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SPACES OF CONSTANT PARA-HOLOMORPHIC SECTIONAL CURVATURE

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We consider the sectional curvatures for metric ($J^4 = 1$)-manifolds, and study particularly the general expression of the metric and almost-product structure in normal coordinates for para-Kaehlerian manifolds of constant para-holomorphic sectional curvature. We also introduce models of such spaces.

1. Introduction. A *metric* ($J^4 = 1$)*-manifold* (cfr. **[3]**, **[11]**) is a pseudo-Riemannian manifold (M^n, g) together with a $(1,1)$ tensor field J such that $J^4 = 1$ and whose characteristic polynomial is $(x-1)^{r_1}(x+1)^{r_2}(x^2+1)^s$ with $r_1 + r_2 + 2s = n$; also, the tensor fields g and J are related by one of the following relations:

(i) $g(JX, Y) + g(X, JY) = 0$ (then g is necessarily pseudo-Riemannian and $r_1 = r_2$);

(ii) g is Riemannian and $g(JX, JY) = g(X, Y)$.

In the first case it is said that g is an aem (adapted in the electromagnetic sense metric), because this situation generalizes in a sense that of Mishra **[8]** and Hlavatý **[4]**; in the second one, g is called arm (adapted Riemannian metric).

In this note we consider, g being an aem, the J-Kaehler manifolds, that is ($J^4 = 1$)-manifolds such that $\nabla J = 0$, where ∇ is the Levi-Civita connection of g, and study the J-sectional curvature which generalizes the usual holomorphic-type sectional curvatures. We define the spaces of constant J-sectional curvature, and prove a lemma of Schur type. Also, we obtain explicitly the models corresponding to the situation of an aem g and $J^2 = 1$.

2. Terminology. We shall use the following terminology:

($J^4 = 1$)*-manifold*: the pair (M^n, J), where J is a $(1,1)$ tensor field such that $J^4 = 1$ and whose characteristic polynomial is $(x-1)^{r_1}(x+1)^{r_2}(x^2+1)^s$ with $r_1 + r_2 + 2s = n$.

e-metric ($J^4 = 1$)*-manifold*: a ($J^4 = 1$)-manifold (M^n, J) together with an aem, that is a pseudo-Riemannian metric g such that $g(JX, Y) + g(X, JY) = 0$.

Riemannian $(J^4 = 1)$*-manifold*: a $(J^4 = 1)$-manifold (M^n, J) with an arm, i.e., a Riemannian metric g such that $g(JX, JY) = g(X, Y)$.

The remaining cases have already their own names:

almost para-Hermitian manifold (see Libermann ([7]): it is an e-metric $(J^4 = 1)$-manifold such that $J^2 = 1$, or in other terms, $s = 0$ (see also Legrand [6]).

Riemannian almost-product manifold: a Riemannian $(J^4 = 1)$-manifold with $J^2 = 1$, or equivalently $s = 0$.

almost-Hermitian manifold: it is the case of $J^2 = -1$ or equivalently $r_1 = r_2 = 0$. In this case there is no distinction between aem and arm.

3. J-sectional curvature. We consider first that (M, J, g) is an e-metric $(J^4 = 1)$-manifold. We have $g(JX, Y) + g(X, JY) = 0$. Then necessarily $r_1 = r_2 = r$ (see [3]). Let ∇ be the Levi-Civita connection of g. The curvature operator $R(X, Y) : \Gamma(\bigotimes TM) \to \Gamma(\bigotimes TM)$ is defined by

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y],$$

and we use the following convention for the Riemann-Christoffel tensor field

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

We shall denote also by R the value of R at a generic point $x \in M$. Then, if $X, Y \in T_xM$, we put

$$\overline{K}(X, Y) = R(X, Y, X, Y).$$

A subspace E of T_xM is said to be *non-degenerate* if $g|E$ is non-degenerate. If $\{X, Y\}$ is a basis of a plane E of T_xM, then E is non-degenerate if and only if

$$g(X, X)g(Y, Y) - g(X, Y)^2 \neq 0.$$

For any non-degenerate plane E of T_xM we define the sectional curvature as

$$K(X, Y) = \frac{\overline{K}(X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2},$$

where $\{X, Y\}$ is any basis of E; $K(X, Y)$ only depends on E.

Since $g(JX, Y) + g(X, JY) = 0$, then $g(X, JX) = 0$. If $X, JX \in T_xM$ are linearly independent, they determine a plane of T_xM that we call the *J-section defined by X*. The sectional curvature of $\{X, JX\}$ is only defined if $g(lX, lX)^2 \neq g(l_3X, l_3X)^2$, where

$$l = \tfrac{1}{2}(1 + J^2), \qquad l_3 = \tfrac{1}{2}(1 - J^2),$$

are, respectively, the projectors upon the almost-product and the almost-complex subbundles of TM defined by J. In that case we put

$$\overline{H}(X) = \overline{K}(X, JX), \qquad H(X) = K(X, JX),$$

and say that $H(X)$ is the *J-sectional curvature determined by X*.

If $\nabla J = 0$ we say that (M, g, J) is an *e-$(J^4 = 1)$-Kaehler manifold.* The characterization of these manifolds is given through the following results, where we put

$$F(X, Y) = g(X, JY) = -F(Y, X).$$

3.1. LEMMA. *Let (M, g, J) be an e-metric $(J^4 = 1)$-manifold. Then:*

$$\begin{aligned} 4g((\nabla_X J)Y, Z) = & -2dF(X, Y, Z) + 2dF(X, J^2Y, J^2Z) \\ & + 2dF(JX, JY, J^2Z) + 2dF(JX, J^2Y, JZ) \\ & - g(N(Y, Z), J^3X) + g(N(JY, JZ), JX) \\ & + g(N(X, JY), J^2Z) + g(N(JZ, X), J^2Y), \end{aligned}$$

where $N(X, Y) = 2\{[JX, JY] + J^2[X, Y] - J[JX, Y] - J[X, JY]\}$ defines the Nijenhuis tensor of J.

