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ON THE FINEST LEBESGUE TOPOLOGY ON THE SPACE OF ESSENTIALLY BOUNDED MEASURABLE FUNCTIONS

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Let (Ω, Σ, μ) be a σ-finite measure space and let $\mathscr{T}_0$ and $\mathscr{T}_\infty$ denote the usual metrizable topologies on L^0 and L^∞, respectively. In this paper the space L^∞ with the mixed topology $\gamma(\mathscr{T}_\infty, \mathscr{T}_0|_{L^\infty})$ is examined. It is proved that $\gamma(\mathscr{T}_\infty, \mathscr{T}_0|_{L^\infty})$ is the finest Lebesgue topology on L^∞, and that it coincides with the Mackey topology $\tau(L^\infty, L^1)$.

1. Introduction. For notation and terminology concerning Riesz spaces and locally solid topologies we refer to [**1**].

Let (Ω, Σ, μ) be a σ-finite measure space, and let L^0 denote the set of equivalence classes of all real valued μ-measurable functions defined and finite a.e. on Ω. Then L^0 is a super Dedekind complete Riesz space under the ordering $x \leq y$, whenever $x(t) \leq y(t)$ a.e. on Ω. The Riesz F-norm

$$\|x\|_0 = \int_\Omega |x(t)|(1 + |x(t)|)^{-1} f(t)\, d\mu \quad \text{for } x \in L^0,$$

where a function $f\colon \Omega \to (0, \infty)$ is μ-measurable with $\int_\Omega f(t)\, d\mu = 1$, determines a Lebesgue topology on L^0, which we will denote by $\mathscr{T}_0$ (see [7, I, §6], [**1**, Theorem 24.67]). This topology generates convergence in measure on the measurable subsets of Ω whose measure is finite. We will denote by $\mathscr{T}_\infty$ the topology on L^∞ generated by the usual B-norm

$$\|x\|_\infty = \operatorname*{ess\,sup}_{t \in \Omega} |x(t)|.$$

Moreover, we denote by $\sigma(L^\infty, L^1)$, $\tau(L^\infty, L^1)$ and $\beta(L^\infty, L^1)$ the weak, Mackey and strong topologies on L^∞ respectively, with respect to the dual pair $(L^\infty, L^1, \langle\ ,\ \rangle)$, where

$$\langle x, y\rangle = \int_\Omega x(t) y(t)\, d\mu \quad \text{for } x \in L^\infty, y \in L^1.$$

In this paper we shall examine the space L^∞ with the mixed topology $\gamma(\mathscr{T}_\infty, \mathscr{T}_0|_{L^\infty})$. This topology is defined as follows. Take a sequence

(ε_n) of positive numbers, a number $r > 0$ and let

$$W((\varepsilon_n), r) = \bigcup_{N=1}^{\infty} \left(\sum_{n=1}^{N} V(\varepsilon_n) \cap nB(r) \right),$$

where $B(r) = \{x \in L^\infty \colon \|x\|_\infty \leq r$ and $V(\varepsilon_n) = \{x \in L^\infty \colon \|x\|_0 \leq \varepsilon_n\}$. Then the family of all such $W((\varepsilon_n), r)$ forms a base of neighbourhoods of zero for $\gamma(\mathscr{T}_\infty, \mathscr{T}_0|_{L^\infty})$ (see **[11**, p. 49]). In view of **[11**, Theorem 2.2.2] $\gamma(\mathscr{T}_\infty, \mathscr{T}_0|_{L^\infty})$ is the finest linear topology on L^∞ which agrees with $\mathscr{T}_0|_{L^\infty}$ on $\|\cdot\|$-bounded sets. Henceforth, we will write briefly γ instead of $\gamma(\mathscr{T}_\infty, \mathscr{T}_0|_{L^\infty})$.

The space of bounded sequences l^∞ with the mixed topology γ has been investigated in **[4]**, where among other things, the results from Theorems 5, 6 and 8 below are obtained. The mixed topology γ on l^∞ is the same as the strict topology β **[3]** on $C(S)$, where $S = N =$ the set of all natural numbers.

2. The mixed topology γ on L^∞. It is well known that the norm topology $\mathscr{T}_\infty$ on L^∞ satisfies both the Fatou property and the Levi property (see [7, IV, §3] and [7, X, §4]), and that $\mathscr{T}_\infty$ does not satisfy the Lebesgue property if Ω does not consist of only finite number of atoms (see [7, IV, §3]). We shall show that the mixed topology γ is the finest Hausdorff Lebesgue topology on L^∞. We start by giving some characterization of sequential convergence in (L^∞, γ).

