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ISOMETRIES OF TRIDIAGONAL ALGEBRAS

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Let $\operatorname{Alg}\mathcal{L}$ be a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson. In this paper it is proved that if $\varphi\colon \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is a linear surjective isometry, then there exist unitary operators W and V such that $\varphi(A) = WAV$ for all $A \in \operatorname{Alg}\mathcal{L}$.

Introduction. The study of reflexive, but not necessarily self-adjoint, algebras of Hilbert space operators has become one of the fastest-growing specialties in operator theory. In this paper we study the linear surjective isometries of a certain class of reflexive algebras, which were introduced by F. Gilfeather, A. Hopenwasser and D. Larson [**5**]. These algebras have been found to be useful counterexamples to a number of plausible conjectures. In particular, these algebras have non-trivial cohomology [**5**], and they admit automorphisms which are not spatially implemented [**2**].

First we introduce the notation which is used in this paper. Let $\{e_1, e_2, \dots, e_{2n}\}$ and $\{e_1, e_2, \dots\}$ be fixed bases of $2n$-dimensional complex Hilbert space and separable infinite dimensional Hilbert space, respectively. If $x_1, x_2, \dots, x_k$ are vectors in some Hilbert space, we denote by $[x_1, x_2, \dots, x_k]$ the closed subspace spanned by the vectors $x_1, x_2, \dots, x_k$.

Let x and y be two vectors in some Hilbert space. Then (x, y) means the inner product of the vectors x and y.

Let H_{2n} be $2n$-dimensional Hilbert space. We denote by $\mathcal{L}_{2n}$ the subspace lattice generated by the subspaces $[e_1], [e_3], [e_5], \dots, [e_{2n-1}]$, $[e_1, e_2, e_3]$, $[e_3, e_4, e_5], \dots, [e_{2n-3}, e_{2n-2}, e_{2n-1}]$, $[e_1, e_{2n-1}, e_{2n}]$.

By $\operatorname{Alg}\mathcal{L}_{2n} = \Phi_{2n}$ we mean the algebra of bounded operators which leave invariant all of the subspaces in $\mathcal{L}_{2n}$. It is easy to see that all such operators have the matrix form

$$\begin{bmatrix} * & * & & & & & * \\ & * & & & & & \\ & * & * & * & & & \\ & & & * & & & \\ & & & * & \cdot & & \\ & & & & & \cdot & \\ & & & & & \cdot & * \\ & & & & & & * \end{bmatrix},$$

where all non-starred entries are zero. Note that all diagonal operators and the identity operator I lie in $\operatorname{Alg}\mathcal{L}_{2n}$.

Let H_∞ represent infinite-dimensional separable Hilbert space, and let $\mathcal{L}_\infty$ be the lattice of subspaces generated by $[e_1], [e_3], [e_5], \ldots,$ $[e_1, e_2, e_3], [e_3, e_4, e_5], \ldots$.

Let $\Phi_\infty = \operatorname{Alg}\mathcal{L}_\infty$ be the algebra of bounded operators leaving every subspace of $\mathcal{L}_\infty$ invariant. Matricially, such operators have the form

$$\begin{bmatrix} * & * & & & & \\ & * & & & & \\ & * & * & * & & \\ & & & * & & \\ & & & * & \cdot & \\ & & & & & \cdot \\ & & & & & & \cdot \end{bmatrix},$$

where all non-starred entries are zero.

By an isometry of an operator algebra Φ we mean a linear map $\varphi \colon \Phi \to \Phi$ such that $\|\varphi(A)\| = \|A\|$ for every A in Φ. We do not assume any algebraic properties for isometries, although the main theorem will imply that such properties may exist.

Let i and j be two non-zero natural numbers. Then E_{ij} is the matrix whose (i, j)-component is 1 and all other entries are zero.

In this paper we will prove the following theorem.

THEOREM. *Let* $\varphi \colon \operatorname{Alg}\mathcal{L}_{2n} \to \operatorname{Alg}\mathcal{L}_{2n}$ *be a surjective isometry and let* $\varphi(I) = U$. *Then* U *and* U^* *are in* $\operatorname{Alg}\mathcal{L}_{2n}$, *and* U *is unitary. Let* $\varphi_1 \colon \operatorname{Alg}\mathcal{L}_{2n} \to \operatorname{Alg}\mathcal{L}_{2n}$ *be the surjective isometry defined by* $\varphi_1(A) = U^*\varphi(A)$ *for all* A *in* $\operatorname{Alg}\mathcal{L}_{2n}$. *Then either* $\varphi_1(\mathcal{L}_{2n}) = \mathcal{L}_{2n}$ *or* $\varphi_1(\mathcal{L}_{2n}) = \mathcal{L}_{2n}^{\perp}$. *If* $\varphi_1(\mathcal{L}_{2n}) = \mathcal{L}_{2n}$, *then there exists a unitary operator* W *such that* $\varphi_1(A) = WAW^*$ *for all* A *in* $\operatorname{Alg}\mathcal{L}_{2n}$. *If* $\varphi_1(\mathcal{L}_{2n}) = \mathcal{L}_{2n}^{\perp}$, *then there exist a conjugation* J *and a unitary operator* W *such that* $\varphi_1(A) =$