Proof. We have

$$4g((\nabla_X J)Y, Z) = 4g(\nabla_X (JY), Z) + 4g(\nabla_X Y, JZ);$$

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y]Z) \\ &\quad + g([Z, X], Y) + g([Z, Y], X); \\ dF(X, Y, Z) &= X(g(Y, JZ)) - Y(g(X, JZ)) + Z(g(X, JY)) \\ &\quad - g([X, Y], JZ) + g([X, Z], JY) - g([Y, Z], JX), \end{aligned}$$

and our claim is obtained directly by application of these formulae. □

3.2. COROLLARY. *In an e-metric $(J^4 = 1)$-manifold (M, J, g), the condition $\nabla J = 0$ is equivalent to the simultaneous verification of the following conditions:*

(a) $N = 0$;

(b) $dF = 0$.

Proof. If $N = 0$ and $dF = 0$, it is obvious by 3.1 that $\nabla J = 0$. If $\nabla J = 0$, then $dF = 0$, because $\nabla g = 0$; also, $N = 0$ as it is easily checked from the expression of N, having in mind that ∇ is torsionless. □

If (M, g, J) is an almost para-Hermitian manifold and $\nabla J = 0$, we then have a *hyperbolic Kaehler manifold* (Raševski [**10**]), also called *para-Kaehler manifold* (Libermann [**7**]). See also Prvanović [**9**] and references therein. We adopt Libermann's terminology. The preceding result implies that an e-$(J^4 = 1)$-Kaehler manifold is locally the product of a para-Kaehler manifold and a Kaehler manifold.

3.3. PROPOSITION. *On an e-$(J^4 = 1)$-Kaehler manifold we have*

$$R(X, Y, Z, JW) + R(X, Y, JZ, W) = 0.$$

Proof. By applying the operator $R(X, Y)$, we have

$$\begin{aligned} R(X, Y)(g(Z, JW)) = 0 &= g(R(X, Y)Z, JW) + g(Z, R(X, Y)JW) \\ &= R(X, Y, Z, JW) - g(JZ, R(X, Y)W) \\ &= R(X, Y, Z, JW) + R(X, Y, JZ, W). \end{aligned} \qquad \square$$

3.4. PROPOSITION. *Let (M, J, g) be an e-$(J^4 = 1)$-Kaehler manifold. Then, if $\overline{H}(X) = 0$ for all $X \in TM$, we have $R = 0$.*

Proof. We consider the following $(0, 4)$ tensor field Q which generalizes that of the Kaehler case (see [**5**]):

$$Q(X, Y, Z, W) = R(X, JY, Z, JW) + R(X, JZ, Y, JW) + R(X, JW, Y, JZ).$$

From 3.3 and the usual symmetries of R we obtain that Q is totally symmetric. But $Q(X, X, X, X) = 3\overline{H}(X)$; whence $Q = 0$. Now, since $\nabla J = 0$, it is immediate to prove that

$$R(X, Y, X, Y) = R(lX, lY, lX, lY) + R(l_3X, l_3Y, l_3X, l_3Y).$$

Since $J^2 l = l$, $J^2 l_3 = -l_3$, the same technique of the Kaehler case (see [**5**]) leads to

$$\begin{aligned} &R(lX, lY, lX, lY) = 0, \\ &R(l_3X, l_3Y, l_3X, l_3Y) = 0. \end{aligned}$$

Thus, $R(X, Y, X, Y) = 0$, whence $R = 0$. $\square$

3.5. COROLLARY. *Let (M, J, g) be an e-$(J^4 = 1)$-Kaehler manifold. If $\tilde{R}$ is a $(0, 4)$ tensor field having the usual symmetries of R and also the one given in* 3.3, *and if*

$$\tilde{R}(X, JX, X, JX) = \overline{H}(X)$$

for all $X \in TM$, then $\tilde{R} = R$.

We now define the $(0, 4)$ tensor field R' on M by

$$\begin{aligned}R'(X, Y, Z, W) = \tfrac{1}{4}\{&g(X, lZ)g(Y, lW) - g(X, lW)g(Y, lZ)\\ &- g(X, JlZ)g(Y, JlW) + g(X, JlW)g(Y, JlZ)\\ &- 2g(X, JlY)g(Z, JlW) + g(X, l_3Z)g(Y, l_3W)\\ &- g(X, l_3W)g(Y, l_3Z) + g(X, Jl_3Z)g(Y, Jl_3W)\\ &- g(X, Jl_3W)g(Y, Jl_3Z)\\ &+ 2g(X, Jl_3Y)g(Z, Jl_3W)\},\end{aligned}$$

whose properties are given in the following

3.6. Proposition. *The field R' has the usual symmetries of the Riemann-Christoffel tensor and also the symmetry of Proposition* 3.3. *The following relations hold*:

$$\begin{aligned}&R'(X, Y, X, Y)\\ &\quad = \tfrac{1}{4}\{g(X, lX)g(Y, lY) - g(X, lY)^2 - 3g(X, JlY)^2\\ &\qquad + g(X, l_3X)g(Y, l_3Y) - g(X, l_3Y)^2 + 3g(X, Jl_3Y)^2\};\\ &R'(X, JX, X, JX) = g(X, l_3X)^2 - g(X, lX)^2.\end{aligned}$$

Proof. Immediate.

