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COMPARISON SURFACES FOR THE WILLMORE PROBLEM

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The infimum of the conformally invariant functional $W = \int H^2$ is estimated for each regular homotopy class of immersed surfaces in $\mathbf{R}^3$. Consequently, we obtain rather sharp bounds on the maximum multiplicity and branching order of a W-minimizing surface. In the case of $\mathbf{R}P^2$ we provide an example of a symmetric W-minimizing Boy's surface ($W = 12\pi$)—as well as symmetric static surfaces of higher index—thereby solving part of the Willmore problem.

0. Introduction. This paper addresses the well-known variational problem posed by T. J. Willmore [WT] in 1965:

Find the minimum for the squared-mean-curvature integral

$$W(M) = \int_M H^2\,da$$

among compact embedded surfaces $M \subset \mathbf{R}^3$ of a given genus.

Willmore noted that $W(M) \geq 4\pi$, with equality only for *round spheres*. He also found a torus $M_1 \subset \mathbf{R}^3$ with $W(M_1) = 2\pi^2$ and conjectured this value to be the minimum among embedded tori. Although Willmore's conjecture remains unresolved, at least his example serves as a *comparison surface*, showing that the infimum of W among embedded tori is not greater than $2\pi^2 < 8\pi$.

This is our starting point: for each genus g we exhibit a comparison surface $M_g \subset \mathbf{R}^3$ with $W(M_g) < 8\pi$. More generally, we consider the Willmore problem for *immersed* surfaces $M \looparrowright \mathbf{R}^3$. A path of immersed surfaces is a *regular homotopy*, and the path component—or *regular homotopy class*—of $M \looparrowright \mathbf{R}^3$ is denoted by $[M]$. We will construct (in §5) appropriate comparison surfaces to deduce the following

MAIN THEOREM. *The infimum $W_{[M]}$ for W over any regular homotopy class $[M]$ of compact immersed surfaces $M \looparrowright \mathbf{R}^3$ satisfies*

$$W_{[M]} < 20\pi,$$

with the best upper estimates known given in §5. In particular, the infimum of W among compact immersed surfaces of a given topological *type* $M \looparrowright \mathbf{R}^3$ *is strictly less than* $(<)$

$$\begin{array}{ll} 8\pi & \textit{if } M \textit{ is orientable,} \\ 12\pi & \textit{if } M \textit{ is non-orientable with } \chi(M) \textit{ even,} \end{array}$$

and

$$16\pi \quad \textit{if } M \textit{ is non-orientable with } \chi(M) \textit{ odd.}$$

(Our result should be compared with the recent work of W. Kuhnel and U. Pinkall **[KWP]**, who do not consider the regular homotopy class problem and who obtain only the weak $(\leq)$ inequality in the latter part of our main theorem. Pinkall has independently observed the strict inequality $\inf W < 8\pi$ for orientable surfaces—see **[SL]** and the appendix to **[KWP]**.)

Next we consider the regularity problem for *W-minimizing* surfaces. Already, L. Simon **[SL]** has employed our comparison surfaces $M_g \subset \mathbf{R}^3$ to obtain results for embedded surfaces minimizing W. In §6 we indicate how his machinery extends to the immersed case, and we shall apply our estimates on $W_{[M]}$ to bound the *local branching order* of a W-minimizing *branched-immersed* surface. A key idea here is the *Li-Yau inequality* **[LY]**

$$4\pi\mu(M) \leq W(M)$$

relating W to the *maximum multiplicity* μ of the surface. We derive (in §1) a sharp version of this inequality valid for branched-immersed surfaces. Our derivation employs a form of the *Gauss-Bonnet formula* for proper branched-immersed surfaces. In §7 we reinterpret this as a *Riemann-Hurwitz formula* and compute a bound on the *total branching order*, using a method introduced by R. Bryant **[Br1]**.

Some further comment on the construction of our comparison surfaces is due. First, an important property of W is its *conformal invariance* (§1). This led us to two sources of surfaces for which W is computable: complete minimal surfaces in $\mathbf{R}^3$ (§2) and compact minimal surfaces in $\mathbf{S}^3$ (§3). Conformal geometry also suggests—using the enumeration of regular homotopy classes (§4) as a guide—the most efficient way to weld these basic surfaces together and obtain the required comparison surfaces (§5).

Second, many of our comparison surfaces are *critical points* for W, and at least one—the *Boy's surface* $P_3 \looparrowright \mathbf{R}^3$ with $W = 12\pi$—achieves the minimum among immersed projective planes! (See §2.) We conjecture that the embedded surface $M_g \subset \mathbf{R}^3$ also achieves the minimum for the original Willmore problem. (The case $g = 1$ is Willmore's conjecture.) A small measure of evidence for this conjecture is provided in the final section (§8) of this paper.

It remains a pleasant task to thank the many people who made important contributions: Robert Bryant and Leon Simon, for introducing the author to this problem; Rick Schoen and S.-T. Yau, for their encouragement and interest; Geoff Mess and Gang Tian, for discussions on regular homotopy classes and line-bundles, respectively; Wu-Yi Hsiang, for inviting the author to lecture on this work at Berkeley in March 1985; Tiny, for her patient support, for postponing physics and supercomputers, and for our son, Woden; and Woden, for helping to get "Daddy's paper" finished, even though he was "only an idea" when the first draft was written (1984). The author also wishes to thank the University of California, Berkeley and San Diego for their financial support and hospitality during the preparation of this paper.

A revised version of this paper appears as a chapter in the author's doctoral dissertation **[KR2]** at the University of California, Berkeley. Some results of this paper have been announced in **[KR1]**. We should also mention the very recent survey article **[PS]** which provides a pictorial introduction to this subject, and the author's article **[KR3]**.

1. Branched-immersed surfaces, conformal invariance, and the Li-Yau inequality. A surface embedded in $\mathbf{R}^3$ can be represented in several (equivalent) ways: as a *subset* of $\mathbf{R}^3$ which is locally the graph of a function $\mathbf{R}^2 \to \mathbf{R}$; as the *image of an embedding*; or, as an *equivalence class of embeddings modulo reparametrizations*. This latter viewpoint reflects the fact that geometric quantities—such as the functional $W = \int H^2\, da$—are independent of parametrization; and it is the easiest to generalize for our purposes.

DEFINITION 1.1. Let Σ be an abstract smooth surface and B a (necessarily) discrete set in Σ. A branched-immersed surface $M \not\ell\, \mathbf{R}^3$ is an equivalence class of branched immersions $f\colon \Sigma \to \mathbf{R}^3$ (with branch locus B), where $f \sim g$ provided there is a diffeomorphism $F\colon \Sigma \to \Sigma$ (preserving B) such that $f = g \circ F$.

We call f a *representative*, and F a *reparametrization* of M, and we use the self-evident terminology: M is *immersed* (if B is empty), M is *embedded* (if any f is an embedding), M is *compact* (if Σ is compact), and so forth.

Note that—unlike embedded surfaces—distinct branched-immersed surfaces may have the same image. However, a $C^{1,\alpha}$ branched-immersed surface can still be represented locally as a union of (multiple) $C^{1,\alpha}$ graphs away from the (image of the) branch locus. Near a branch point $y \in \Sigma$ we may choose a representative

$$f(r,\theta) = (r\cos m\theta, r\sin m\theta, h(r,\theta))$$

where $m = m(y)$ is a positive integer, $h \in C^{1,\alpha}$ with $|h| \le Cr^{1+\alpha}$, $|Dh| \le cr^{\alpha}$, and where (r,θ) is a polar coordinate centered at y. (Cf. **[GOR]**, **[SL]**.)

We introduce

$$\mu(x) = \sum_{y \in f^{-1}(x)} m(y)$$

for the *multiplicity* of M at $x \in \mathbf{R}^3$. Define the *local branching order*

$$\beta(x) = \sum_{y \in f^{-1}(x)} (m(y) - 1)$$

and observe $\beta(x) \neq 0$ iff x is in the image of the branch locus. Also set

$$\mu(M) = \max_{x \in M} \mu(x)$$

and define the *total branching order* by

$$\beta(M) = \sum_{x \in M} \beta(x).$$

Henceforth we adopt the abbreviation *surface*, and note that the concept (and notation) extends naturally to any ambient manifold. In particular, it will be convenient to work also in $\mathbf{S}^3$, which we regard as $\mathbf{R}^3 \cup \infty$ via stereographic projection.