JWA^*W^*J for all A in $\mathrm{Alg}\mathscr{L}_{2n}$. Let $\varphi\colon \mathrm{Alg}\mathscr{L}_\infty \to \mathrm{Alg}\mathscr{L}_\infty$ be a surjective isometry and let $\varphi(I) = U$; then U and U^* are in $\mathrm{Alg}\mathscr{L}_\infty$ and U is unitary. Let $\varphi_1\colon \mathrm{Alg}\mathscr{L}_\infty \to \mathrm{Alg}\mathscr{L}_\infty$ be the surjective isometry defined by $\varphi_1(A) = U^*\varphi(A)$ for all A in $\mathrm{Alg}\mathscr{L}_\infty$. Then $\varphi_1(I) = I$, $\varphi_1(E_{ii}) = E_{ii}$ for all i ($i = 1, 2, \dots$), $\varphi_1(\mathscr{L}_\infty) = \mathscr{L}_\infty$, and there are diagonal unitary operators W and V such that $\varphi_1(A) = WAV$ for all A in $\mathrm{Alg}\mathscr{L}_\infty$.

1. Examples of isometries.

EXAMPLE 1. Let the Hilbert space be separable with an orthonormal basis $\{e_k\colon k = 1, 2, \dots\}$ and let U be a diagonal unitary operator whose (i,i)-component is u_{ii} such that $|u_{ii}| = 1$ for all i. Define $\varphi\colon \mathrm{Alg}\mathscr{L}_\infty \to \mathrm{Alg}\mathscr{L}_\infty$ by $\varphi(A) = U^*AU$ for all A in $\mathrm{Alg}\mathscr{L}_\infty$. Then φ is a surjective isometry such that $\varphi(I) = I$, the (i,i)-component of $\varphi(A)$ is the same as the (i,i)-component of A and if $A = (a_{ij})$ is in $\mathrm{Alg}\mathscr{L}_\infty$, then the $(2i+1, 2i+1)$-component of $\varphi(A)$ is $u_{2i+1,2i+1}a_{2i+1,2i}a_{2i,2i}$ and the $(2i+1, 2i+2)$-component of $\varphi(A)$ is $u_{2i+1,2i+1}a_{2i+1,2i+2}a_{2i+2,2i+2}$.

In Examples 2 and 3, the Hilbert space is $2n$-dimensional with an orthonormal basis $\{e_1, e_2, \dots, e_{2n}\}$.

EXAMPLE 2. Let D_n be the $n \times n$ matrix with 1 the $(i, n-i+1)$-component ($i = 1, 2, \dots, n$) and 0 elsewhere. Let $U_{2i+1} = D_{2i+1} \oplus D_{2n-2i-1}$. Define $\varphi\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ by $\varphi(A) = U_{2i+1}AU_{2i+1}^*$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$. It is straightforward to show that $U_{2i+1}AU_{2i+1}^*$ and $U_{2i+1}^*AU_{2i+1}$ are in $\mathrm{Alg}\mathscr{L}_{2n}$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$. So φ is a surjective isometry such that $\varphi(I) = I$, $\varphi(E_{11}) = E_{2i+1,2i+1}$, $\varphi(E_{22}) = E_{2i,2i}, \dots$, $\varphi(E_{2i-1,2i-1}) = E_{33}$, $\varphi(E_{2i,2i}) = E_{22}$, $\varphi(E_{2i+1,2i+1}) = E_{11}$, $\varphi(E_{2i+2,2i+2}) = E_{2n,2n}$, $\varphi(E_{2i+3,2i+3}) = E_{2n-1,2n-1}, \dots$, $\varphi(E_{2n,2n}) = E_{2i+2,2i+2}$. Moreover, it is easy to check that $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}$.

EXAMPLE 3. We denote the identity on n-dimensional Hilbert space by I_n. Let

$$V_{2i+1} = \begin{bmatrix} 0 & I_{2i} \\ I_{2n-2i} & 0 \end{bmatrix}.$$

Then V_{2i+1} is a unitary operator. Define $\varphi\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ by $\varphi(A) = V_{2i+1}AV_{2i+1}^*$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$. It is straightforward to show that $V_{2i+1}AV_{2i+1}^*$ and $V_{2i+1}^*AV_{2i+1}$ are in $\mathrm{Alg}\mathscr{L}_{2n}$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$. So φ is a surjective isometry such that $\varphi(I) = I$, $\varphi(E_{11}) = E_{2i+1,2i+1}$, $\varphi(E_{22}) = E_{2i+2,2i+2}, \dots, \varphi(E_{2n-2i,2n-2i}) = E_{2n,2n}$, $\varphi(E_{2n-2i+1,2n-2i+1}) = E_{11}$, $\varphi(E_{2n-2i+2,2n-2i+2}) = E_{22}, \dots, \varphi(E_{2n,2n}) = E_{2i,2i}$. Moreover, it is easy to check that $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}$.

EXAMPLE 4. Let $\varphi\colon \operatorname{Alg}\mathscr{L}_4 \to \operatorname{Alg}\mathscr{L}_4$ be defined by $\varphi(A) = A_f$ for every A in $\operatorname{Alg}\mathscr{L}_4$, where if

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}; \quad \text{then } A_f = \begin{bmatrix} a_{44} & a_{34} & 0 & a_{14} \\ 0 & a_{33} & 0 & 0 \\ 0 & a_{32} & a_{22} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{bmatrix}.$$

Define $J\colon \mathbf{C}^4 \to \mathbf{C}^4$ by $J(x_1, x_2, x_3, x_4)^t = (\overline{x_4}, \overline{x_3}, \overline{x_2}, \overline{x_1})^t$ for every $(x_1, x_2, x_3, x_4)^t$ in $\mathbf{C}^4$.