From this, we deduce the

3.7. Proposition. *Let (M, J, g) be an e-$(J^4 = 1)$-Kaehler manifold such that for each $x \in M$, there exists $c_x \in \mathbf{R}$ satisfying $H(X) = c_x$ for every $X \in T_xM$ such that $g(X, X)g(JX, JX) \neq 0$. Then $R = cR'$, where c is the function defined by $x \to c_x$. And conversely.*

Proof. Since $g(X, X)g(JX, JX) = g(X, l_3X)^2 - g(X, lX)^2$, we deduce from 3.6 that

$$\overline{H}(X) = cR'(X, JX, X, JX).$$

Hence $(R - cR')(X, JX, X, JX) = 0$ for all X such that

$$g(X, X)g(JX, JX) \neq 0.$$

Now, if X verifies $g(X, X)g(JX, JX) = 0$, then we can choose a sequence $\{X_m\}$ such that $X_m \to X$ and

$$g(X_m, X_m)g(JX_m, JX_m) \neq 0.$$

In fact, $g(X, X)g(JX, JX)$ is a polynomial in the components of X whose set of zeros does not contain any open subset. Since

$(R-cR')(X_m, JX_m, X_m, JX_m) = 0$ for each index m, we have by continuity that $(R-cR')(X, JX, X, JX) = 0$. Then, by 3.5 we have $R = cR'$. The converse is obvious. □

If the e-$(J^4 = 1)$-Kaehler manifold (M, J, g) satisfies the conditions of the above proposition, we say that it is of *constant J-sectional curvature c*. We have the following result of Schur type.

3.8. THEOREM. *Let (M, J, g) be an e-$(J^4 = 1)$-Kaehler manifold of constant J-sectional curvature c. If $r, s > 0$, or if $r = 0$, $s > 1$, or if $r > 1$, $s = 0$, then c is a constant function.*

Proof. We first choose an orthogonal basis of T_xM, $\{U_i, V_i, W_j, JW_j\}$ $(i = 1, \ldots, r; j = 1, \ldots, s)$ such that $\{U_i, V_i\}$ is a basis of lT_xM, $\{W_j, JW_j\}$ is a basis of l_3T_xM, $g(U_i, U_j) = -\delta_{ij}$, $g(V_i, V_j) = g(W_i, W_j) = g(JW_i, JW_j) = \delta_{ij}$, $(i, j = 1, \ldots, r$ or $i, j = 1, \ldots, s)$. If S is the Ricci tensor field, we have

$$\begin{aligned} S(X, Y) = &-\sum_{i=1}^{r} R(U_i, X, U_i, Y) + \sum_{i=1}^{r} R(V_i, X, V_i, Y) \\ &+ \sum_{i=1}^{s} R(W_i, X, W_i, Y) + \sum_{i=1}^{s} R(JW_i, X, JW_i, Y). \end{aligned}$$

From this, and applying 3.7, we obtain after a calculation

$$S(X, Y) = \frac{c}{2}\{g(X, Y) + rg(X, lY) + sg(X, l_3Y)\}. \tag{1}$$

Since $R = cR'$ and $\nabla R' = 0$, we have $\nabla_X R = X(c)R'$. Now, if $\{e_i\}$ is any orthonormal basis of T_xM in the sense that $g(e_i, e_j) = a_i\delta_{ij}$ with $a_i \in \{-1, 1\}$, we have by direct application of the second Bianchi identity

$$\sum_i \{X(c)S(a_ie_i, e_i) - 2e_i(c)S(X, a_ie_i)\} = 0. \tag{2}$$

Now,

$$\sum_i S(X, a_ie_i)e_i = \frac{c}{2}(X + rlX + sl_3X),$$

because of (1). Therefore, from (2):

$$(r^2 + s^2 + r + s - 1)X(c^2) - rlX(c^2) - sl_3X(c^2) = 0.$$

If $X = lX$, then

$$(r^2 + s^2 + s - 1)X(c^2) = 0;$$

If $X = l_3 X$, then

$$(r^2 + s^2 + r - 1)X(c^2) = 0.$$

Then, if $r, s > 0$, or if $s = 0$, $r > 1$, or if $s > 1$, $r = 0$, we obtain

$$X(c^2) = lX(c^2) + l_3 X(c^2) = 0.$$

Thus c^2, and therefore c, are constants. □

In the conditions of the preceding Theorem, the scalar curvature is given by the function

$$\rho = c\{r(r+1) + s(s+1)\}.$$

Thus, if $r = s = 1$, we have $\rho = 4c$.

3.9. THEOREM. *Let (M, J, g) be an e-$(J^4 = 1)$-Kaehler manifold of constant J-sectional curvature c. Then:*

(i) *if $X, Y \in l_3 T_x M$ we have*

$$\begin{aligned} c/4 \le K(X,Y) \le c, &\quad \text{if } c > 0; \\ c \le K(X,Y) \le c/4, &\quad \text{if } c < 0; \end{aligned}$$

(ii) *Let us denote by K_L the restriction of K to the planes of lTM. Then:*

$$\begin{aligned} &K_L(X,Y) = c \quad \text{if } r = 1; \\ &K_L \text{ is unbounded} \quad \text{if } r > 1,\ c \ne 0. \end{aligned}$$

Proof. (i) The restriction of g to $l_3 TM$ is Riemannian. Then if we choose $\{X, Y\}$ orthonormal, we have:

$$K(X,Y) = \frac{c}{4}(1 + 3g(X, JY)^2) = \frac{c}{4}(1 + 3\cos^2\alpha),$$

where α is the angle between the plane $\{X, Y\}$ and the plane $\{JX, JY\}$, and the claim is obvious;

(ii) If $r = 1$ we can choose a basis $\{X, JX\}$ of $lT_x M$; thus $K(X, JX) = H(X) = c$. Now assume that $c \ne 0$, $r > 1$. Let $(U_1, V_1) \in l_1 T_x M$, $(U_2, V_2) \in l_2 T_x M$ be such that $g(U_1, U_2) = g(V_1, V_2) = 1$, $g(U_1, V_2) = g(U_2, V_1) = 0$. Here, l_1 and l_2 are the projectors on $lT_x M$ given by the eigenvalues $+1$ and -1 of $J|lT_x M$. We take first

$$\begin{aligned} X &= U_1 + V_1 - U_2 + \frac{1}{2}V_2, \\ Y &= U_1 + (1-\lambda)V_1 + \frac{\lambda}{2}U_2 + \frac{1}{2}V_2. \end{aligned}$$

Then $g(X, X) = -1$, $g(Y, Y) = 1$, $g(X, JY) = -(1+\lambda)$, $g(X, Y) = 0$. Hence $K(X, Y) = (c/4)(1 + 3(1 + \lambda)^2)$.

