of signs $\varepsilon_j = \pm 1$ and $h_j \in H_j$, $j = 1, 2, \dots$, with finitely many of the h_j's non-zero. The smallest possible λ so that this holds is the *unconditional decomposition constant* of $(H_j)_{j=1}^{\infty}$. It is easily seen that in case the H_j's are all finite-dimensional, the bi-FDD constant of $(H_j)_{j=1}^{\infty}$ is no greater than its unconditional decomposition constant. We note that in the setting of Theorems 2.1 and 2.3, if the H_j's form an unconditional FDD, it suffices for the mapping $\tau\colon \mathscr{T}^{\#} \to \mathbf{N}$ to be an injection, since we can use the unconditionality to re-order the H_j's if desired.

Proposition 2.8. *Let X be a Banach space, K a closed convex subset of X, and let $(H_j)_{j=1}^{\infty}$ be a sequence of closed linear subspaces of X so that (H_j) is an unconditional decomposition for its closed linear span; let λ be the unconditional decomposition constant of (H_j) and let $\delta > 0$. Let $\varepsilon_j = \alpha/2^j$ and $\eta_j = \alpha/(2^j 4)$ for $j = -1, 0, 1, \dots$, where $\alpha > 0$ is chosen so that $\sum_{j=-1}^{\infty} \varepsilon_j < \delta/(2\lambda)$ and $2\sum_{j=0}^{\infty} \eta_j \le \delta$. Let $\mathscr{T}$ be a finitely branching tree, $(w_\alpha)_{\alpha\in\mathscr{T}}$ a K-valued approximate 2δ-bush corresponding to $(\eta_j/3)_{j=0}^{\infty}$ and $\tau\colon \mathscr{T}^{\#} \to \mathbf{N}$ an injection so that for all $\alpha \in \mathscr{T}^{\#}$ and $\beta \in \mathscr{S}_\alpha$*

$$d(d_\beta, H_{\tau(\alpha)}) < \eta_{|\alpha|}/3.$$

Then K fails the KMP.

We sketch the proof. Let $(w_\alpha)_{\alpha\in\mathscr{T}}$ have bush function λ. First, we pare down $\mathscr{T}$ to produce a finitely branching tree $\mathscr{T}'$, and a δ-bush $(\bar{w}_\alpha)_{\alpha\in\mathscr{T}'}$ along with the pared down version of our approximate 2δ-bush $(w_\alpha)_{\alpha\in\mathscr{T}'}$ which now share a (new) bush function σ. We proceed in a lexicographic order. At the top-most node there is little to do; we select $\bar{d}_0 = \bar{w}_0 \in H_{\tau(-1)}$ so that

$$\|d_0 - \bar{d}_0\| < \frac{\eta_{|-1|}}{3} < \eta_{|-1|},$$

and add the 0 node to $\mathcal{T}'$. Say we are now at a node $\alpha_1 \in \mathcal{T}'$ so that the successors to α_1 have not yet been defined. We know that

$$\left\| w_{\alpha_1} - \sum_{\beta \in \mathcal{S}_{\alpha_1}} \lambda(\beta) w_\beta \right\| < \frac{\eta_{|\alpha_1|}}{3}$$

where $\mathcal{S}_{\alpha_1}$ denotes the successors to α_1 in $\mathcal{T}$. For $\beta \in \mathcal{S}_{\alpha_1}$ we select $d'_\beta \in H_{\tau(\alpha_1)}$ satisfying

$$\|d_\beta - d'_\beta\| < \frac{\eta_{|\alpha_1|}}{3}.$$

Following the procedure outlined following Equation (10), we set $d''_\beta = d'_\beta - \sum_{\beta \in \mathcal{S}_{\alpha_1}} \lambda(\beta) d'_\beta$ for $\beta \in \mathcal{S}_{\alpha_1}$ and pare down the set $\mathcal{S}_{\alpha_1}$ to obtain $\mathcal{S}'_{\alpha_1}$, the successors to α_1 in $\mathcal{T}'$, and the values of the bush function $\sigma(\beta)$ for $\beta \in \mathcal{S}'_{\alpha_1}$ along with the corresponding $(\bar{d}_\beta)_{\beta \in \mathcal{S}_{\alpha_1}}$ and differences of the original approximate 2δ-bush $(d_\beta)_{\beta \in \mathcal{S}'_{\alpha_1}}$, for which the following estimates now hold for $\beta \in \mathcal{S}'_{\alpha_1}$:

(i) $\|d_\beta - \bar{d}_\beta\| < \eta_{|\alpha_1|}$,
(ii) $\bar{d}_\beta \in H_{\tau(\alpha_1)}$,
(iii) $\{\bar{d}_\beta : \beta \in \mathcal{S}'_{\alpha_1}\}$ is affinely independent,
(iv) $\|\sum_{\beta \in \mathcal{S}'_{\alpha_1}} \sigma(\beta) d_\beta\| \le \eta_{|\alpha_1|}$.

Continuing in this fashion, we obtain the desired pared down tree $\mathcal{T}'$ and δ-bush $(\bar{w}_\alpha)_{\alpha \in \mathcal{T}'}$ along with $(w_\alpha)_{\alpha \in \mathcal{T}'}$ so that (i)–(iii) hold for all $\alpha_1 \in \mathcal{T}'^{\#}$ and all $\beta \in \mathcal{S}_{\alpha_1}$, and (iv) holds for all $\alpha_1 \in \mathcal{T}'$. Since η_j satisfies $\eta_{j+k} \le \varepsilon_j/(4 \cdot 2^k)$ for $j = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$, as noted following Lemma 2.7, the "averaged back" bush $(\tilde{w}_\alpha)_{\alpha \in \mathcal{T}'}$ will satisfy (i)–(iii) with ε_j replacing η_j. Thus the necessary conclusions of Theorem 2.1 are met. By the arguments of Theorems 2.3 and 2.4, $(\tilde{w}_\alpha)_{\alpha \in \mathcal{T}'}$ is well-separated and a strong martingale representation for its closed convex hull, which therefore fails to have extreme points. Since $(\tilde{w}_\alpha)_{\alpha \in \mathcal{T}'} \subset K$, K fails the KMP.

REMARKS. We conclude with some open questions suggested by our work. Let K be a closed bounded convex subset of a Banach space X.

Let X have an unconditional skipped-blocking decomposition.

1. Evidently our results yield that if K fails the PCP, K admits a K-valued well-separated δ-bush which is a martingale representation

for its closed convex hull. Can this bush be chosen to be a martingale *coordinatization*? If X has an unconditional skipped-blocking decomposition, can the bush be chosen to be a *strong* martingale coordinatization? Our results show that every ε-bubble of K contains such a coordinatization. (See the comments at the beginning of §2.) Hence in particular, the answer is yes if K is the unit ball of X and X fails the PCP.

2. Does every K failing the PCP admit a K-valued δ-bush which is a strong martingale representation for its closed convex hull? In view of Schachermayer's results [S] and Theorem 1.2 an affirmative answer would, of course, solve the RNP/KMP equivalence problem.

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