Then J is a conjugation; that is,

(1) J is bijective.

(2) $J(x+y) = Jx + Jy$ for x, y in $\mathbf{C}^4$.

(3) $J(\alpha x) = \bar{\alpha} Jx$ for every α in $\mathbf{C}$ and every x in $\mathbf{C}^4$.

(4) $J^2 = I$.

(5) $(Jx, y) = (Jy, x)$ for x, y in $\mathbf{C}^4$.

It is easy to check that $\varphi(A) = JA^*J$; φ is a surjective isometry by (5) and $\varphi(I) = I$. This isometry is not implemented by any unitary operator. The algebra $\operatorname{Alg}\mathscr{L}_{2n}$ admits this kind of isometry for other values of n. Note that in this example, if E is in $\mathscr{L}_{2n}$, then $\varphi(E)^\perp$ is in $\mathscr{L}_{2n}$, that is, $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^\perp$.

2. General theorems. We want to show that every surjective isometry on $\operatorname{Alg}\mathscr{L}_{2n}$ or $\operatorname{Alg}\mathscr{L}_\infty$ is a composition of the types mentioned in the examples. Our first task is to show that the image of the identity under a surjective isometry of $\operatorname{Alg}\mathscr{L}_{2n}$ (or $\operatorname{Alg}\mathscr{L}_\infty$) must be a unitary operator.

Let x and y be two non-zero vectors in a Hilbert space H. Then $x^* \otimes y$ is a rank one operator defined by $x^* \otimes y(h) = (h, x)y$ for every h in H.

LEMMA 1 (*Longstaff* [9]). *Let $\mathscr{L}$ be a commutative lattice and let x and y be two vectors. Then $x^* \otimes y$ is in* $\operatorname{Alg}\mathscr{L}$ *if and only if there exists E in $\mathscr{L}$ such that y is in E and x is in $E_-^\perp$ ($E_-^\perp$ means $(E_-)^\perp$), where $E_- = \vee\{F : F \text{ is in } \mathscr{L} \text{ and } F \not\geq E\}$.*

The following lemma appears in an unpublished paper. We include the proof for the convenience of the reader.

LEMMA 2 (*Moore and Trent* [10]). *Let $\varphi\colon \operatorname{Alg}\mathscr{L}_{2n} \to \operatorname{Alg}\mathscr{L}_{2n}$ be a linear surjective isometry. If $A = \varphi(I)$ and if $x^* \otimes x$ is in* $\operatorname{Alg}\mathscr{L}_{2n}$, *then* $\|Ax\| = \|x\|$.

Proof. Without loss of generality, we may assume that $\|x\| = 1$. Since $x^* \otimes Ax = A(x^* \otimes x)$, the operator $x^* \otimes Ax$ lies in $\mathrm{Alg}\mathscr{L}_{2n}$, and there is an operator R in $\mathrm{Alg}\mathscr{L}_{2n}$ for which $\varphi(R) = x^* \otimes Ax$. For any complex α,

$$\begin{aligned}\|I + \alpha R\|^2 &= \|A + \alpha(x^* \otimes Ax)\|^2 \\ &= \|(A + \alpha(x^* \otimes Ax))(A^* + \bar{\alpha}((Ax)^* \otimes x))\| \\ &= \|AA^* + (2\,\mathrm{Re}\,\alpha + |\alpha|^2)((Ax)^* \otimes Ax)\| \\ &\leq 1 + \|Ax\|^2 |2\,\mathrm{Re}\,\alpha + |\alpha|^2|.\end{aligned}$$

By choosing $\alpha = -it$ purely imaginary, and by letting $R = H + iK$ and $\delta \in \sigma(K)$, we find that $|1+t\delta|^2 \leq 1+t^2\|Ax\|^2$, or $(\|Ax\|^2 - \delta^2)t^2 - 2\delta t \geq 0$ for all real t, and it is easy to see that this condition implies that $\delta = 0$. Thus, $\sigma(K) = \{0\}$, $K = 0$, and R is Hermitian. Now let $\tau \in \sigma(R)$ and let $\alpha = t$ be real and deduce that $|1 + t\tau|^2 \leq 1 + \|Ax\|^2|2t + t^2|$, or $2t\tau + t^2\tau^2 \leq \|Ax\|^2|2t + t^2|$. Choose $t = -2$ to get $\tau^2 \leq \tau$, which means that $\tau \geq 0$ (and hence R is a positive operator). Finally, let $t \to 0^+$ and conclude that $\tau \leq \|Ax\|^2$, and, consequently, that $\|R\| \leq \|Ax\|^2$. But $\|R\| = \|\varphi(R)\| = \|x^* \otimes Ax\| = \|x\|\,\|Ax\| = \|Ax\|$. Thus, $\|Ax\| \leq \|Ax\|^2$ and it follows that $\|Ax\| \geq 1$. On the other hand, $\|A\| = 1$, so $\|Ax\| = 1$ and we are done.

In particular, since $e_i^* \otimes e_i$ is in $\mathrm{Alg}\mathscr{L}_{2n}$, $\|Ae_i\| = \|e_i\| = 1$ by Lemma 2 for every $1 \leq i \leq 2n$.