Now, we take

$$X = U_1 + V_1 + U_2 - \frac{1}{2}V_2,$$
$$Y = \frac{\lambda^2}{2}U_1 + (\lambda^2 - \lambda + 1)V_1 - \lambda U_2 + \frac{\lambda + 1}{2}V_2.$$

Then $g(X, X) = g(Y, Y) = 1$, $g(X, Y) = 0$, $g(X, JY) = \lambda - 1$. Hence $K(X, Y) = (c/4)(1 - 3(\lambda - 1)^2)$, and this proves our claim. □

3.10. Definition. We say that two metric $(J^4 = 1)$-manifolds (M, J, g) and (M', J', g') are *J-isometric* if there exists an isometry $f: M \to M'$ such that $f_* \circ J = J' \circ f_*$.

It is clear that in the case of almost Hermitian manifolds this definition is the usual one for holomorphically isometric manifolds. Also we can generalize Theorem 7.9 of **[5]**, Vol. II to obtain

3.11. Proposition. *Two complete, connected and simply connected e-$(J^4 = 1)$-Kaehler manifolds of constant and equal J-sectional curvature c are J-isometric (we assume that c is a constant function).*

Proof. It is enough to apply Proposition 2.5 which furnishes the expression of R in terms of J and g in the case of spaces of constant J-sectional curvature. □

4. The models of constant J-sectional curvature. Let (M, J, g) be an e-$(J^4 = 1)$-Kaehler manifold; then it is locally the product of a para-Kaehler manifold and a Kaehler manifold. Since the latter, in the case of constant holomorphic sectional curvature, is well known (see **[5]**), we are interested in the para-Kaehler case.

Thus, let (M, J, g) be a para-Kaehler space of constant J-sectional curvature c, and assume $r > 1$. Then c is a constant function. We have $J^2 = 1$ and $g(X, JY) + g(JX, Y) = 0$.

Let $x_0 \in M$, and $\{e_i, e_{i+r}\}$ be an orthonormal basis of $T_{x_0}M$, i.e.:

$$g(e_i, e_j) = -\delta_{ij}, \quad g(e_{i+r}, e_{j+r}) = \delta_{ij}, \quad g(e_i, e_{j+r}) = 0,$$
$$Je_i = e_{i+r}, \quad Je_{i+r} = e_i.$$

If we put $R_{ABCD} = R(e_A, e_B, e_C, e_D)$, $A, B, C, D \in \{1, \dots, 2r\}$, then

$$R_{ABCD} = \frac{c}{4}(g_{AC}g_{BD} - g_{AD}g_{BC} - g_{AC\pm r}g_{BD\pm r} + g_{AD\pm r}g_{BC\pm r} - 2g_{AB\pm r}g_{CD\pm r}),$$

where

$$E \pm r = \begin{cases} E + r & \text{if } 1 \leq E \leq r, \\ E - r & \text{if } r+1 \leq E \leq 2r. \end{cases}$$

Prvanović [9] obtains this expression in a different way.

Now, we apply the structural equations in polar coordinates in order to obtain g and J in these coordinates (see [1], [12]).

For doing that, let I be an interval of $\mathbf{R}$ containing 0 and 1, U a neighbourhood of 0 in $T_{x_0}M$ and V a neighbourhood of x_0 in M such that $\exp\colon U \to V$ is a diffeomorphism and such that the map $\Phi\colon I \times U \to M$ given by $\Phi(t, X) = \exp(tX)$ is well defined. If $\{\gamma^A\}$ is the dual of $\{e_A\}$, we have coordinates (t, t^A) on $I \times U$ given by $t(t_0, X) = t_0$, $t^A(t_0, X) = \gamma^A(X)$.

By parallel transport of $\{e_A\}$ along the geodesics starting at x_0 we obtain a frame $\{e_A\}$ on V with dual $\{\gamma^A\}$. If we define the 1-forms ϑ^A on $I \times U$ by

$$\vartheta^A = \phi^* \gamma^A - t^A\, dt,$$

then $i(\partial/\partial t)\vartheta^A = 0$, and we have the conditions

$$\vartheta^A_{(0,X)} = 0, \quad \frac{\partial \vartheta^A}{\partial t}\Big|_{(0,X)} = dt^A|_{(0,X)},$$
$$\frac{\partial^2 \vartheta^A}{\partial t^2} = (R^A_{BCD} \circ \phi) t^B t^C \vartheta^D.$$