THEOREM 1. *For a sequence (x_n) in L^∞, $x_n \to 0$ for γ if and only if $\|x_n\|_0 \to 0$ and $\|x_n\|_\infty < M$ for some $M > 0$ and all $n = 1, 2, \ldots$.*

Proof. Since the balls $B(r) = \{x \in L^\infty \colon \|x\|_\infty \leq r\}$, $r > 0$ are closed in $\mathscr{T}_0$ (see [7, IV, §3, Lemma 5]) the result follows from **[11**, Theorem 2.3.1].

We now are able to prove the basic property of γ.

THEOREM 2. *The mixed topology γ is the finest Hausdorff Lebesgue topology on L^∞.*

Proof. Using **[1**, Theorem 1.2] it is easy to show that γ is a locally solid topology. In order to show that γ is a Lebesgue topology, let us assume that $x_\alpha \downarrow 0$ holds in L^∞ and let (ε_n) be a sequence of positive numbers and $r > 0$. Then there exists an increasing sequence of indices $\{\alpha_n\} \subset \{\alpha\}$ such that $x_{\alpha_n} \downarrow 0$ holds in L^∞, because

L^∞ has the countable sup property (see [**9**, Proposition 5.20]). Since $\mathscr{T}_0$ is a Lebesgue topology, we have $x_{\alpha_n} \to 0$ for γ by Theorem 1. Then there exists a natural number n_0 such that $x_{\alpha_{n_0}} \in W((\varepsilon_n), \tau)$, so $x_\alpha \in W((\varepsilon_n), r)$ for $\alpha \geq \alpha_{n_0}$, and hence $x_\alpha \to 0$ for γ. Now let ξ be a Hausdorff Lebesgue topology on L^∞. Then by [**1**, Theorem 12.9] we have $\xi_{[-x,x]} = \mathscr{T}_0|_{[-x,x]}$ for every $0 < x \in L^\infty$. Hence, by [**11**, Theorem 2.2.2] the inclusion $\xi \subset \gamma$ holds, and thus the proof is finished.

REMARK. It is known that L^∞ has no minimal topology, if the measure μ is atomless [**2**].

We now consider the problem of separableness of the space (L^∞, γ). First, we recall some definition. Let $\sim$ be the following equivalence relation in Σ: $A \sim B$ if and only if $\mu(A \dot{-} B) = 0$ ($\dot{-}$ denotes the symmetric difference). Denote by $\Sigma/\sim$ the set of equivalence classes and by $[A]$ the equivalence class of A. Then on $\Sigma/\sim$ one can define a metric function $\rho([A],[B]) = \|\chi_A - \chi_B\|_0$. ($\chi_A$ denotes the characteristic function of the set A.) The measure μ is said to be separable if the metric space $(\Sigma/\sim, \rho)$ is separable (see [**7**, I, §6]).

THEOREM 3. *The space (L^∞, γ) is separable if and only if the measure μ is separable.*

Proof. Assume that the space (L^∞, γ) is separable and let $0 < x \in L^0$. Let $x_n = x \wedge ne$, where e denotes the constant function one. Then $0 \leq x_n \uparrow x$ holds in L^0, so $x_n \to x$ for $\mathscr{T}_0$. Thus L^∞ is dense in $(L^0, \mathscr{T}_0)$, hence $(L^0, \mathscr{T}_0)$ is separable by hypothesis [**7**, 1, §6]. By [**7**, I, §6, Theorem 16] the measure μ is separable.

Next, assume that the measure μ is separable. Let

$$\mathscr{P} = \left\{ \sum_{k=1}^{m} c_k \chi_{A_k} : A_k \in \Sigma, \mu(A_k) < \infty, \right.$$
$$\left. A_{k_1} \cap A_{k_2} = \emptyset \text{ for} k_1 \neq k_2, c_k \in \mathbf{R}, m \in \mathbf{N} \right\}$$

where $\mathbf{R}$ denotes the set of real numbers. Then $\mathscr{P} \subset L^\infty$ and using Theorem 1, by usual argument one can show that the set $\mathscr{P}$ is dense in (L^∞, γ). Let Σ_0 be a countable subset of $\Sigma/\sim$, which is dense in $(\Sigma/\sim, \rho)$. Let $\mathscr{P}_0 = \{\sum_{k=1}^m r_k \chi_{A_k} \in \mathscr{P} : [A_k] \in \Sigma_0,\ r_k \in \mathbf{Q}\}$, where $\mathbf{Q}$ denotes the set of rational numbers. Let $0 \leq x = \sum_{k=1}^m c_k \chi_{A_k} \in \mathscr{P}$. Then, by hypothesis, for every $k = 1, \ldots, m$ there exist a sequence