If $\dot{M} \not\subset \mathbf{R}^3$ denotes the proper surface obtained from a compact surface $M \not\subset \mathbf{S}^3$ via stereographic projection, we reserve the notation

$$\eta(\dot{M}) = \mu(\infty)$$

for the *multiplicity of* $\dot{M}$ at ∞. Notice that the *number of ends* of $\dot{M}$ (that is, the difference in *Euler numbers* $\chi(M) - \chi(\dot{M})$) equals $\eta(\dot{M})$ iff each end of $\dot{M}$ is embedded iff M is unbranched at ∞. In general, we have the relation

$$\chi(\dot{M}) + \eta(\dot{M}) + \beta(\dot{M}) = \chi(M) + \beta(M).$$

We are now prepared to state a useful version of the *Gauss-Bonnet formula*, and apply it to study the functional $W = \int H^2\,da$. (The derivation is quite simple, but we postpone it to §7, where we reinterpret it as a *Riemann-Hurwitz formula* for branched-immersed surfaces.)

LEMMA 1.2. *Let $\dot{M}$ be a proper surface obtained from a compact $C^{1,\alpha}$ surface $M \looparrowright \mathbf{S}^3$ via stereographic projection. Then the curvature $\dot{K}$ of $\dot{M} \looparrowright \mathbf{R}^3$ satisfies*

$$\int_{\dot{M}} \dot{K}\, \dot{da} = 2\pi(\chi(\dot{M}) - \eta(\dot{M}) + \beta(\dot{M})).$$

This formula was known for a complete minimal surface of finite total curvature in $\mathbf{R}^3$: in fact, such a surface always conformally compactifies to a $C^{1,\alpha}$ surface in $\mathbf{S}^3$ **[KR1]**, **[KR2]**. Of course, for a compact surface $M \looparrowright \mathbf{R}^3$ we have

$$\int_M K\, da = 2\pi(\chi(M) + \beta(M)).$$

Using the *Gauss equation* one obtains a similar formula for the *extrinsic curvature* $\overline{K}$ of a compact surface $\overline{M} \looparrowright \mathbf{S}^3$:

$$\int_{\overline{M}} (1 + \overline{K})\, \overline{da} = 2\pi(\chi(\overline{M}) + \beta(\overline{M})).$$

Recall that the basic conformal invariant of a submanifold **[TG]**, **[CBY]** is the *traceless-second-fundamental-form*

$$\mathring{A} = (h^\circ_{ij}) = (h_{ij} - H g_{ij}).$$

On a surface $M \looparrowright \mathbf{N}^3$ this yields the *conformally invariant density*

$$(H^2 - K)\, da = \tfrac{1}{4}(\kappa_1 - \kappa_2)^2\, da = \tfrac{1}{2}|\mathring{A}|^2\, da.$$

Thus, if $\mathbf{N} \to \dot{\mathbf{N}}$ is a conformal map carrying $M \looparrowright \mathbf{N}$ to $\dot{M} \looparrowright \dot{\mathbf{N}}$, then

$$\int_M (H^2 - K)\, da = \int_{\dot{M}} (\dot{H}^2 - \dot{K})\, \dot{da}.$$

In particular, if $M \looparrowright \mathbf{R}^3$ is a compact surface, and if there is a Möbius transformation carrying M to another compact surface $\dot{M} \looparrowright \mathbf{R}^3$, then Gauss-Bonnet implies

$$W(M) = \int_M H^2\, da = \int_{\dot{M}} \dot{H}^2\, \dot{da} = W(\dot{M}),$$

and in this sense W is itself a conformal invariant. In fact, this is a special case of a formula which holds when a Möbius transformation carries a compact surface M to a *proper* surface $\dot{M} \looparrowright \mathbf{R}^3$; we compute

$$\begin{aligned}
W(M) &= \int_M (H^2 - K)\, da + \int_M K\, da \\
&= \int_{\dot{M}} (\dot{H}^2 - \dot{K})\, \dot{d}a + 2\pi(\chi(M) + \beta(M)) \\
&= \int_{\dot{M}} \dot{H}^2\, \dot{d}a - 2\pi(\chi(\dot{M}) + \beta(\dot{M}) - \eta(\dot{M})) \\
&\quad + 2\pi(\chi(\dot{M}) + \beta(\dot{M}) + \eta(\dot{M})) \\
&= W(\dot{M}) + 4\pi\eta(\dot{M}).
\end{aligned}$$

Here we have used our Gauss-Bonnet formulas (and the relation preceding them) to handle the Gauss curvature terms. We can now derive a sharp version of the *Li-Yau inequality* **[LY]**:

PROPOSITION 1.3. *Let $M \looparrowright \mathbf{R}^3$ be a compact surface with maximum multiplicity $\mu(M)$. Then*

$$W(M) = \int_M H^2\, da \geq 4\pi\mu(M).$$

Equality holds iff there exists a complete minimal surface $\dot{M} \looparrowright \mathbf{R}^3$ of finite total curvature with $\eta(\dot{M}) = \mu(M)$, and a Möbius transformation carrying M to $\dot{M}$.

Proof. Let x be a point of maximum multiplicity for M. Then the proper surface $\dot{M}$ obtained from M by a Möbius transformation carrying x to ∞ satisfies the previous formula with $\eta(\dot{M}) = \mu(M)$. Obviously $\dot{M}$ is complete, with finite total curvature, and $W(\dot{M}) \geq 0$ with equality iff $\dot{M}$ is minimal. □

We apply this result in §6 to estimate the maximum multiplicity of a W-minimizing surface. To this end, we need to exhibit comparison surfaces for which W is computable. The preceding discussion suggests that complete minimal surfaces in $\mathbf{R}^3$ provide a natural source:

Fact 1.4. The compact immersed surface $M \looparrowright \mathbf{R}^3$ obtained (via a Möbius transformation) from a complete immersed minimal surface $\dot{M} \looparrowright \mathbf{R}^3$, with finite total curvature and p (separately) embedded ends, satisfies

$$W(M) = 4\pi p.$$

Examples and further discussion will be given in the next section.

Another source of comparison surfaces is compact minimal surfaces in $\mathbf{S}^3$. Indeed, if $\overline{M} \nearrow \mathbf{S}^3$ is a compact surface (which avoids ∞) and $M \nearrow \mathbf{R}^3$ is the compact surface obtained by stereographic projection, then a computation similar to the one above (using now the Gauss-Bonnet formula for $\int_{\overline{M}}(1+\overline{K})\overline{da}$) yields

$$W(\overline{M}) = \int_{\overline{M}} (1+\bar{H}^2)\,\overline{da},$$

from which we deduce

Fact 1.5. The compact surface $M \nearrow \mathbf{R}^3$ obtained (by stereographic projection) from a compact minimal surface $\overline{M} \nearrow \mathbf{S}^3$ satisfies

$$W(M) = \operatorname{area}(\overline{M}).$$

The areas of Lawson's **[LHB]** minimal surfaces in $\mathbf{S}^3$ will be estimated in §4.

Finally, we remark that Proposition 1.3 can be used to show that ∞ *is the point of maximum multiplicity for a complete minimal surface in* $\mathbf{R}^3$ **[KR1]**, **[KR2]**, a fact we will use in §7.

2. Complete minimal surfaces in $\mathbf{R}^3$. In this section we describe a new family of complete minimal surfaces of finite total curvature (which were announced in **[KR1]**), and we indicate their relationship to the Willmore problem.

THEOREM 2.1. *For each odd $p \geq 3$ there is a complete immersed minimal surface $M_p \nearrow \mathbf{R}$ with the following properties*:

(i) *M_p has p ends, each of which is embedded and flat.*

(ii) *M_p is non-orientable. The conformal compactification $\overline{M}_p \nearrow \mathbf{S}^3$ is a $C^{2,\alpha}$ (in fact, real algebraic) immersed real projective plane $\mathbf{R}P^2$.*

(iii) *The total curvature of M_p equals $-2\pi(2p-1)$.*

(iv) *There is a flat plane $\mathbf{R}^2 \subset \mathbf{R}^3$ such that $M_p \cap \mathbf{R}^2$ contains p straight lines meeting at equal angles. The dihedral group of order $2p$ acts on M_p by reflections around these lines.*

Proof. We use the classical *Weierstrass representation*, suitably modified for the non-orientable setting **[SM]**, **[MW1]**. Let z denote a meromorphic coordinate on $\mathbf{S}^2$, the orientation-double-covering-space of $\mathbf{R}P^2$. Then the Weierstrass representatives are

$$g_p(z) = z^{p-1}(z^p - s)/(sz^p + 1)$$

and

$$f_p(z) = i(sz^p + 1)^2/(z^{2p} + rz^p - 1)^2$$

where $s = \sqrt{2p-1}$ and $r = 2s/(p-1)$. These values are chosen so that the Weierstrass 1-forms

$$\varphi = f(1 - g^2, i(1 + g^2), 2g)\,dz$$

are exact. It follows (after a long, but straightforward computation) that M_p is represented by the conformal harmonic map Re Φ_p where

$$\begin{aligned}\Phi_p(z) &= \int^z \varphi_p \\ &= \frac{i}{z^{2p} + rz^p - 1}\left(z^{2p-1} - z, -i(z^{2p-1} + z), \frac{p-1}{p}(z^{2p} + 1)\right).\end{aligned}$$

In terms of the coordinate z, the antipodal map $*\colon \mathbf{S}^2 \to \mathbf{S}^2$ is given by

$$z \to z* = -1/z,$$

and one readily checks that

$$\Phi_p(z*) = \overline{\Phi_p(z)}$$

so that Re Φ_p is well defined on $\mathbf{R}P^2$; equivalently (see **[SM]**, **[MW1]**), one checks

$$g(z*) = (g(z))*$$

and

$$f(z*) = -\overline{(zg(z))^2 f(z)}.$$

Properties (i), (ii), and (iii) can now be verified.