THEOREM 3. *If* $\varphi\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ *is a surjective isometry, then* $\varphi(I)$ *is a unitary operator in* $\mathrm{Alg}\mathscr{L}_{2n}$.

Proof. Let $\varphi(I) = A = (a_{ij})$. Then $|a_{ii}| = 1$ by the above statement for all odd numbers i; $1 \leq i \leq 2n$. But $\|A\| = \|I\| = 1$, so $a_{12} = a_{1,2n} = 0$, $a_{32} = a_{34} = 0$, $a_{54} = a_{56} = 0, \ldots$, $a_{2n-1,2n-2} = a_{2n-1,2n} = 0$. Thus, $\varphi(I) = A$ is a diagonal matrix whose components have absolute value 1 and hence $A = \varphi(I)$ is a unitary operator in $\mathrm{Alg}\mathscr{L}_{2n}$.

Similarly, we can get the following theorem.

THEOREM 4. *If* $\varphi\colon \mathrm{Alg}\mathscr{L}_\infty \to \mathrm{Alg}\mathscr{L}_\infty$ *is a surjective isometry, then* $\varphi(I)$ *is a unitary operator in* $\mathrm{Alg}\mathscr{L}_\infty$.

Let $\varphi(I) = U$. Then UA and U^*A are in $\mathrm{Alg}\mathscr{L}_{2n}$ (resp. $\mathrm{Alg}\mathscr{L}_\infty$) if A is in $\mathrm{Alg}\mathscr{L}_{2n}$ (resp. $\mathrm{Alg}\mathscr{L}_\infty$). Define $\hat{\varphi}\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ by $\hat{\varphi}(A) = U^*\varphi(A)$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$ or $\hat{\varphi}\colon \mathrm{Alg}\mathscr{L}_\infty \to \mathrm{Alg}\mathscr{L}_\infty$ by

$\hat{\varphi}(A) = U^*\varphi(A)$ for every A in Alg$\mathscr{L}_\infty$. Then $\hat{\varphi}$ is a surjective isometry such that $\hat{\varphi}(I) = I$.

Let $\Omega = \{A\colon A$ is a diagonal matrix in Alg$\mathscr{L}_{2n}$ (or Alg$\mathscr{L}_\infty$)$\}$. Then it is easy to check that Ω is the smallest von Neumann algebra containing $\mathscr{L}_{2n}$ (or $\mathscr{L}_\infty$) and $\Omega = \text{Alg}\mathscr{L}_{2n} \cap (\text{Alg}\mathscr{L}_{2n})^*$ (or $\Omega = \text{Alg}\mathscr{L}_\infty \cap (\text{Alg}\mathscr{L}_\infty)^*$).

We will require the following facts, first proved by Kadison.

LEMMA 5 (*Kadison* [8]). *A linear map φ of one C^*-algebra into another which carries the identity into the identity and is isometric on normal elements preserves adjoints, i.e., $\varphi(A^*) = (\varphi(A))^*$.*

DEFINITION 6. Let Φ_1 and Φ_2 be C^*-algebras. A Jordan isomorphism or C^*-isomorphism $\varphi\colon \Phi_1 \to \Phi_2$ is a bijective linear map such that if A is self-adjoint in Φ_1, then $\varphi(A)$ is also self-adjoint in Φ_2 and $\varphi(A^n) = (\varphi(A))^n$.

LEMMA 7 (*Kadison* [8]). (a) *A linear bijection φ of one C^*-algebra Φ_1 onto another Φ_2 which is isometric is a C^*-isomorphism followed by left multiplication by a fixed unitary operator, viz, $\varphi(I)$.*

(b) *A C^*-isomorphism φ of a C^*-algebra Φ_1 onto a C^*-algebra Φ_2 is isometric and preserves commutativity.*

LEMMA 8. $\hat{\varphi}(\Omega) = \Omega$, (*where $\hat{\varphi}$ and Ω are defined above*).

Proof. Since $\hat{\varphi}|\Omega$ preserves adjoints by Lemma 5, $\hat{\varphi}(\Omega)$ is contained in Ω. Similarly, $\hat{\varphi}^{-1}(\Omega)$ is contained in Ω. Hence $\hat{\varphi}(\Omega) = \Omega$.

Since $\hat{\varphi}\colon \text{Alg}\mathscr{L}_{2n} \to \text{Alg}\mathscr{L}_{2n}$ (or $\text{Alg}\mathscr{L}_\infty \to \text{Alg}\mathscr{L}_\infty$) is a surjective isometry, just like φ, and since the main theorem would be true of φ if it were true of $\hat{\varphi}$, we now work exclusively with $\hat{\varphi}$ and drop the "ˆ" symbol. Equivalently we assume that $\varphi(I) = I$.

Then we can get the following corollary.