$([A_k^n])$ in Σ_0 and a sequence (r_k^n) of positive rational numbers such that $\|\chi_{A_k^n} - \chi_{A_k}\|_0 \to 0$ as $n \to \infty$ and $0 \le r_k^n \uparrow_n c_k$ for $k = 1, \ldots, m$. Putting $x_n = \sum_{k=1}^m r_k^n \chi_{A_k^n}$ for $n = 1, 2, \ldots$, we have $\|x_n - x\|_0 \to 0$ and $|x_n(t)| \le \max_{1 \le k \le m} c_k$ a.e. on Ω. Thus, by Theorem 1, $x_n \to x$ for γ. It follows that the set $\mathscr{P}_0$ is dense in $(\mathscr{P}, \gamma|_{\mathscr{P}})$, so $\mathscr{P}_0$ is dense also in (L^∞, γ). Thus the space (L^∞, γ) is separable, because the set $\mathscr{P}_0$ is countable.

The next theorem describes the topological dual of (L^∞, γ).

THEOREM 4. *For a linear functional f on L^∞ the following statements are equivalent*:

(i) *f is continuous for γ.*
(ii) *f is sequentially continuous for γ.*
(iii) *There exists a unique $y \in L^1$ such that*

$$f(x) = \int_\Omega x(t) y(t)\, d\mu \quad \text{for } x \in L^\infty.$$

Proof. (i) $\Leftrightarrow$ (ii) It follows from [**11**, Theorem 2.6.1].

(ii) $\Leftrightarrow$ (iii) By Theorem 1, the functional f is sequentially continuous for γ if and only if it is sequentially order star-continuous, and if and only if it is sequentially order continuous (cf. [**6**, VII, §2]). Thus, in view of [**7**, VI, §2, Theorem 1] the proof is finished.

As an application of Theorems 2 and 4 we get the following important property of γ.

THEOREM 5. *The mixed topology γ on L^∞ is a Mackey topology, i.e., $\gamma = \tau(L^\infty, L^1)$.*

Proof. Since the Mackey topology $\tau(L^\infty, L^1)$ is a Lebesgue topology (see [**1**, Ex. 4, p. 163] and [**1**, Theorem 9.1]), by Theorem 2 we have $\tau(L^\infty, L^1) \subset \gamma$. According to Theorem 4, it suffices to show that γ is a locally convex topology. Indeed, let us put $x_n(t) = n$ for $t \in \Omega$ and $n = 1, 2, \ldots$. Let $\mathscr{T}_I$ be the generalized inductive limit topology of $(L^\infty, \tau(L^\infty, L^1), j_n, [-x_n, x_n])$ (see [**5**, p. 2]), i.e., $\mathscr{T}_I$ is the finest of all locally convex topologies ξ on L^∞ under which the inclusion maps

$$j_n \colon ([-x_n, x_n], \tau(L^\infty, L^1)|_{[-x_n, x_n]}) \to (L^\infty, \xi)$$

are continuous for $n = 1, 2, \ldots$. By [**5**, Proposition 5] $\mathscr{T}_I$ is also the finest of all linear topologies ξ on L^∞ under which each of the maps j_n

is continuous. Since γ and $\tau(L^\infty, L^1)$ are Hausdorff Lebesgue topologies, by [**1**, Theorem 12.9] we have

$$\gamma_{[-x_n,x_n]} = \tau(L^\infty, L^1)|_{[-x_n,x_n]} \quad \text{for } n = 1, 2, \ldots.$$

Thus $\gamma \subset \mathscr{T}_I$. On the other hand, since

$$\mathscr{T}_I|_{[-x_n,x_n]} \subset \tau(L^\infty, L^1)|_{[-x_n,x_n]} = \mathscr{T}_0|_{[-x_n,x_n]} \quad \text{for } n = 1, 2, \ldots,$$

by [**11**, Theorem 2.2.2] we get $\mathscr{T}_I \subset \gamma$. Thus $\mathscr{T}_I = \gamma$; hence γ is locally convex. Therefore, we have $\gamma \subset \tau(L^\infty, L^1)$. Thus the proof is finished.