To establish property (iv) use the identities

$$\Phi_p(\bar{z}) = \Phi_p(z)A, \qquad \Phi_p(e^{2\pi i/p}z) = \Phi_p(z*)B^2$$

where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} \cos\pi/p & -\sin\pi/p & 0 \\ \sin\pi/p & \cos\pi/p & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

We write P_p for the compact surface in $\mathbf{R}^3$ gotten from $\overline{M}_p$ by a Möbius transformation. By Fact 1.4 we see

$$W(P_p) = 4\pi p.$$

Now every immersed $\mathbf{R}P^2 \looparrowright \mathbf{R}^3$ must have a triple-point, so by Proposition 1.3 we have the immediate

COROLLARY 2.2 [**KR1**]. *The surface $P_3 \looparrowright \mathbf{R}^3$ minimizes W among all immersed real projective planes, with value*

$$W(P_3) = 12\pi.$$

We remark that the other surfaces P_p are *static surfaces*—also known as "Willmore surfaces"—that is, they are critical points for W; in fact, P_p minimizes W among competing surfaces with the same symmetry group (see below). It is also interesting to note [**KR1**], [**KR2**] that for *even* values of p the corresponding formula for Φ_p gives rise to a static sphere $S_p \looparrowright \mathbf{R}^3$ with conformal dihedral symmetry and with

$$W(S_p) = 8\pi p.$$

The surface S_2 has a quadruple-point and represents the midpoint of an *eversion* of the sphere.

In a sense this entire procedure can be reversed. By choosing the Möbius transformation carefully we may assume P_p has p-fold rotation symmetry. Now any immersed $\mathbf{R}P^2 \looparrowright \mathbf{R}^3$ with this symmetry must have a point of multiplicity greater than p [**FG**], and so by Proposition 1.3, P_p minimizes W among all such surfaces. But R. Bryant [**BR1**] has shown that any static sphere arises from a complete minimal surface in the manner described. By double-covering, the same is true for static real projective planes, and we are led to a complete minimal surface $M_p \looparrowright \mathbf{R}^3$ with the indicated total curvature, asymptotics, and symmetries. This information is then used to *derive* the Weierstrass representatives. An additional consequence [**KR1**], [**KR2**]: *P_p is the unique surface possessing conformal dihedral symmetry* (*of order* $2p$) *in the moduli space Π_p of immersed static $\mathbf{R}P^2 \looparrowright \mathbf{R}^3$* (*modulo Möbius transformations*) *with $W = 4\pi p$.*

One can show (using, for example, the Weierstrass representation) [**KR1**], [**KR2**] that Π_p is a noncompact complex variety of dimension $p-2$. By a different method, Bryant [**BR2**] has explicitly computed Π_3 to be a *closed half-plane* $\subset \mathbf{C}$; he also observed that Meeks' minimal Möbius strip [**MW1**] provides a natural compactification of Π_3 to a *closed disk.*

A point we want to emphasize is this: the compact surface obtained from Meeks' minimal Möbius strip is also a W-minimizing $\mathbf{R}P^2$ with $W = 12\pi$, but with a single branch point of order $\beta = 2$. Thus branch points are an essential feature of the general Willmore problem.

3. Compact minimal surfaces in $\mathbf{S}^3$. In 1970 Lawson [LHB] constructed three families of minimal surfaces—denoted ξ, τ, and η—thereby showing that every compact surface (save $\mathbf{R}P^2$, which is prohibited) can be minimally immersed in $\mathbf{S}^3$. The areas of these surfaces will be estimated in this section.

Recall the construction [LHB] of the minimal surface $\xi_{m,k} \subset \mathbf{S}^3$. Fix integers $m, k \geq 0$ and let

$$A_p = (e^{ip\pi/k+1}, 0), \qquad B_q(0, e^{iq\pi/m+1})$$

in $\mathbf{S}^3$ (which we regard as the unit sphere in $\mathbf{C}^2 = \mathbf{R}^4$). These are the endpoints of a (unique) shortest geodesic arc A_pB_q in $\mathbf{S}^3$. The geodesic polygon

$$\Gamma_{m,k} = A_0B_0 \cup B_0A_1 \cup A_1B_1 \cup B_1A_0$$

bounds an area minimizing disk $\delta_{m,k} \subset \mathbf{S}^3$. Repeated reflection of $\delta_{m,k}$ around the geodesics $\{A_pB_q | 0 \leq p \leq k,\ 0 \leq q \leq m\}$ produces the compact surface $\xi_{m,k}$.

Now it is an elementary matter to check that $\xi_{m,k}$ is the union of $2(k+1)(m+1)$ disks, each congruent to $\delta_{m,k}$, and the genus $(\xi_{m,k}) = mk$. Furthermore, $\xi_{m,0} \subset \mathbf{S}^3$ is an equator, and $\xi_{m,k}$ is congruent to $\xi_{k,m}$, so we may assume (without loss of generality) that $m \geq k \geq 1$.

LEMMA 3.1. *We have the strict inequality*

$$\operatorname{area}(\delta_{m,k}) < 2\pi/m + 1.$$

Proof. We compare $\delta_{m,k}$ with the pair of totally geodesic simplices

$$\gamma_{m,k} = A_0B_0B_1 \cup A_1B_1B_0.$$

Since $\delta_{m,k}$ is area-minimizing, since the angle $\gamma_{m,k}$ makes along the geodesic arc B_0B_1 is $\pi/k + 1 < \pi$, and since

$$\partial(\delta_{m,k}) = \Gamma_{m,k} = \partial(\gamma_{m,k}),$$

we conclude that

$$\operatorname{area}(\delta_{m,k}) < \operatorname{area}(\gamma_{m,k}).$$

However,

$$\begin{aligned}\operatorname{area}(\gamma_{m,k}) &= \operatorname{area}(\text{domain on } \mathbf{S}^2 \text{ between longitudes } 0 \text{ and } \pi/m+1) \\ &= \operatorname{area}(\mathbf{S}^2)/2(m+1) = 2\pi/m + 1\end{aligned}$$

which proves the lemma. □

Combining Lemma 3.1 with the preceding observations gives the following

PROPOSITION 3.2. *Let $\xi_{m,k} \subset \mathbf{S}^3$ be Lawson's genus mk minimal surface* $(m \geq k \geq 1)$. *Then*

$$\operatorname{area}(\xi_{m,k}) < 4\pi(k+1).$$

We remark that this estimate is sharp in the following sense: if we fix k, then as $m \to \infty$,

$$\operatorname{area}(\xi_{m,k}) \to 4\pi(k+1).$$

(The rate of convergence can also be estimated **[KR2]**, **[KR3]**.)