COROLLARY 9. *If $\varphi\colon \text{Alg}\mathscr{L}_{2n} \to \text{Alg}\mathscr{L}_{2n}$ (or $\text{Alg}\mathscr{L}_\infty \to \text{Alg}\mathscr{L}_\infty$) is a surjective isometry such that $\varphi(I) = I$, then $\varphi(\Omega) = \Omega$.*

LEMMA 10. *Let $\varphi\colon \text{Alg}\mathscr{L}_{2n} \to \text{Alg}\mathscr{L}_{2n}$ (or $\text{Alg}\mathscr{L}_\infty \to \text{Alg}\mathscr{L}_\infty$) be a surjective isometry such that $\varphi(I) = I$. Then E is a projection in Ω if and only if $\varphi(E)$ is a projection in Ω.*

Proof. First, suppose that E is a projection in Ω. Since $\varphi|\Omega$ is a Jordan isomorphism, $\varphi(E) = \varphi(E^*) = \varphi(E)^*$ and $\varphi(E) = \varphi(E^2) =$

$\varphi(E)^2$. So $\varphi(E)$ is a projection in Ω because $\varphi(\Omega) = \Omega$. Suppose that $\varphi(E)$ is a projection in Ω. Then since $\varphi^{-1}|\Omega$ is a Jordan isomorphism, by the above argument $\varphi^{-1}\varphi(E) = E$ is a projection in Ω.

LEMMA 11 (*Kadison* [**8**]). *If φ is a Jordan isomorphism from a C^*-algebra Φ_1 onto a C^*-algebra Φ_2, then $\varphi(BAB) = \varphi(B)\varphi(A)\varphi(B)$ with A and B in Φ_1.*

THEOREM 12. *Let $\varphi\colon \mathrm{Alg}\mathscr{L}_\infty \to \mathrm{Alg}\mathscr{L}_\infty$ be a surjective isometry such that $\varphi(I) = I$. Let $\{e_i\colon i = 1, 2, \dots\}$ be the orthonormal basis for which the generators of the lattice are $[e_1], [e_3], \dots, [e_{2n-1}], \dots, [e_1, e_2, e_3]$, $[e_3, e_4, e_5], \dots, [e_{2n-3}, e_{2n-2}, e_{2n-1}], \dots$. Then $\varphi([e_i])$ is rank-one for each i; $i = 1, 2, \dots$.*

Proof. Let $E_k = \varphi^{-1}([e_k])$ for each k; $k = 1, 2, \dots$, that is, $\varphi(E_k) = [e_k]$. Then E_k is a projection in Ω by Lemma 10. If E_k is not a rank 1 projection, then $E_k = E + F$ with E, F on $\mathrm{Alg}\mathscr{L}_\infty$, both non-zero projections. But then $[e_k] = \varphi^{-1}(E) + \varphi^{-1}(F)$ expresses $[e_k]$ as a sum of 2 non-zero projections.

With the same proof as Theorem 12, we can get the following theorem.

THEOREM 13. *Let $\varphi\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ be a surjective isometry such that $\varphi(I) = I$. Then $\varphi([e_i])$ is rank-one in Ω for each i; $i = 1, 2, \dots, 2n$.*

LEMMA 14. *Let R be an operator and suppose that there is a non-negative number M and a positive number N such that, for all complex numbers α with $|\alpha| \geq N$, we have $\|R + \alpha I\|^2 \leq M^2 + |\alpha|^2$. Then $R = 0$.*

Proof. Choose x in the Hilbert space H, with $\|x\| = 1$. We have $\|Rx + \alpha x\|^2 \leq M^2 + |\alpha|^2$, or $\|Rx\|^2 + |\alpha|^2 + 2\,\mathrm{Re}\,\bar{\alpha}(Rx, x) \leq M^2 + |\alpha|^2$, or $2\,\mathrm{Re}\,\bar{\alpha}(Rx, x) \leq M^2 - \|Rx\|^2$. Choosing $\alpha = t\overline{(Rx, x)}$ for positive t, we get $2t|(Rx, x)|^2 \leq M^2 - \|Rx\|^2$ for all $t > N$. This is impossible unless $(Rx, x) = 0$. The fact that this equation holds for all x means that $R = 0$.

LEMMA 15 (*Moore and Trent* [**10**]). *Let $\varphi\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ (or $\mathrm{Alg}\mathscr{L}_\infty \to \mathrm{Alg}\mathscr{L}_\infty$) be a surjective isometry such that $\varphi(I) = I$. Let P be* *a projection in Ω and let T be in* $\operatorname{Alg}\mathscr{L}_{2n}$ *(or* $\operatorname{Alg}\mathscr{L}_\infty$*) with $T = PTP^\perp$. Then we have $\varphi(T) = \varphi(P)\varphi(T)\varphi(P)^\perp + \varphi(P)^\perp\varphi(T)\varphi(P)$.*

Proof. We begin by writing $\varphi(T)$ as 2×2 matrix, using the decomposition $I = \hat{P} + \hat{P}^\perp$:

$$\varphi(T) = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}, \quad \hat{P} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \hat{P}^\perp = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

where $\hat{P} = \varphi(P)$. Then, for all complex α,

$$\|T + \alpha P\| = \|\varphi(T) + \alpha\hat{P}\| = \left\| \begin{bmatrix} R_1 + \alpha & R_2 \\ R_3 & R_4 \end{bmatrix} \right\|.$$

On the other hand, T, written using "$I = P + P^\perp$", is the matrix $T = \left[\begin{smallmatrix} 0 & S \\ 0 & 0 \end{smallmatrix}\right]$. So

$$\begin{aligned} \|T + \alpha P\|^2 &= \left\| \begin{bmatrix} \alpha & S \\ 0 & 0 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} \alpha & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & S \\ 0 & 0 \end{bmatrix}^* \right\| \\ &= \left\| \begin{bmatrix} 0 & 0 \\ 0 & |\alpha|^2 + SS^* \end{bmatrix} \right\| = |\alpha|^2 + \|S\|^2 \end{aligned}$$

since SS^* is a positive operator. Thus, $\|R_1 + \alpha\|^2 \leq |\alpha|^2 + \|S\|^2$, and Lemma 14 tells us that $R_1 = 0$. Similarly, by considering $\|t + \alpha P^\perp\|$, we can show that $R_4 = 0$. So $\varphi(T) = \hat{P}\varphi(T)\hat{P}^\perp + \hat{P}^\perp\varphi(T)\hat{P}$.