Thus

$$\begin{aligned} \frac{\partial^2 \vartheta^i}{\partial t^2} &= -(R_{iBCD} \circ \phi) t^B t^C \vartheta^D \\ &= -\frac{c}{4}\{t^j(t^i\vartheta^j - t^j\vartheta^i - t^{i+r}\vartheta^{j+r} + t^{j+r}\vartheta^{i+r}) \\ &\qquad + t^{j+r}(t^{j+r}\vartheta^i - t^i\vartheta^{j+r} + t^{i+r}\vartheta^j - t^j\vartheta^{i+r}) \\ &\qquad\qquad + 2t^{i+r}(t^{j+r}\vartheta^j - t^j\vartheta^{j+r})\}, \\ \frac{\partial^2 \vartheta^{i+r}}{\partial t^2} &= (R_{i+rBCD} \circ \phi) t^B t^C \vartheta^D \\ &= -\frac{c}{4}\{t^{j+r}(t^i\vartheta^j - t^j\vartheta^i - t^{i+r}\vartheta^{j+r} + t^{j+r}\vartheta^{i+r}) \\ &\qquad + t^j(t^{j+r}\vartheta^i - t^i\vartheta^{j+r} + t^{i+r}\vartheta^j - t^j\vartheta^{i+r}) \\ &\qquad\qquad + 2t^i(t^{j+r}\vartheta^j - t^j\vartheta^{j+r})\}. \end{aligned}$$

To simplify this, we introduce on $I \times U$ new coordinates $\{a^i, b^i\}$ and new 1-forms μ^i, ν^i by:

$$a^i = \frac{t^i + t^{i+r}}{\sqrt{2}}, \quad b^i = \frac{t^i - t^{i+r}}{\sqrt{2}}, \quad \mu^i = \frac{\vartheta^i + \vartheta^{i+r}}{\sqrt{2}}, \quad \nu^i = \frac{\vartheta^i - \vartheta^{i+r}}{\sqrt{2}}.$$

Then

$$\frac{\partial^2 \mu^i}{\partial t^2} = \frac{c}{4}(a^j b^j \mu^i + a^i b^j \mu^j - 2a^i a^j \nu^j),$$
$$\frac{\partial^2 \nu^i}{\partial t^2} = \frac{c}{4}(a^j b^j \nu^i + b^i a^j \nu^j - 2b^i b^j \mu^j).$$

By putting $\langle a, b\rangle = a^j b^j$, etc., this can be written

$$\frac{\partial^2 \mu}{\partial t^2} = \frac{c}{4}(\langle a, b\rangle \mu + \langle b, \mu\rangle a - 2\langle a, \nu\rangle a),$$
$$\frac{\partial^2 \nu}{\partial t^2} = \frac{c}{4}(\langle a, b\rangle \nu + \langle a, \nu\rangle b - 2\langle b, \mu\rangle a).$$

If we put $\rho^2 = -\frac{1}{2}c\langle a, b\rangle$, these equations read

$$\frac{\partial^2 \mu}{\partial t^2} + \rho^2 \mu = -\frac{\rho^2}{\langle a, b\rangle}\langle b, \mu\rangle a + \frac{2\rho^2}{\langle a, b\rangle}\langle a, \nu\rangle a, \tag{3}$$
$$\frac{\partial^2 \nu}{\partial t^2} + \rho^2 \nu = -\frac{\rho^2}{\langle a, b\rangle}\langle a, \nu\rangle b + \frac{2\rho^2}{\langle a, b\rangle}\langle b, \mu\rangle b. \tag{4}$$

If we multiply (3) by b and (4) by a, we obtain

$$\left\langle b, \frac{\partial^2 \mu}{\partial t^2}\right\rangle + \rho^2\langle b, \mu\rangle = -\rho^2\langle b, \mu\rangle + 2\rho^2\langle a, \nu\rangle, \tag{5}$$
$$\left\langle a, \frac{\partial^2 \nu}{\partial t^2}\right\rangle + \rho^2\langle a, \nu\rangle = -\rho^2\langle a, \nu\rangle + 2\rho^2\langle b, \mu\rangle. \tag{6}$$

By adding and subtracting (5) and (6), we get

$$\frac{\partial^2}{\partial t^2}(\langle b, \mu\rangle + \langle a, \nu\rangle) = 0, \tag{7}$$
$$\frac{\partial^2}{\partial t^2}(\langle b, \mu\rangle - \langle a, \nu\rangle) + 4\rho^2(\langle b, \mu\rangle - \langle a, \nu\rangle) = 0, \tag{8}$$

with the initial conditions

$$\mu_{(0)} = \nu_{(0)} = 0, \quad \frac{\partial \mu}{\partial t}\Big|_0 = da, \quad \frac{\partial \nu}{\partial t}\Big|_0 = db. \tag{9}$$

The solution of the system (7), (8), (9) is obviously

$$\langle b, \mu\rangle = \frac{\langle b, da\rangle - \langle a, db\rangle}{4\rho}\sin 2\rho t + \tfrac{1}{2}(\langle b, da\rangle + \langle a, db\rangle)t,$$
$$\langle a, \nu\rangle = \frac{\langle a, db\rangle - \langle b, da\rangle}{4\rho}\sin 2\rho t + \tfrac{1}{2}(\langle b, da\rangle + \langle a, db\rangle)t.$$

By substitution in (3), we get

$$\frac{\partial^2\mu}{\partial t^2} + \rho^2\mu = -\frac{3\rho}{4\langle a,b\rangle}(\langle b,da\rangle - \langle a,db\rangle)(\sin 2\rho t)a + \frac{\rho^2}{2\langle a,b\rangle}(\langle b,da\rangle + \langle a,db\rangle)\,ta.$$

It we call

$$\eta = \mu - \frac{1}{2\langle a,b\rangle}(\langle b,da\rangle + \langle a,db\rangle)\,ta,$$

then this equation reads

$$\frac{\partial^2\eta}{\partial t^2} + \rho^2\eta = -\frac{3\rho}{4\langle a,b\rangle}(\langle b,da\rangle - \langle a,db\rangle)(\sin 2\rho t)a.$$