Next we estimate the areas of the second family of surfaces. The minimal surface $\tau_{m,k}$ may be constructed in the same way as $\xi_{m,k}$, replacing $\Gamma_{m,k}$ with

$$A_0B_0 \cup B_0B_1 \cup B_1A_1 \cup A_1A_0.$$

However, its area can be estimated more efficiently if we note (as Lawson did **[LHB]**) that $\tau_{m,k}$ is ruled, and so can be parametrized by the doubly-periodic mapping

$$\psi_{m,k}\colon [0,2\pi] \times [0,\pi] \subset \mathbf{R}^2 \to \mathbf{S}^3 \subset \mathbf{C}^2$$

given by the formula

$$\psi_{m,k}(x,y) = (e^{imx}\cos y, e^{ikx}\sin y).$$

PROPOSITION 3.3. *Let $\tau_{m,k}$ be Lawson's compact ruled minimal surface in $\mathbf{S}^3$* $(m \geq k \geq 1)$. *Then*

$$\pi^2(m+k) \leq \operatorname{area}(\tau_{m,k}) < 4\pi(m+k).$$

Equality holds (*on the left*) *if and only if $m = k$.*

Proof. Observe **[LHB]** that $\operatorname{area}(\tau_{m,k})$ is exactly π times the elliptic integral

$$I_{m,k} = \int_0^{2\pi} \sqrt{m^2\cos^2 t + k^2\sin^2 t}\, dt.$$

Clearly $I_{m,k} < 4(m+k)$, since for $0 < a, b$ we have $\sqrt{(a^2+b^2)} < a+b$, and since

$$\int_0^{2\pi} |\sin t|\, dt = \int_0^{2\pi} |\cos t|\, dt = 4.$$

To show $I_{m,k} \geq \pi(m+k)$ notice that *equality* holds when $m = k$, and that the (strict) inequality holds for $m > k$ because (for fixed k)

$$\frac{\partial}{\partial m} I_{m,k} = \int_0^{2\pi} m\cos^2 t / \sqrt{m^2\cos^2 t + k^2 \sin^2 t}\, dt$$
$$> \int_0^{2\pi} \cos^2 t\, dt = \pi. \qquad \square$$

We should point out that $\tau_{1,1} = \xi_{1,1}$ is the familiar *Clifford minimal torus* with area $2\pi^2$. The surface $\tau_{2,1}$ is a *minimal Klein bottle* with area strictly less than 12π. (In fact, numerical integration yields an area between 9π and 10π.)

Finally, we compute the area of the Lawson surface $\eta_{m,k}$ in the same way as we did for $\xi_{m,k}$. For k odd (which we will assume) $\eta_{m,k}$ is a non-orientable surface with Euler number $1 - mk$, and it is a union of $2(m+1)(k+1)$ disks each congruent to the unique area-minimizing disk spanning

$$(-B_1)A_0B_1 \cup B_1A_1 \cup A_1B_0 \cup B_0(-B_1).$$

Again assuming $m \geq k \geq 1$, and using a totally geodesic comparison, deduce

PROPOSITION 3.4. *Lawson's non-orientable minimal surface $\eta_{m,k}$ with Euler number $1 - mk$ satisfies*

$$\operatorname{area}(\eta_{m,k}) < 2\pi(m+2)(k+1).$$

We note that $\eta_{1,1} = \tau_{2,1}$ is Lawson's Klein bottle. In §5 we will make use of surfaces from all three of Lawson's families.

Not only is the value of W easily computable for the stereographic projection of a compact minimal surface in $\mathbf{S}^3$ (by Fact 1.5), but also we find (by applying a *first variation* argument to Fact 1.5—or see **[TG]**, **[BR1]**) that the projected surface in $\mathbf{R}^3$ is indeed a critical point for W—a static surface! We will comment on this further in §8.

4. Enumerating the regular homotopy classes of immersed surfaces. The classification of immersed surfaces in $\mathbf{R}^3$ up to regular homotopy —part of the "folklore" of low-dimensional topology—has recently been completed **[HH]**, **[PU]**. Here we organize the results into a convenient format. This section is essentially expository: the reader may refer to **[HH]**, **[HJ]**, and **[PU]** for background.

Two immersed surfaces are said to be *regularly homotopic* provided they have regularly homotopic representatives. This is easily seen to be an equivalence relation. We write $[M]$ for the *regular homotopy class* of the immersed surface $M \looparrowright \mathbf{R}^3$. It may be useful to think of $[M]$ as the path component of M in the space of immersed surfaces.

We proceed to state the basic facts about regular homotopy classes of immersed surfaces in $\mathbf{R}^3$.

Fact 4.1. Given immersed surfaces $M, N \looparrowright \mathbf{R}^3$, there is a well-defined connected-sum operation which produces an immersed surface $M\#N \looparrowright \mathbf{R}^3$.

We give one geometric construction of this operation (the "conformal connected-sum") in the next section. The important point is that—up to regular homotopy—connected-sum is unique, and so we may write $[M] + [N]$ for the class $[M\#N]$. In this way the set of all regular homotopy classes of compact surfaces in $\mathbf{R}^3$ becomes an *abelian semigroup* $\mathscr{S}$. The (unique) regular homotopy class of the sphere provides the *zero* element $\mathbb{O}$ for $\mathscr{S}$.

Fact 4.2. $\mathscr{S}$ has four generators (in the notation of [**PU**]):

$$\mathbb{S} = [\text{a standard torus}], \quad \mathbb{B} = [\text{a (right-handed) Boy's surface}],$$
$$\mathbb{T} = [\text{a twisted torus}], \quad \overline{\mathbb{B}} = [\text{a (left-handed) Boy's surface}].$$

Examples of each are provided in the next section. As the preceding notation suggests, we will write $[\overline{M}]$ for the class of the *mirror image* $M^- \looparrowright \mathbf{R}^3$ of the surface M.

Fact 4.3. There is a function ϑ from $\mathscr{S}$ to the real numbers modulo 2π. This function is additive

$$\vartheta_{[M]+[N]} = \vartheta_{[M]} + \vartheta_{[N]}$$

and involutive

$$\vartheta_{[\overline{M}]} = -\vartheta_{[M]}.$$

We call ϑ the *twist* since (for M in general position) $\vartheta_{[M]}$ can be viewed geometrically as (half) the angle in the rotation group SO(3) through which the *double-point locus* of M twists [**HH**]. (An algebraic interpretation is that $e^{i\vartheta_{[M]}}$ is the multiplicative *Arf invariant* of a certain inner product space associated to M [**PU**].)

We come now to the main fact, which was first published in [**PU**]:

Fact 4.4. Immersed surfaces $M, N \looparrowright \mathbf{R}^3$ are regularly homotopic if and only if

(i) their domains are diffeomorphic, and
(ii) they have the same twist $\vartheta_{[M]} = \vartheta_{[N]}$.

TABLE 4.6
(The "geometrically distinct" regular homotopy classes.)

$\mathbb{O}$	$\mathbb{S}$	$2\mathbb{S}$	$3\mathbb{S}$	$4\mathbb{S}$	$5\mathbb{S}$
$\mathbb{T}$	$\mathbb{T}+\mathbb{S}$	$\mathbb{T}+2\mathbb{S}$	$\mathbb{T}+3\mathbb{S}$	$\mathbb{T}+4\mathbb{S}$	$\mathbb{T}+5\mathbb{S}$

		$\iddots$			$\iddots$
	$\mathbb{K}_0+\mathbb{S}$	$\mathbb{B}+2\mathbb{S}$	$\mathbb{K}+2\mathbb{S}$	$\mathbb{B}+\mathbb{K}+2\mathbb{S}$	$\iddots$
$\mathbb{K}_0$	$\mathbb{B}+\mathbb{S}$	$\mathbb{K}+\mathbb{S}$	$\mathbb{B}+\mathbb{K}+\mathbb{S}$	$2\mathbb{K}+\mathbb{S}$	
$\mathbb{B}$	$\mathbb{K}$	$\mathbb{B}+\mathbb{K}$	$2\mathbb{K}$		

It is interesting to note that if one drops condition (i), then Fact 4.4 is modified to read

$$\text{“}\ldots M, N \not\hookrightarrow \mathbf{R}^3 \text{ are cobordant } \ldots\text{”} \quad \text{[HJ] [WR]}.$$

Since the surfaces M and M^- contain the same geometric information (save chirality) we will ignore the distinction between the classes $[M]$ and $[\overline{M}]$, and can therefore use the *real Arf invariant* $\cos \vartheta_{[M]}$ to enumerate the regular homotopy classes of "geometrically distinct" immersed surfaces in $\mathbf{R}^3$. This is an elementary matter if we decompose

$$[M] = m\mathbb{S} + n\mathbb{T} + p\mathbb{B} + q\overline{\mathbb{B}}$$

and use the following

Fact 4.5. The twist on the generating classes is (modulo 2π)

$$\vartheta_{\mathbb{S}} = 0, \qquad \vartheta_{\mathbb{B}} = \pi/4,$$
$$\vartheta_{\mathbb{T}} = \pi, \qquad \vartheta_{\overline{\mathbb{B}}} = -\pi/4.$$

Here we have used the notations

$$\mathbb{K} = 2\mathbb{B} \quad \text{and} \quad \mathbb{K}_0 = \mathbb{B} + \overline{\mathbb{B}}$$

for the right-handed and amphichiral Klein-bottle classes, respectively. (The latter is the familiar version with a plane of reflection symmetry.) We have also used the identity

$$\mathbb{B} + \mathbb{K}_0 = \mathbb{B} + \mathbb{S};$$

another important identity is

$$\mathbb{B} + \mathbb{K} = \overline{\mathbb{B}} + \mathbb{T}.$$

Both follow from Fact 4.4 and the fact that (abstract) connected-sum of three copies of $\mathbf{R}P^2$ is diffeomorphic to the sum of $\mathbf{R}P^2$ with a torus. In the next section we use the identity

$$2\mathbb{K} = \mathbb{K}_0 + \mathbb{T}$$

which follows from the previous one by adding $\mathbb{B}$ to both sides.