THEOREM 16. *Let φ:* $\operatorname{Alg}\mathscr{L}_\infty \to \operatorname{Alg}\mathscr{L}_\infty$ *be a surjective isometry such that $\varphi(I) = I$. Let $\varphi(E_{2i-1,2i-1}) = E_{jj}$ and let $\varphi(E_{2i,2i}) = E_{kk}$. Then $|k - j| = 1$.*

Proof. Since

$$\begin{aligned} E_{2i,2i}^\perp E_{2i-1,2i} E_{2i,2i} &= E_{2i-1,2i} \quad \text{and} \\ E_{2i-1,2i-1} E_{2i-1,2i} E_{2i-1,2i-1}^\perp &= E_{2i-1,2i}, \end{aligned}$$

Lemma 15 tells us that

$$\begin{aligned} &\varphi(E_{2i,2i})^\perp \varphi(E_{2i-1,2i}) \varphi(E_{2i,2i}) \\ &\qquad + \varphi(E_{2i,2i}) \varphi(E_{2i-1,2i}) \varphi(E_{2i,2i})^\perp = \varphi(E_{2i-1,2i}) \end{aligned}$$

and

$$\begin{aligned} &\varphi(E_{2i-1,2i-1}) \varphi(E_{2i-1,2i}) \varphi(E_{2i-1,2i-1})^\perp \\ &\qquad + \varphi(E_{2i-1,2i-1})^\perp \varphi(E_{2i-1,2i}) \varphi(E_{2i-1,2i-1}) = \varphi(E_{2i-1,2i}). \end{aligned}$$

Then

$$(*) \qquad E_{kk}^\perp \varphi(E_{2i-1,2i}) E_{kk} + E_{kk} \varphi(E_{2i-1,2i}) E_{kk}^\perp = \varphi(E_{2i-1,2i})$$

and

$$E_{jj}\varphi(E_{2i-1,2i})E_{jj}^{\perp} + E_{jj}^{\perp}\varphi(E_{2i-1,2i})E_{jj} = \varphi(E_{2i-1,2i}).$$

So we can get the following from the second equation of $(*)$;

(1) If j is 1, then $\varphi(E_{2i-1,2i})$ is a matrix all of whose entries are zero except for the $(1,2)$-component and the $(1,2n)$-component.

(2) If j is an odd number and $j \neq 1$, then $\varphi(E_{2i-1,2i})$ is a matrix all of whose entries are zero except for the $(j, j-1)$-component and the $(j, j+1)$-component.

(3) If j is 2, then $\varphi(E_{2i-1,2i})$ is a matrix all of whose entries are zero except for the $(1,2)$-component and the $(3,2)$-component.

(4) If j is an even number and $j \neq 2$, then $\varphi(E_{2i-1,2i})$ is a matrix all of whose entries are zero except for the $(j-1, j)$-component and the $(j+1, j)$-component.

(α) From the first equation of $(*)$ we know the following: If k is 1, then $\varphi(E_{2i-1,2i})$ is a matrix all of whose entries are zero except for the $(1,2)$-component.

(β) If k is an odd number and $k \neq 1$, then $\varphi(E_{2i-1,2i})$ is a matrix all of whose entries are zero except for the $(k, k-1)$-component and the $(k, k+1)$-component.

(τ) If k is 2, then $\varphi(E_{2i-1,2i})$ is a matrix all of whose entries are zero except for the $(1,2)$-component and the $(3,2)$-component.

(δ) If k is an even number and $k \neq 2$, then $\varphi(E_{2i-1,2i})$ is a matrix all of whose entries are zero except for the $(k-1, k)$-component and the $(k+1, k)$-component.

Then the following cannot happen at the same time;
(1) and (α) because $j \neq k$.
(1) and (β) because $j = 1$ and $k \geq 3$.
(1) and (δ) because $k > 2$.
(2) and (α) because $j \neq 1$.
(2) and (β) because $j \neq k$.
(3) and (τ) because $j \neq k$.
(3) and (δ) because $k > 2$.
(4) and (α) because $j > 2$.
(4) and (τ) because $j > 2$.
(4) and (δ) because $j \neq k$.
Then the following can happen at the same time;
(1) and (τ) if $|k - j| = 1$.
(2) and (τ) if $j = 3$ and so $|j - k| = 1$.
(2) and (δ) if $|j - k| = 1$.

(3) and (α) if $|j-k|=1$.
(3) and (β) if $k=3$ and so $|j-k|=1$.
(4) and (τ) if $|j-k|=1$.
So we can get the result of the theorem.

Note that in all cases, $\varphi(E_{2i-1,2i})$ is a scalar multiple of E_{kj} or E_{jk}. From this theorem, we can get the following corollary.