For a linear topology $\mathscr{T}$ on L^∞, we will denote by $\mathrm{Bd}(\mathscr{T})$ the collection of all $\mathscr{T}$-bounded subsets of L^∞.

Additional properties of γ are included in the next theorem.

THEOREM 6. *The space L^∞ endowed with γ is complete.*

Proof. Since γ is a Lebesgue topology, in view of [**1**, Theorem 13.9] it suffices to show that γ is a Levi topology. But $\mathrm{Bd}(\gamma) = \mathrm{Bd}(\mathscr{T}_\infty)$ [**11**, Theorem 2.4.1], so γ is a Levi topology, because we know that $\mathscr{T}_\infty$ is a Levi topology.

COROLLARY 7. *The mixed topology γ is not metrizable.*

Locally convex Hausdorff space (X, ξ) is called sequentially barreled if every $\sigma(X^*, X)$-convergent to zero sequence in the topological dual $X^* = (X, \xi)^*$ is equicontinuous [**10**].

THEOREM 8. *The space (L^∞, γ) is sequentially barreled.*

Proof. Combining Theorem 4 and Theorem 5, we have $\gamma = \tau(L^\infty, (L^\infty, \gamma)^+)$, where $(L^\infty, \gamma)^+$ denotes the sequential topological dual of (L^∞, γ). Since the space (L^∞, γ) is complete, according to [**10**, Proposition 4.3] the space (L^∞, γ) is sequentially barreled.

Since L^∞ is the norm dual of L^1 we have $\beta(L^\infty, L^1) = \mathscr{T}_\infty$. Therefore, according to Theorem 4 and Corollary 7 we obtain that the space (L^∞, γ) is not barreled.

Additional characterizations of sequential convergence in (L^∞, γ) are included in the next theorem.

THEOREM 9. *For a sequence (x_n) in L^∞ the following statements are equivalent*:

(i) $x_n \to 0$ *for* γ.
(ii) $x_n \to 0$ *for the absolutely weak topology* $|\sigma|(L^\infty, L^1)$.
(iii) $\int_\Omega |x_n(t)y(t)|\,d\mu \to 0$ *for every* $y \in L^1$.

Proof. (i) $\Leftrightarrow$ (ii) Since $|\sigma|(L^\infty, L^1) \subset \tau(L^\infty, L^1)$ (see [**1**, Theorem 6.7], assume that $x_n \to 0$ for $|\sigma|(L^\infty, L^1)$. By [**1**, Theorem 12.9] we have $|\sigma|(L^\infty, L^1)|_{[-x,x]} = \mathscr{T}_0|_{[-x,x]}$ for every $0 < x \in L^\infty$, because $|\sigma|(L, L^1)$ is a Hausdorff Lebesgue topology. Since the set $\{x_n\}$ is $\sigma(L, L^1)$-bounded and $\mathrm{Bd}(\sigma(L^\infty, L^1)) = \mathrm{Bd}(\tau(L^\infty, L^1)) = \mathrm{Bd}(\tau_\infty)$ we obtain that $\{x_n\} \subset [-x, x]$ for some $0 < x \in L^\infty$. Thus $\|x_n\|_0 \to 0$, and in view of Theorem 1 we have $x_n \to 0$ for γ.

(ii) $\Leftrightarrow$ (iii) Obvious.

The next theorem gives criteria for the compactness of sets in (L^∞, γ).

THEOREM 10. *For a subset Z of L^∞ the following statements are equivalent*:

(i) *Z is relatively compact for $\mathscr{T}_0$ and $\|x\|_\infty < M$ for some $M > 0$ and every $x \in Z$.*
(ii) *Z is relatively compact for γ.*
(iii) *Z is relatively compact for $|\sigma|(L^\infty, L^1)$.*

Proof. (i) $\Leftrightarrow$ (ii) Obvious, because we know that $\mathrm{Bd}(\mathscr{T}_\infty) = \mathrm{Bd}(\gamma)$ and the topologies γ and $\mathscr{T}_0$ coincide on order intervals of L^∞.

(ii) $\Rightarrow$ (iii) Obvious, because $|\sigma|(L^\infty, L^1) \subset \gamma$.

(iii) $\Rightarrow$ (ii) Combining [**8**, I, §3, Lemma 11] and Theorem 9, Z is relatively compact for γ.

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