Finally, we note that the numbers p and q count (*modulo* 4) the right-handed and left-handed Möbius strips in the immersed surface, provided we use a decomposition with $m = n = 0$.

5. Constructing comparison surfaces. Let $W_{[M]}$ denote the infimum of W over the regular homotopy class $[M]$ of a compact immersed surface $M \looparrowright \mathbf{R}^3$. It is a simple matter to estimate $W_{[M]}$ from below, using Proposition 1.3 and well-known facts about multiple points of immersed surfaces **[BT]**, **[HJ]**. (See Table 5.14). To estimate $W_{[M]}$ from *above*, we construct a global comparison surface—for which W is computable—in the class $[M]$.

We begin with a technical result which will permit us to "weld" comparison surfaces together.

PROPOSITION 5.1. *Let $M \subset \mathbf{R}^3$ be a compact embedded surface and let $S \subset \mathbf{R}^3$ be a round sphere (or plane). Let*

$$X = \{x \in M \cap S \mid T_x M = T_x S\}$$

where T_x denotes the tangent plane at x. Then for any $\varepsilon > 0$ there is a neighborhood U of X in S, and a $C^{1,\alpha}$-close surface $N \subset \mathbf{R}^3$ such that

$$W(N) \le W(M) + \varepsilon$$

and with

$$U \subset N \cap S.$$

In particular, U is contained in the umbilic locus of N.

Proof. There is a neighborhood Y of X in S so that (locally) M is represented as a graph of a function $y \colon Y \to \mathbf{R}$ with $y = |\nabla y| = 0$ on X. Let $\varepsilon > 0$. Then a simple mollifier argument (see, for example, **[GT]**, Chapter 7) shows that there are neighborhoods $U \Subset V \Subset Y$ of X and a function $u \colon Y \to \mathbf{R}$ with $u = 0$ on U, $u - y = 0$ on $Y \setminus V$, and

$$|u - y|_{C^{1,\alpha}} + |u - y|_{W^{2,2}} \leq \varepsilon.$$

The approximating surface N is given by the graph of u. □

COROLLARY 5.2. *Any immersed surface can be approximated* (*in the above sense*) *by one which is umbilic in a neighborhood of a point.*

Proof. Let $x \in M$. By an initial approximation we may assume that a neighborhood of x is embedded. Now apply the proof of Proposition 5.1 to $S = T_x M$. □

We use this to give an explicit construction of the connected-sum of two immersed surfaces $M, N \looparrowright \mathbf{R}^3$. As above, we may assume $x \in M$ has an embedded, umbilic neighborhood. Let M be the proper surface obtained from M by a Möbius transformation x to ∞. Then M has one end ($\eta(\dot{M}) = 1$) and this end (outside a large ball) is planar! $\dot{N}$ is obtained similarly. We weld together the planar ends of $\dot{M}$ and $\dot{N}$ in the obvious smooth way, so the resulting surface $\dot{M}\#\dot{N}$ also has one, planar end. By another Möbius transformation we recover the compact surface $M\#N \looparrowright \mathbf{R}^3$, a *conformal connected-sum* of M and N.

If we observe that $W(\dot{M}\#\dot{N}) = W(\dot{M}) + W(\dot{N})$ and apply the formula preceding Proposition 1.3 (where $\eta = 1$) we find

$$W(M\#N) = W(M) + W(N) - 4\pi.$$

We can now derive an important consequence of this discussion.

PROPOSITION 5.3.

$$W_{[M\#N]} \leq W_{[M]} + W_{[N]} - 4\pi.$$

Proof. For any $\varepsilon > 0$ we may choose M, N such that

$$W(M) \leq W_{[M]} + \varepsilon/2, \qquad W(M) \leq W_{[N]} + \varepsilon/2,$$

and we may assume both have umbilic neighborhoods at the connected-sum (by Proposition 5.1). So the preceding remarks imply

$$\begin{aligned} W_{[M\#N]} &\leq W(M\#N) = W(M) + W(M) - 4\pi \\ &\leq W_{[M]} + W_{[N]} - 4\pi + \varepsilon. \end{aligned} \qquad \square$$

Next we consider the basic surfaces to be welded together. Recall from §3 the compact minimal surface $\xi_{g,1} \subset \mathbf{S}^3$. Let $M_g \subset \mathbf{R}^3$ denote its compact stereographic image. Clearly M_1 is a standard torus and, in fact, $[M_g] = g\mathbb{S}$. By Fact 1.5 and Proposition 3.2 we deduce immediately

PROPOSITION 5.4.

$$W_{\mathbb{O}} = 4\pi \quad \textit{and} \quad W_{g\mathbb{S}} \leq W(M_g) = \operatorname{area}(\xi_{g,1}) < 8\pi.$$

The minimal surface $\tau_{2,2} \looparrowright \mathbf{S}^3$ is a diagonally double-covered Clifford torus. Let $T \looparrowright \mathbf{R}^3$ denote the compact surface obtained via stereographic projection. Then T can be perturbed to a general position immersed torus whose double-locus corresponds to a diagonal (or "1, 1") curve on a standard torus. The torus makes a full twist around this curve, so $[T] = \mathbb{T}$. Therefore

PROPOSITION 5.5.

$$W_{\mathbb{T}} \leq W(T) = 2\operatorname{area}(\tau_{1,1}) = 4\pi^2.$$

Comparison surfaces for the remaining orientable regular homotopy classes $\mathbb{T} + g\mathbb{S}$ can be represented by the conformal connected-sum $T\#M_g$. However, for *even* values of g, this is not the most efficient procedure. Instead, for $g = 2(k-1)$, one can diagonally double-cover the surface M_k (in complete analogy to the way T covers M_1). We conclude

PROPOSITION 5.6.

$$W_{\mathbb{T}+g\mathbb{S}} \leq W(T) + W(M_g) - 4\pi = 4\pi^2 + \operatorname{area}(\xi_{g,1}) - 4\pi < 4\pi(1+\pi);$$

and for odd genus,

$$W_{\mathbb{T}+2(k-1)\mathbb{S}} \leq 2W(M_k) = 2\operatorname{area}(\xi_{k,1}) < 16\pi.$$

(*Of course,* $W_{\mathbb{T}+\mathbb{S}} \leq 6\pi^2 - 4\pi < 16\pi$; *a numerical estimate of* $\operatorname{area}(\xi_{3,1})$ *shows also that* $W_{\mathbb{T}+3\mathbb{S}} < 16\pi$.)

In the non-orientable case we have the W-minimizing Boy's surface P_3 constructed in §2. The family with twist $\vartheta = \pi/4$ is obtained by taking conformal connected-sum of P_3 with M_g; these surfaces satisfy

PROPOSITION 5.7.
$$W_{\mathbb{B}} = W(P_3) = 12\pi$$
and
$$W_{\mathbb{B}+g\mathbb{S}} \leq W(P_3) + W(M_g) - 4\pi = 8\pi + \operatorname{area}(\xi_{g,1}) < 16\pi.$$

Now we need a little topological

LEMMA 5.8. *The minimal surface $\eta_{m,1} \not\hookrightarrow \mathbf{S}^3$ contains $m+1$ (disjoint) Möbius strips, all with the same chirality. Therefore, the surface $N_p \not\hookrightarrow \mathbf{R}^3$ arising from $\eta_{p-1,1}$ (via a stereographic projection) represents the class $p\mathbb{B}$, and in particular*
$$[N_2] = \mathbb{K} \quad \textit{and} \quad [N_3] = \mathbb{B} + \mathbb{K}.$$

Proof. One Möbius strip in $\mathbf{S}^3$ is gotten by reflecting the basic piece around the geodesic $(-B_1)A_0B_1$ and reflecting the result around the geodesic $(-A_1)B_1A_1$ (see §3). The other Möbius strips come from rotating this one successively by $2\pi/m+1$ about the geodesic $(-A_1)A_0A_1$. □

We remark that this result can also be obtained easily for the Lawson Klein bottle $\eta_{1,1} = \tau_{2,1}$ by using the parametrization $\psi_{2,1}$.