COROLLARY 17. *Let* $\varphi\colon \mathrm{Alg}\mathscr{L}_\infty \to \mathrm{Alg}\mathscr{L}_\infty$ *be a surjective isometry such that* $\varphi(I)=I$. *Then* (1) $\varphi(E_{ii})=E_{ii}$ *for all* i; $i=1,2,3,\ldots$ *and* (2) $\varphi(\mathscr{L}_\infty)=\mathscr{L}_\infty$.

Proof. Suppose that $\varphi(E_{11})=E_{ii}$ for $i\neq 1$. Then $\varphi(E_{22})=E_{i-1,i-1}$ or $\varphi(E_{22})=E_{i+1,i+1}$ by Theorem 16. If $\varphi(E_{22})=E_{i-1,i-1}$, then $\varphi(E_{33})=E_{i-2,i-2}$, and by continuing we get $\varphi(E_{ii})=E_{11}$. Let $\varphi(E_{i+1,i+1})=E_{kk}$. Then since $k\geq i+1$, $k-1\neq 1$, contradicting Theorem 16. If $\varphi(E_{22})=E_{i+1,i+1}$, then by Theorem 16 $\varphi(E_{33})=E_{i+2,i+2},\ldots,\varphi(E_{kk})=E_{i+k-1,i+k-1},\cdots(*)$. But since φ is a surjective isometry, $\varphi(E_{jj})=E_{11}$ for some j. But $\varphi(E_{jj})=E_{i+j-1,i+j-1}$ by $(*)$. Then $i+j-1=1$. So $j=2-i$, which is impossible because $i\geq 2$. Thus $\varphi(E_{11})=E_{11}$ and hence $\varphi(E_{ii})=E_{ii}$ for all i by Theorem 16. By (1) $\varphi(\mathscr{L}_\infty)=\mathscr{L}_\infty$.

LEMMA 18. *Let* $\varphi\colon \mathrm{Alg}\mathscr{L}_{2n}\to\mathrm{Alg}\mathscr{L}_{2n}$ *be a surjective isometry such that* $\varphi(I)=I$. *Let* $\varphi(E_{11})=E_{ii}$ *and let* $\varphi(E_{22})=E_{kk}$. *If* $1<i<2n$, *then* $|i-k|=1$.

Proof. Since $E_{11}E_{12}E_{11}^{\perp}=E_{12}$ and $E_{22}^{\perp}E_{12}E_{22}=E_{12}$, $E_{ii}\varphi(E_{12})E_{ii}^{\perp}+E_{ii}^{\perp}\varphi(E_{12})E_{ii}=\varphi(E_{12})$ and $E_{kk}^{\perp}\varphi(E_{12})E_{kk}+E_{kk}\varphi(E_{12})E_{kk}^{\perp}=\varphi(E_{12})$.

(1) If i is an odd number, then $\varphi(E_{12})$ is a $2n\times 2n$ matrix whose entries are zero except for the $(i,i-1)$-component and the $(i,i+1)$-component.

(2) If i is an even number, then $\varphi(E_{12})$ is a $2n\times 2n$ matrix whose entries are zero except for the $(i-1,i)$-component and the $(i+1,i)$-component.

(α) If k is an odd number, the $\varphi(E_{12})$ is a $2n\times 2n$ matrix whose entries are zero except for the $(k,k-1)$-component and the $(k,k+1)$-component.

(β) If k is an even number, then $\varphi(E_{12})$ is a $2n\times 2n$ matrix whose entries are zero except for the $(k-1,k)$-component and the $(k+1,k)$-component.

Then the following combinations are impossible;

(1) and (α) because $i \neq k$.
(2) and (β) because $i \neq k$.
The following combinations are possible;
(1) and (β) if $|i-k|=1$.
(2) and (α) if $|i-k|=1$.
By an argument similar to Lemma 18, we can get the following lemma.

LEMMA 19. *Let* $\varphi\colon \operatorname{Alg}\mathscr{L}_{2n} \to \operatorname{Alg}\mathscr{L}_{2n}$ *be a surjective isometry such that* $\varphi(I)=I$. *Let* $\varphi(E_{2i-1,2i-1})=E_{jj}$ *and let* $\varphi(E_{2i,2i})=E_{kk}$. *If* $1<j<2n$, *then* $|j-k|=1$.

From Lemma 18 and Lemma 19, we can get the following corollary.

COROLLARY 20. *Let* $\varphi\colon \operatorname{Alg}\mathscr{L}_{2n} \to \operatorname{Alg}\mathscr{L}_{2n}$ (*or* $\operatorname{Alg}\mathscr{L}_{\infty} \to \operatorname{Alg}\mathscr{L}_{\infty}$) *be a surjective isometry such that* $\varphi(I)=I$, $\varphi(E_{2i-1,2i-1})=E_{jj}$ *and* $\varphi(E_{2i,2i})=E_{kk}$. *If* $1<j<2n$, *then* $\varphi(E_{2i-1,2i-2})$ *and* $\varphi(E_{2i-1,2i})$ *have the form*

$$\begin{bmatrix} \cdot & & & & \\ & \cdot & & & \\ & & 0 & * & \\ & & & 0 & \\ & & & & \cdot \\ & & & & & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cdot & & & & \\ & \cdot & & & \\ & & 0 & & \\ & & * & 0 & \\ & & & & \cdot \\ & & & & & 0 \end{bmatrix}.$$

In particular, if $\varphi(E_{ii})=E_{ii}$ for each i ($i=1,2,\ldots,2n$), then there exists a complex number α_{ij} such that $\varphi(E_{ij})=\alpha_{ij}E_{ij}$ for each E_{ij} in $\operatorname{Alg}\mathscr{L}_{2n}$ (or E_{ij} in $\operatorname{Alg}\mathscr{L}_{\infty}$).