We seek a particular solution of the type $\eta = (D/\rho\langle a,b\rangle)(\sin 2\rho t)a$. Then we get the condition

$$D = \tfrac{1}{4}(\langle b,da\rangle - \langle a,db\rangle).$$

Thus the solution is

$$\mu = \frac{1}{2\langle a,b\rangle}(\langle b,da\rangle + \langle a,db\rangle)ta + \frac{A}{\langle a,b\rangle}\sin(\rho t) + \frac{\langle b,da\rangle - \langle a,db\rangle}{4\rho\langle a,b\rangle}\sin(2\rho t)a.$$

And the initial conditions imply

$$\mu = \frac{\langle a,b\rangle da - \langle b,da\rangle a}{\langle a,b\rangle\rho}\sin\rho t + \frac{(\langle b,da\rangle - \langle a,db\rangle)a}{4\langle a,b\rangle\rho}\sin 2\rho t + \frac{\langle b,da\rangle + \langle a,db\rangle}{2\langle a,b\rangle}\,dt,$$

$$\nu = \frac{\langle a,b\rangle db - \langle a,db\rangle b}{\langle a,b\rangle\rho}\sin\rho t + \frac{(\langle a,db\rangle - \langle b,da\rangle)b}{4\langle a,b\rangle\rho}\sin 2\rho t + \frac{\langle b,da\rangle + \langle a,db\rangle}{2\langle a,b\rangle}\,bt.$$

Now, we define 1-forms α^i, β^i $(i = 1,\dots,r)$ on U by

$$\alpha^i = \mu^i(1), \quad \beta^i = \nu^i(1),$$

and also define a metric on U, $\tilde{g}$, by

$$\tilde{g} = -\alpha^i \otimes \beta^i - \beta^i \otimes \alpha^i,$$

and a tensor field $\tilde{J}$ on U by

$$\tilde{J} = u_i \otimes \alpha^i - v_i \otimes \beta^i,$$

where $\{u_i, v_i\}$ is the dual of $\{\alpha^i, \beta^i\}$. Then, the map exp: $U \to V$ is a J-isometry as it is easily checked. Thus, we compute $\tilde{g}$ and $\tilde{J}$. First we have

$$\alpha^i = \frac{\sin\rho}{\rho}\, da^i + \frac{\sin 2\rho - 4\sin\rho + 2\rho}{4\langle a, b\rangle \rho} b^k a^i\, da^k + \frac{2\rho - \sin 2\rho}{4\langle a, b\rangle\rho} a^k a^i\, db^k;$$

$$\beta^i = \frac{\sin\rho}{\rho}\, db^i + \frac{\sin 2\rho - 4\sin\rho + 2\rho}{4\langle a, b\rangle \rho} a^k b^i\, db^k + \frac{2\rho - \sin 2\rho}{4\langle a, b\rangle\rho} b^k b^i\, da^k.$$

Therefore, by substitution

$$\begin{aligned}\tilde{g} = -\Bigg\{ & \frac{\sin^2\rho}{\rho^2}(da^i \otimes db^i + db^i \otimes da^i) \\ & + \frac{4\rho^2 - \sin^2 2\rho}{8\langle a, b\rangle \rho^2}(a^i a^k db^i \otimes db^k + b^i b^k da^i \otimes da^k) \\ & + \frac{4\rho^2 + \sin^2 2\rho - 8\sin^2\rho}{8\langle a, b\rangle\rho^2} a^i b^k (db^i \otimes da^k + da^k \otimes db^i)\Bigg\}.\end{aligned}$$

Note that even in the case of $\rho^2 < 0$, the above result is a real tensor field, and it is C^∞ also in the points where $\rho = 0$.

As for the dual base, we have

$$\begin{aligned} u_j = & \frac{\rho}{\sin\rho}\frac{\partial}{\partial a^j} + \frac{\sin 2\rho - 2\rho}{2\langle a, b\rangle \sin 2\rho} b^j b^l \frac{\partial}{\partial b^l} \\ & + \frac{\sin\rho\sin 2\rho + 2\rho\sin\rho - 2\rho\sin 2\rho}{2\langle a, b\rangle \sin\rho\sin 2\rho} b^j a^l \frac{\partial}{\partial a^l}, \\ v_j = & \frac{\rho}{\sin\rho}\frac{\partial}{\partial b^j} + \frac{\sin 2\rho - 2\rho}{2\langle a, b\rangle \sin 2\rho} a^j a^l \frac{\partial}{\partial a^l} \\ & + \frac{\sin\rho\sin 2\rho + 2\rho\sin\rho - 2\rho\sin 2\rho}{2\langle a, b\rangle \sin\rho\sin 2\rho} a^j b^l \frac{\partial}{\partial b^l}. \end{aligned}$$

Therefore, we have by substitution

$$\begin{aligned}\tilde{J} = & \frac{\partial}{\partial a^i}\otimes da^i - \frac{\partial}{\partial b^i}\otimes db^i \\ & + \frac{(2\rho - \sin 2\rho)^2}{4\langle a, b\rangle\rho\sin 2\rho} a^i b^k \left(\frac{\partial}{\partial a^i}\otimes da^k - \frac{\partial}{\partial b^k}\otimes db^i\right) \\ & + \frac{4\rho^2 - \sin^2 2\rho}{4\langle a, b\rangle\rho\sin 2\rho}\left(a^i a^k \frac{\partial}{\partial a^i}\otimes db^k - b^i b^k \frac{\partial}{\partial b^i}\otimes da^k\right).\end{aligned}$$

The expression of $\tilde{g}$ and $\tilde{j}$ give the space form in normal coordinates for the para-Kaehler manifolds of constant J-sectional curvature and $r > 1$. If $r = 1$, we have automatically $N = 0$, $dF = 0$, $\nabla J = 0$, (cfr.

3.1) and the space is of constant J-sectional curvature c, but c may not be a constant. However if c were a constant, the above formulae for normal coordinates are also valid. Thus, we will say in the following that an almost para-Hermitian manifold with $r = 1$ is a *para-Kaehler manifold of constant J-sectional curvature* if the above function c is constant.