Combining this lemma with Propositions 3.3, 3.4, and again using conformal connected-sum, deduce:

PROPOSITION 5.9.
$$W_{\mathbb{K}} \leq W(N_2) = \operatorname{area}(\tau_{2,1}) < 12\pi,$$
$$W_{\mathbb{B}+\mathbb{K}} \leq W(N_3) = \operatorname{area}(\eta_{2,1}) < 16\pi,$$
$$W_{\mathbb{K}+g\mathbb{S}} \leq W(N_2) + W(M_g) - 4\pi = \operatorname{area}(\tau_{2,1}) + \operatorname{area}(\xi_{g,1}) - 4\pi < 16\pi$$
and
$$W_{\mathbb{B}+\mathbb{K}+g\mathbb{S}} \leq W(N_3) + W(M_g) - 4\pi = \operatorname{area}(\eta_{2,1}) + \operatorname{area}(\xi_{g,1}) - 4\pi < 20\pi.$$

(In fact, $W_{\mathbb{K}+\mathbb{S}} < 12\pi$ by the numerical integration of $\operatorname{area}(\tau_{2,1})$ noted in §3.)

To complete the program we need another construction, also based on Proposition 5.1.

LEMMA 5.10. *For any $\varepsilon > 0$ and any parallel planes $P, P' \subset \mathbf{R}^3$ there exists a (rotation symmetric) cylinder $Z \subset \mathbf{R}^3$ with $W(Z) < \varepsilon$, and a ball $b \subset \mathbf{R}^3$ such that*

$$Z \backslash b = (P \cup P') \backslash b.$$

Proof. Simply apply the argument of Proposition 5.1 to each end of the catenoid

$$\{x^2 + y^2 = \cosh^2 z\} \subset \mathbf{R}^3,$$

which has $W = 0$. □

(One could also apply Corollary 5.2 directly to the *inverted* catenoid, then invert the result, using Proposition 1.3 to keep track of W.)

We use this construction to attach "orientation-reversing handles" to an immersed surface: at the level of regular homotopy classes this will add $\mathbb{K}_0$.

PROPOSITION 5.11.

$$W_{2\mathbb{K}+g\mathbb{S}} \leq 4\pi^2 < 16\pi.$$

Proof. Consider first the case $g = 0$. Recall the identity $2\mathbb{K} = \mathbb{K}_0 + \mathbb{T}$. Represent $\mathbb{T}$ by the diagonally double-covered torus $T \looparrowright \mathbf{R}^3$ (as in Proposition 5.5). Use Corollary 5.2 to flatten out *both* sheets of T in a small neighborhood. We may assume the sheets are parallel and slightly separated. Use Lemma 5.10 to insert a cylinder Z between the sheets, and weld this up to obtain a surface $M \looparrowright \mathbf{R}^3$. For any $\varepsilon > 0$, we can obviously arrange $W(M) \leq W(T) + \varepsilon = 4\pi^2 + \varepsilon$.

We claim that $[M] = \mathbb{K}_0 + \mathbb{T}$. To see this, slide one end of Z around a meridian of the standard torus which T double-covers. The two ends of the cylinder now attach to the same sheet of T, looking much like a "drain-trap". This drain-trap reverses orientation and has a plane of reflection symmetry; hence it represents connected-sum with an amphichiral Klein bottle.

In case $g \geq 1$, we simply weld in $g + 1$ of these cylinders between the sheets of T, and recall that

$$\mathbb{T} + \mathbb{K}_0 = 2\mathbb{K} \qquad \mathbb{K} + \mathbb{K}_0 = \mathbb{K} + \mathbb{S}$$

which implies that the resulting surface represents

$$\mathbb{T} + (g + 1)\mathbb{K}_0 = 2\mathbb{K} + g\mathbb{K}_0 = 2\mathbb{K} + g\mathbb{S}$$

as required. □

Only the non-orientable surfaces with $\vartheta = 0$ remain.

PROPOSITION 5.12. $W_{\mathbb{K}_0+g\mathbb{S}} \le 2\pi^2 + 4\pi < 12\pi$.

Proof. First assume $g \ge 1$. Consider M_1 as torus of revolution "standing on its end"—it has 2 vertical planes of symmetry and a height function with four critical points which we order according to their height. Choose a round sphere S which is tangent to M_1 at the *first* and *third* critical points. Now flatten M_1 and S at these points of tangency, and shrink S slightly towards its center. As in the proof of Proposition 5.11, we insert *one* cylinder at the first, and g cylinders at the third level. For any $\varepsilon > 0$, we can ensure that for this surface

$$W \le W(M_1) + W(S) + \varepsilon = 2\pi^2 + 4\pi + \varepsilon.$$

One easily checks that this surface is non-orientable, and that the construction can be done while preserving one vertical plane of symmetry. Therefore, the surface is amphichiral, and must represent $\mathbb{K}_0 + g\mathbb{S}$.

Surprisingly, the most familiar non-orientable surface—the amphichiral Klein bottle (the case $g = 0$)—requires the most elaborate construction. Choose $\varepsilon > 0$. Begin again with $M_1 \subset \mathbf{R}^3$ and consider the (two) circles $X = M_1 \cap P$, where P is a vertical plane of symmetry. By Proposition 5.1, we flatten a neighborhood of X (preserving the symmetry plane). Use a Möbius transformation to carry a point of X to ∞ (also preserving P). The resulting proper surface $N \subset \mathbf{R}^3$ has one (horizontal) planar end. In fact N resembles an "underpass" with a planar "road" passing beneath an umbilic "bridge". Note that

$$W(N) \le 2\pi^2 - 4\pi + \varepsilon.$$

Similarly, we take a cylinder Z (from Proposition 5.10) with a symmetry plane Q cutting its "neck", a neighborhood of which we have made umbilic. We scale Z so that its neck matches the bridge of N.

Now separate Z along Q, and weld the planar ends together, creating a surface Y with *one* planar end and two umbilic "holes", one facing left, and the other, right. Obviously, $W(Y) = W(Z) \le \varepsilon$. Also separate N along P and weld each half of the bridge onto the corresponding hole of Y; then weld in a planar strip to fill the split in the road.

The proper surface $\dot{M} \looparrowright \mathbf{R}^3$ so obtained has two planar ends (one horizontal, one vertical) and

$$W(\dot{M}) = W(N) + W(Y) \le 2\pi^2 - 4\pi + 2\varepsilon.$$

Apply another Möbius transformation to get a compact $M \looparrowright \mathbf{R}^3$ with

$$W(M) = W(\dot{M}) + 8\pi \le 2\pi^2 + 4\pi + 2\varepsilon.$$

(Here we have used the formula preceding Proposition 1.3, setting $\eta = 2$.)

Observe that M resembles a sphere with a bulging drain-trap welded in, so (as in the proof of Proposition 5.11) $[M] = \mathbb{K}_0$ which completes the proof. □

In summarizing the results of this section (and in applying these in the next sections) it will be convenient to write $\mu^{[M]}$ for the greatest integer in $W_{[M]}/4\pi$. Our upper bounds on $W_{[M]}$ imply those on $\mu^{[M]}$ (upper number in Table 5.14). A *lower* bound for $\mu^{[M]}$ follows from the Li-Yau inequality (Proposition 1.3); indeed, if we write $\mu_{[M]}$ (lower number in Table 5.14) for the *infimum* of the multiplicity $\mu(M)$ over $[M]$, then:

TABLE 5.14
(Multiplicity bounds.)

1 $\mathbb{O}$ 1	1 $\mathbb{S}$ 1	1 $2\mathbb{S}$ 1	1 $3\mathbb{S}$ 1	1 $4\mathbb{S}$ 1	1 $5\mathbb{S}$ 1
3 $\mathbb{T}$ 2	3 $\mathbb{T}+\mathbb{S}$ 2	3 $\mathbb{T}+2\mathbb{S}$ 2	3 $\mathbb{T}+3\mathbb{S}$ 2	3 $\mathbb{T}+4\mathbb{S}$ 2	4 $\mathbb{T}+5\mathbb{S}$ 2

		. . .			. . .
	2 $\mathbb{K}_0+\mathbb{S}$ 2	3 $\mathbb{B}+2\mathbb{S}$ 3	3 $\mathbb{K}+2\mathbb{S}$ 2	4 $\mathbb{B}+\mathbb{K}+2\mathbb{S}$ 3	. . .
2 $\mathbb{K}_0$ 2	3 $\mathbb{B}+\mathbb{S}$ 3	2 $\mathbb{K}+\mathbb{S}$ 2	3 $\mathbb{B}+\mathbb{K}+\mathbb{S}$ 3	3 $2\mathbb{K}+\mathbb{S}$ 2	
3 $\mathbb{B}$ 3	2 $\mathbb{K}$ 2	3 $\mathbb{B}+\mathbb{K}$ 3	3 $2\mathbb{K}$ 2		

The upper and lower numbers are remarkably close! The author does not know whether the "gaps" detect a gap in our knowledge of comparison surfaces or an invariant related to the twist ϑ. (The gaps occur for $\cos\vartheta \leq 0$, corresponding to the "more twisted" surfaces.)

We conclude by observing that the theorem stated in the Introduction follows directly from the results of this section.

6. Partial regularity for W-minimizing surfaces. Here we use our estimates on $W_{[M]}$ together with regularity results of L. Simon **[SL]** to bound the multiplicity and local branching order of a W-minimizing surface.

DEFINITION 6.1. Let $M \looparrowright \mathbf{R}^3$ be a compact immersed surface. Denote by $[[M]]$ the closure of the regular homotopy class $[M]$ in the space of branched immersed surfaces with the following topology:

A $C^{1,\alpha}$ surface $M' \looparrowright \mathbf{R}^3$ (with branch locus B) is the limit of a sequence $\{M_i\} \subset [M]$ provided there are representatives $f'\colon \Sigma' \to \mathbf{R}^3$ of M', and $f_i\colon \Sigma \to \mathbf{R}^3$ of M_i, as well as embeddings $F_i\colon \Sigma' \backslash B \to \Sigma$ onto open sets U_i such that

(i) $f_i \circ F_i \Rightarrow f'$ in $C^2_{\mathrm{loc}}(\Sigma' \backslash B, \mathbf{R}^3)$, and

(ii) For any $\delta > 0$, and large enough i, $f_i(\Sigma \backslash U_i) \subset \bigcup_{x \in f(B)} b_\delta(x)$, where $b_\delta(x)$ is the ball of radius δ about a point x in the (image of the) branch locus.

We remark that if $N \in [[M]]$ is *immersed*, then *either* $N \in [M]$, or N is of *lower topological type*, by which we mean $\chi(M) < \chi(N)$. (We use the fact that $\mathbb{O}$ is the *sole* regular homotopy class of the sphere in $\mathbf{R}^3$; the corresponding statement for immersed two-spheres in $\mathbf{R}^4$ is false **[HJ]**, **[KR3]**.) In general, $\beta(N) + \chi(N) \geq \chi(M)$, and in particular (the domain of) N may become disconnected (as a neck pinches).

Now Simon's regularity theorem **[SL]** applies to minimizing functionals of the form

$$V(M) = \int_M (v + |A|^2)\, da \qquad (v > 0)$$

for M immersed in a compact manifold. The main—and rather nontrivial—idea used in proving this theorem is that a surface minimizing V is well-approximated locally by biharmonic graphs, except on a finite set of "bad points" where the limit surface has necks pinch or even branch points develop. (See **[KR2]** for further discussion.) If we work in $\mathbf{S}^3$, then by the proof of Fact 1.5 (and the Gauss-Bonnet formula and Gauss equation)

$$W(M) = \int_M (1 + H^2)\, da = \frac{1}{4} \int_M (2 + |A|^2)\, da + \pi(\chi(M) + \beta(M)),$$

so Simon's theorem applies to minimizing W as well, since χ or β can only *increase* in the limit.

THEOREM 6.2. *Let $\{M_i\} \subset [M]$ be a W-minimizing sequence ($W(M_i) \Rightarrow W_{[M]}$). Then there is a subsequence $\{M_j\}$, a sequence of Möbius transformations $\{G_j\}$, and a surface $N \in [[M]]$ such that* (*in the sense of Definition* 6.1) *$G_j(M_j) \Rightarrow N$. Moreover,*

$$W(N) \leq W_{[M]} < 20\pi \quad \text{and so} \quad \mu(N) \leq \mu^{[M]} \leq 4.$$

It follows that the local branching order N is no more than $\mu^{[M]} - 1 \leq 3$. (*The best values of $\mu^{[M]}$ known appear in Table* 5.14.)

Proof. By the preceding discussion, the only hypothesis to check in Simon's regularity theorem **[SL]** is that the diameter of M_j does not shrink to zero: the Möbius transformation G_j prevents this. The remaining statements follow from lower-semicontinuity of W in the topology of Definition 6.1 and from results in previous sections. □

It is an interesting open problem to determine those regular homotopy classes which contain a W-minimizing surface. The only classes for which the answer is known (and affirmative!) are $\mathbb{O}$ and $\mathbb{B}$ (by explicit examples), and $\mathbb{S}$ (by an argument of L. Simon—see §8); partial results are available for $\mathbb{K}$ and $\mathbb{K}_0$ using methods developed in the next section.

Again we emphasize that for $N \in [[M]]$ the equality $W(N) = W_{[M]}$ does *not* guarantee that N is regular. The compactified Meeks' minimal Möbius strip (with a branch point of order $\beta = 2$) provides a counterexample (see §2) with

$$W = W_{\mathbb{B}} = 12\pi.$$

7. A bound on the total branching order of a static surface. In the previous sections we deduced a bound on the local branching order of a W-minimizing surface. Here—using a technique of R. Bryant **[BR1]**—we bound the *total* branching order. In fact, we shall prove the following

THEOREM 7.1. *Let $N \not\nearrow \mathbf{R}^3$ be a* (*compact, connected*) *static surface with total branching order $\beta(N)$. Then at least one of these alternatives holds*:

(i) $\beta(N) \leq -\chi(N)$; *or*

(ii) $\beta(N) < 2\mu(N) - \chi(N)$, *and there is a Möbius transformation carrying N to a complete* (*branched*) *minimal surface $\dot{N} \not\nearrow \mathbf{R}^3$ with* *finite total curvature*; *or*

(iii) $\beta(N) = 2\mu(N) - \chi(N)$, *and* N *is a* $\mu(N)$-*fold branched cover of a round sphere* $S \subset \mathbf{R}^3$.

Proof. We represent N by a branched *conformal* immersion $f\colon \Sigma \to \mathbf{R}^3$. By a standard double-cover argument we can (and will) assume that Σ is oriented and given a complex structure compatible with the orientation and conformal structure induced by f. Consider the following complex line-bundles (see, for example **[GH]**, Chapter 2) over Σ:

$\mathrm{K} =$ the canonical line-bundle of Σ;

$\mathrm{L} =$ the pullback of the line-bundle $(T^*N \otimes \mathbf{C})^{1,0}$ via f, whose fiber over $y \in \Sigma$ is the $(1,0)$ part of the complexified cotangent plane of N at $f(y)$;

$\|(B)\|$ = the line-bundle corresponding to the branching divisor $(B) = \sum_{y\in B}(m(y) - 1)y$ of f. (Here our notation differs from **[GH]**.)

These line-bundles satisfy the following

Formula 7.2.

$$\mathrm{K} = \|(B)\| \otimes \mathrm{L}.$$

To check this, compare with the equivalent statement about the *degrees*

$$d^\circ(\mathrm{K}) = d^\circ(\|(B)\|) + d^\circ(\mathrm{L});$$

using the most elementary form of the Chern-Weil theorem **[GH]** this becomes

$$-\chi(N) = \beta(N) - \frac{1}{2\pi}\int_N K\, da,$$

and the latter is just the Gauss-Bonnet formula (Lemma 1.2) for the compact surface N.

We may view Formula 7.2 as a generalized *Riemann-Hurwitz formula* for branched immersed surfaces. Indeed, if N is represented by the branched cover $f\colon \Sigma \to \Sigma'$ onto an embedded surface $\Sigma' \subset \mathbf{R}^3$, then $\mathrm{L} = f^*\mathrm{K}'$ is the pullback of the canonical line-bundle K' of Σ', and the degree version of the formula can be written in the more familiar way **[GH]**

$$-\chi(\Sigma) = \beta(f) - d^\circ(f)\chi(\Sigma').$$

Now we use our assumption that N is static. For in this event, R. Bryant **[BR1]** has shown how to construct a *holomorphic quartic differential* on N, which we denote by Q. This means that Q is a holomorphic section of the *fourth power* of L, and so—*assuming* $Q \neq 0$—$\|(Q)_0\| = \mathrm{L}^4$ where $(Q)_0$ denotes the *vanishing divisor* of Q. This is a positive divisor, so

$$-\chi(N) - \beta(N) = d^{\circ}(\mathrm{K}) - d^{\circ}(\|B\|) = d^{\circ}(\mathrm{L}) \geq 0$$

which implies alternative (i).