Now, let B be the vector space $\mathbf{R}^2$ with the product $(a, b)(a', b') = (aa', bb')$; then B is a commutative algebra. If we define the conjugate $\overline{w}$ of an element $w = (a, b) \in B$ by $\overline{w} = (b, a)$, then an element $w \in B$ is *real* if $w = \overline{w}$, and is invertible if $w\overline{w} \neq 0$. We put $B_+ = \{(a, b) \in B | a > 0, b > 0\}$; then B_+ is a Lie group. Let

$$B_0^{r+1} = \{z = (z^\alpha) \in B^{r+1} | \langle z, \overline{z}\rangle > 0\},$$

where

$$\langle z, \overline{z}\rangle = \sum_{\alpha=0}^{r} z^\alpha \overline{z}^\alpha.$$

We denote by $\mathfrak{gl}(B; r + 1)$ the algebra of $(r + 1) \times (r + 1)$-matrices with elements in B. Then $\mathfrak{gl}(B; r+1) = \mathfrak{gl}(\mathbf{R}; r+1) \times \mathfrak{gl}(\mathbf{R}; r+1)$. We have the Lie group

$$U(B; r + 1) = \{Z \in \mathfrak{gl}(B; r + 1) | \langle Zz, \overline{Z}\overline{z}\rangle = \langle z, \overline{z}\rangle \text{ for all } z \in B^{r+1}\}.$$

Let $P_r(B)$ be the quotient of B_0^{r+1} under the equivalence given by $(z^\alpha) = (qz^\alpha)$, $q \in B_+$. Then, if $\pi\colon B_0^{r+1} \to P_r(B)$ is the natural projection, we can identify $\pi(z)$ with the unique element $w = qz$ such that $\langle w, \overline{w}\rangle = 1$, $\langle w, w\rangle = \langle \overline{w}, \overline{w}\rangle$, where $q = (a, b) \in B_+$. Indeed, if $z = (z^\alpha) = ((u^\alpha, v^\alpha))$, we have

$$\langle w, \overline{w}\rangle = (ab\langle u, v\rangle, ab\langle u, v\rangle), \quad \langle w, w\rangle = (a^2\langle u, u\rangle, b^2\langle v, v\rangle),$$
$$\langle \overline{w}, \overline{w}\rangle = (b^2\langle v, v\rangle, a^2\langle u, u\rangle).$$

Then

$$a = \frac{\langle v, v\rangle^{1/4}}{\langle u, u\rangle^{1/4}\langle u, v\rangle^{1/2}}, \quad b = \frac{\langle u, u\rangle^{1/4}}{\langle v, v\rangle^{1/4}\langle u, v\rangle^{1/2}}.$$

Thus

$$P_r(B) \simeq \{(u, v) \in \mathbf{R}^{r+1} \times \mathbf{R}^{r+1} | \langle u, u\rangle = \langle v, v\rangle, \ \langle u, v\rangle = 1\}.$$

Since $Z(qz) = qZ(z)$ for all $Z \in U(B; r + 1)$, $z \in B_0^{r+1}$, $q \in B_+$, it is clear that the action of $U(B; r + 1)$ pass to the quotient $P_r(B)$.

4.1. PROPOSITION. *$P_r(B)$ is diffeomorphic to TS^r; therefore it is connected and if $r > 1$ it is simply connected. The group $U(B; r+1)$ acts transitively on $P_r(B)$.*

Proof. We consider the map $\varphi\colon P_r(B) \to TS^r$ given by $\varphi(u,v) = (\|u+v\|^{-1}(u+v), u-v)$. Since $\langle u,u\rangle = \langle v,v\rangle$, we have that $\langle \|u+v\|^{-1}(u+v), u-v\rangle = 0$, then $u-v$ can be considered as a vector tangent to S^r at the point $\|u+v\|^{-1}(u+v)$. It is immediate to prove that φ is a diffeomorphism. Now, let $(u,v) \in P_r(B)$; if $\{e_\alpha\}$ is the canonical basis of $\mathbf{R}^{r+1}$ and $\{\vartheta^\alpha\}$ its dual, let γ^i $(i = 1, \dots, r)$ be a linearly independent set of 1-forms such that $\gamma^i(u) = 0$. If $\gamma^i = \gamma^i_\alpha \vartheta^\alpha$, and $v = v^\alpha e_\alpha$, we define $P \in \mathrm{Gl}(r+1; \mathbf{R})$ by putting $\vartheta^0(Pe_\alpha) = v^\alpha$, $\vartheta^i(Pe_\alpha) = \gamma^i_\alpha$. Then

$$Pu = u^\alpha P e_\alpha = u^\alpha \vartheta^0(Pe_\alpha)e_0 + u^\alpha \vartheta^i(Pe_\alpha)e_i = u^\alpha v^\alpha e_0 + u^\alpha \gamma^i_\alpha e_i = e_0;$$
$${}^tPe_0 = \vartheta^\alpha({}^tPe_0)e_\alpha = \vartheta^0(Pe_\alpha)e_\alpha = v^\alpha e_\alpha = v.$$

Therefore $(P, {}^tP^{-1})(u,v) = (e_0, e_0)$ and since $(P, {}^tP^{-1}) \in U(B; r+1)$, it is clear that $U(B; r+1)$ acts transitively on $P_r(B)$. □

We consider on B_0^{r+1} the covariant tensor field $(0 \neq c \in \mathbf{R})$:

$$\tilde{g} = \frac{2}{c\langle u,v\rangle}\left\{du^\alpha \otimes dv^\alpha + dv^\alpha \otimes du^\alpha \right.$$
$$\left. - \frac{1}{\langle u,v\rangle} u^\alpha v^\beta (dv^\alpha \otimes du^\beta + du^\beta \otimes dv^\alpha)\right\}.$$