From now on assume $Q \equiv 0$. Then Bryant **[BR1]** shows that N arises from a complete (branched) minimal surface $\dot{N} \not\hookrightarrow \mathbf{R}^3$ via a Möbius transformation. We could then argue as before, using instead the *Hopf differential*, and another Riemann-Hurwitz formula. However, assertions (ii) and (iii) follow directly from the Gauss-Bonnet formula, since the Gauss curvature of a minimal surface is negative. In fact, from Lemma 1.2 we have

$$\chi(\dot{N}) - \eta(\dot{N}) + \beta(\dot{N}) \leq 0$$

with equality if and only if $\dot{N}$ is *flat.* But $\mu(\dot{N}) \leq \eta(\dot{N})$ for a complete minimal surface of finite total curvature **[KR]**, **[KR1]**, **[KR2]** so $\mu(N) = \eta(\dot{N})$. Using this and the formula preceding Lemma 1.2 we rewrite the above inequality

$$\chi(N) - 2\mu(N) + \beta(N) \leq 0$$

with equality if and only if N is *totally umbilic*, which yields (ii) or (iii). □

COROLLARY 7.3. *A static torus or Klein bottle is either immersed, or it arises from a complete (branched) minimal surface with finite total curvature in* $\mathbf{R}^3$.

We remark that in the W-minimizing case, the only way the second alternative can hold (for the classes $\mathbb{S}, \mathbb{K}$, and $\mathbb{K}_0$) is with N (branched) covering a *round sphere* or *union of two round spheres.* L. Simon (see the next section) showed that for $\mathbb{S}$ the first alternative of Corollary 7.3 *can* hold! (We believe the same is true for $\mathbb{K}$. However, the comparison surface we constructed for $\mathbb{K}_0$ suggests that the second alternative may hold here: the W-minimizing surface would appear to be the union of two orthogonal spheres!)

We conclude this section with a proof of the Gauss-Bonnet formula.

Proof of Lemma 1.2. We reduce our version to the standard one for a compact immersed surface $M \not\hookrightarrow \mathbf{R}^3$ with boundary ∂M whose geodesic curvature is κ:

$$\int_M K\, da + \int_{\partial M} \kappa\, ds = 2\pi\chi(M).$$

By an obvious induction argument, it is sufficient to treat the case where the proper surface $\dot{M} \not\subset \mathbf{R}^3$ has *one end* and *one branch point* (at 0). Let $M_r = \dot{M} \cap (b_r \backslash b_{1/r})$ where $b_t = b_t(0)$. Also set $s_t = \partial b_t$. Then

$$\int_{M_r} \dot{K}\, da = -\int_{M \cap s_r} \kappa\, ds - \int_{M \cap s_{1/r}} \kappa\, ds + 2\pi\chi(M_r).$$

The terms on the right converge (as $r \Rightarrow \infty$) to

$$\int_{M \cap s_r} \kappa\, ds \Rightarrow 2\pi\mu(\infty) = 2\pi\eta(\dot{M}),$$

$$\int_{M \cap s_{1/r}} \kappa\, ds \Rightarrow 2\pi\mu(0) = 2\pi(\beta(\dot{M}) + 1), \quad \text{and}$$

$$\chi(M_r) \Rightarrow \chi(\dot{M} \backslash \{0\}) = \chi(\dot{M}) = 1.$$

Adding these together, and letting $r \Rightarrow \infty$ on the left hand side too, gives us the desired formula. □

8. Remarks on the embedded case. The only explicitly known W-minimizing surfaces are the round sphere (for $\mathbb{O}$) and the Boy's surface P_3—and its deformations **[KR1]**, **[KR2]**, **[BR2]**—described in §2 (for $\mathbb{B}$). By Theorem 6.2 and Table 5.14, we see for $[M] = g\mathbb{S}$ the W-minimizing surface N in $[[M]]$ must be *regular* ($\beta(N) = 0$) and *embedded* ($\mu(N) = 1$).

L. Simon has observed that for $g \geq 1$ one can always choose a sequence of Möbius transformations G_j (as in the statement of Theorem 6.2) so that the limit N is *not* a round sphere **[SL]**. This implies

THEOREM 8.1 **[SL]**. *There exists an embedded torus $M \subset \mathbf{R}^3$ with*

$$W(M) = W_{\mathbb{S}} \leq 2\pi^2,$$

and therefore we have also a strict lower bound

$$4\pi < W_{\mathbb{S}}.$$

More generally, if one can show that *strict* inequality holds in Proposition 5.3 (for non-trivial connected-sums) then Simon's argument implies that there exists an embedded genus g surface $M \subset \mathbf{R}^3$ satisfying

$$W(M) = W_{g\mathbb{S}}.$$

Since we have shown $W_{g\mathbb{S}} < 8\pi$, this hypothesis will obviously be true provided one can obtain the lower bound $6\pi \leq W_{g\mathbb{S}}$.

Certainly a *necessary* condition on an embedded W-minimizing surface is that it be static. Our comparison surfaces $M_g \subset \mathbf{R}^3$ are static (as are any compact surfaces stereographically projected from minimal surfaces in $\mathbf{S}^3$ **[GT]**, **[BR1]**) and there is some evidence **[KR2]**, **[KR3]** that the corresponding minimal surface $\xi_{g,1} \subset \mathbf{S}^3$ has the *least* area among those of genus g.

Another consideration is the *second variation* of W. If we write Λ for the spectrum of the *Jacobi operator* **[SM]**

$$a'' = \Delta + 2 + |A|^2$$

on the minimal surface $\xi \,\ell\, \mathbf{S}^3$, then its stereographic projection $M \,\ell\, \mathbf{R}^3$ is *W-stable* if and only if $\Lambda \cap (-2, 0)$ is *empty*. (This can be seen quite simply by setting

$$W(M) = \int_\xi (1 + H^2)\, da = a + \frac{1}{4}(a')^2.$$

Since $a' = 0$ when ξ is minimal, one computes the Hessian

$$W''(M) = \frac{1}{2}a''(2 + a'').$$

Therefore, $W''(M) \geq 0$ if and only if a'' has no eigenvalues between -2 and 0.) For example, the Clifford torus $\xi_{1,1} \subset \mathbf{S}^3$ is a flat square torus, with $a'' = \Delta + 4$, so it is easy to check the following (which was earlier discovered by J. Weiner **[WJ]**)

PROPOSITION 8.2. *The stereographic image $M_1 \subset \mathbf{R}^3$ of $\xi_{1,1}$ is W-stable.*

(It also appears that M_g is W-stable.)

Finally, we observe that—if the W-minimizing surface of genus g exists—then there must be a path of embeddings connecting it to the unknotted surface $M_g \subset \mathbf{R}^3$:

PROPOSITION 8.3. *Suppose $M \subset \mathbf{R}^3$ satisfies $W(M) < 8\pi$. Then M is unknotted.*

Proof. Let $M_+ = \{x \in M \mid K(x) \geq 0\}$ be the region of non-negative Gauss curvature. We have the inequalities

$$\begin{aligned} 8\pi > \int_M H^2\, da &= \int_{M_+} H^2\, da + \int_{M\setminus M_+} H^2\, da \\ &\geq \int_{M_+} (H^2 - K)\, da + \int_{M_+} K\, da \geq \int_{M_+} K\, da. \end{aligned}$$

But the average (over the sphere of directions) number of *local maxima* for linear height functions on M equals

$$\frac{1}{4\pi}\int_{M_+} K\,da < 2,$$

so there is a height function with exactly *one* local maximum on M. It follows [**MW2**] that M is standardly embedded. $\square$

We hope that the preceding remarks make plausible the following

Conjecture 8.4. Up to Möbius transformation $M_g \subset \mathbf{R}^3$ is the unique W-minimizing surface in its regular homotopy class. In particular, $W_{g\mathbb{S}} = W(M_g)$.

Of course, the case $g = 1$ is Willmore's conjecture [**WT**]. We also conjecture that the stereographic projections of the surfaces $\tau_{2,1}$ and $\eta_{2,1}$ are the W-minimizers for the regular homotopy classes $\mathbb{K}$ and $\mathbb{B} + \mathbb{K}$, respectively.

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