Then $\tilde{g}$ is invariant by $U(B; r+1)$ as it is easily proved. If $i\colon P_r(B) \to B_0^{r+1}$ is the inclusion, we have by direct computation that $(i \cdot \pi)^*\tilde{g} = \tilde{g}$. Hence, the tensor field $g = i^*\tilde{g}$, which is a pseudo-Riemannian metric on $P_r(B)$, is also invariant by $U(B; r+1)$. We have for $P_r(B)$ the charts $(\varphi^\alpha, U^\pm_\alpha)$, where

$$U^+_\alpha = \{(u,v)\} \in P_r(B) | u^\alpha > 0, v^\alpha > 0\},$$
$$U^-_\alpha = \{(u,v)\} \in P_r(B) | u^\alpha < 0, v^\alpha < 0\},$$

and

$$\varphi^\alpha(u,v) = \left(\frac{u^0}{u^\alpha}, \dots, \frac{\hat{u}^\alpha}{u^\alpha}, \dots, \frac{u^r}{u^\alpha}; \frac{v^0}{v^\alpha}, \dots, \frac{\hat{v}^\alpha}{v^\alpha}, \dots, \frac{v^r}{v^\alpha}\right).$$

If we call (x^i, y^i) to the coordinates of any one of these charts, say $x^i = u^i/u^0$, $y^i = v^i/v^0$, then by direct computation or well by an argument similar to the one used in [5, vol. II, p. 160], we have that

$$(10)\quad g=\frac{2}{c(1+\langle x,y\rangle)}\Big(dx^i\otimes dy^i+dy^i\otimes dx^i\\ -\frac{1}{1+\langle x,y\rangle}x^iy^j(dy^i\otimes dx^j+dx^j\otimes dy^i)\Big).$$

Also, we have on B_0^{r+1} the almost-product structure given by

$$\tilde{J}=\frac{\partial}{\partial u^\alpha}\otimes du^\alpha-\frac{\partial}{\partial v^\alpha}\otimes dv^\alpha,$$

and it defines an almost-product structure on $P_r(B)$, J, by the relation $\pi_*\circ\tilde{J}=J\circ\pi_*$, which in the same chart is given by

$$(11)\qquad J=\frac{\partial}{\partial x^i}\otimes dx^i-\frac{\partial}{\partial y^i}\otimes dy^i.$$

Then

4.2. THEOREM. *$P_r(B)$ admits a para-Kaehler structure of constant J-sectional curvature $c\neq 0$ given by* (10) *and* (11). *Then $P_r(B)$ is connected and complete, and if $r>1$, it is also simply connected.*

Proof. The 2-form $F(X,Y)=g(X,JY)$ is given by

$$F=\frac{2}{c(1+\langle x,y\rangle)}\left(dy^i\wedge dx^i-\frac{1}{1+\langle x,y\rangle}x^jdy^j\wedge y^idx^i\right).$$

Then $dF=0$. Since evidently $N=0$, we have that $P_r(B)$ is a para-Kaehler manifold. Since $\nabla J=0$, we have $\nabla_{\partial/\partial x^i}(\partial/\partial y^j)=0$. Also

$$g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^j}\right)=\frac{2}{c}\frac{\partial}{\partial x^i}\frac{x^j}{1+\langle x,y\rangle}.$$

Hence

$$g\left(\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^j},\frac{\partial}{\partial y^k}\right)=\frac{\partial}{\partial x^i}g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^k}\right)=\frac{2}{c}\frac{\partial^2}{\partial x^i\partial x^j}\frac{x^k}{1+\langle x,y\rangle}\\ =-\frac{2}{c}\left\{\frac{\delta_{ik}y^j+\delta_{jk}y^i}{(1+\langle x,y\rangle)^2}-\frac{2x^ky^iy^j}{(1+\langle x,y\rangle)^3}\right\}.$$

Therefore

$$\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^k}=-\frac{1}{1+\langle x,y\rangle}\left(y^k\frac{\partial}{\partial x^i}+y^i\frac{\partial}{\partial x^k}\right).$$

And if 0 is the point of $P_r(B)$ with coordinates $x^i = y^i = 0$, we have

$$\left(\nabla_{\partial/\partial y^j}\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^k}\right)_0 = -\left(\delta_{kj}\frac{\partial}{\partial x^i} + \delta_{ij}\frac{\partial}{\partial x^k}\right)_0.$$

Therefore

$$\begin{aligned} R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}, \ \frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^l}\right)_0 &= g\left(\nabla_{\partial/\partial y^j}\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^l}\right)_0 \\ &= -\delta_{kj} g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^l}\right)_0 - \delta_{ij} g\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^l}\right)_0 \\ &= -\frac{2}{c}(\delta_{kj}\delta_{il} + \delta_{kl}\delta_{ij}), \end{aligned}$$

$$\begin{aligned} R'\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^l}\right)_0 &= \frac{1}{c^2}(-\delta_{il}\delta_{jk} - \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}) \\ &= -\frac{2}{c^2}(\delta_{kj}\delta_{il} + \delta_{kl}\delta_{ij}). \end{aligned}$$

Hence $R = cR'$ at 0. Since R and R' are invariant by $U(B; r+1)$ we conclude that the J-sectional curvature is c, and that $(P_r(B), g)$ is complete. □

As for the problem of finding a complete, connected and simply connected para-Kaehler manifold of constant J-sectional curvature in the case $r = 1$, it is enough to extend the above structure on $P_1(B)$ up to the universal covering of $P_1(B) = S^1 \times \mathbf{R}$, which is $\mathbf{R}^2$.

We shall study the spaces $P_r(B)$ as symmetric spaces in a forthcoming paper.

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