In the following, we will investigate $\varphi(\mathscr{L}_{2n})$ case by case.

LEMMA 21. *If* $\varphi\colon \operatorname{Alg}\mathscr{L}_{2n} \to \operatorname{Alg}\mathscr{L}_{2n}$ *is a surjective isometry such that* $\varphi(I)=I$ *and if* $\varphi(E_{11})=E_{11}$, *then* $\varphi(\mathscr{L}_{2n})=\mathscr{L}_{2n}$.

Proof. Since $E_{11}E_{12}E_{11}^{\perp}=E_{12}$, $E_{11}\varphi(E_{12})E_{11}^{\perp}+E_{11}^{\perp}\varphi(E_{12})E_{11}=\varphi(E_{12})$. So $\varphi(E_{12})$ is a $2n\times 2n$ matrix whose entries are zero except for the $(1,2)$-component and the $(1,2n)$-component. Set $\varphi(E_{22})=E_{kk}$. Since $E_{22}^{\perp}E_{12}E_{22}=E_{12}$, $E_{kk}^{\perp}\varphi(E_{12})E_{kk}+E_{kk}\varphi(E_{12})E_{kk}^{\perp}=\varphi(E_{12})$. So the only possibility is $k=2$ or $k=2n$. Assume that $k=2$. Then $\varphi(E_{ii})=E_{ii}$ for all i by Lemma 19; $i=1,2,\ldots,2n$. In this case, $\varphi(\mathscr{L}_{2n})=\mathscr{L}_{2n}$. Assume that $k=2n$. Since $E_{22}^{\perp}E_{32}E_{22}=E_{32}$ and $E_{33}E_{32}E_{33}^{\perp}=E_{32}$, $E_{2n,2n}^{\perp}\varphi(E_{32})E_{2n,2n}+E_{2n,2n}\varphi(E_{32})E_{2n,2n}^{\perp}=\varphi(E_{32})$

and $E_{jj}^{\perp}\varphi(E_{32})E_{jj} + E_{jj}\varphi(E_{32})E_{jj}^{\perp} = \varphi(E_{32})$, where $E_{jj} = \varphi(E_{33})$. We know that $j \neq 1$ and $j \neq 2n$. By the first equation, $\varphi(E_{32})$ is a $2n \times 2n$ matrix whose entries are zero except for the $(1, 2n)$-component and the $(2n-1, 2n)$-component. If j is an odd number, then $\varphi(E_{32})$ is a $2n \times 2n$ matrix whose entries are zero except for the $(j, j-1)$-component and the $(j, j+1)$-component. If j is an even number, then $\varphi(E_{32})$ is a $2n \times 2n$ matrix whose entries are zero except for the $(j-1, j)$-component and the $(j+1, j)$-component. So the only possibility is $j = 2n-1$, that is, $\varphi(E_{33}) = E_{2n-1,2n-1}$. By Lemma 19, $\varphi(E_{44}) = E_{2n-2,2n-2}, \ldots, \varphi(E_{2n,2n}) = E_{22}$. In this case, if $\varphi(E_{kk}) = E_{jj}$, then k and j have the same parity and it is straightforward to see that $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}$.

COROLLARY 22. *If* $\varphi\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ *is a surjective isometry such that* $\varphi(I) = I$ *and* $\varphi(E_{11}) = E_{2n,2n}$, *then* $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^{\perp}$.

Proof. Let $\varphi_1\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ be the surjective isometry in Example 4. Then $\varphi_1 \circ \varphi\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ is a surjective isometry such that $\varphi_1 \circ \varphi(I) = I$ and $\varphi_1 \circ \varphi(E_{11}) = \varphi_1(E_{2n,2n}) = E_{11}$. So $\varphi_1 \circ \varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}$ by Lemma 21. Since $\varphi_1(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^{\perp}$, $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^{\perp}$.

LEMMA 23. *Let* $\varphi\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ *be a surjective isometry such that* $\varphi(I) = I$ *and* $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^{\perp}$. *Then* $\varphi(\mathscr{L}_{2n}^{\perp}) = \mathscr{L}_{2n}$.

Proof. $\varphi(\mathscr{L}_{2n}^{\perp}) = \varphi(\mathscr{L}_{2n})^{\perp} = (\mathscr{L}_{2n}^{\perp})^{\perp} = \mathscr{L}_{2n}$.

COROLLARY 24. *Let* $\varphi\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ *be a surjective isometry such that* $\varphi(I) = I$. *Let* $\varphi(E_{11}) = E_{ii}$; $i \neq 1$ *and* $i \neq 2n$. *If* i *is an odd number, then* $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}$. *If* i *is an even number, then* $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^{\perp}$.

Proof. First, let $i = 2k-1$, for some k. Let φ_1 be the surjective isometry in Example 2; that is, $\varphi_1(E_{11}) = E_{2k-1,2k-1}$. Then $\varphi_1 \circ \varphi(E_{11}) = \varphi_1(E_{2k-1,2k-1}) = E_{11}$. By Lemma 21, $\varphi_1 \circ \varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}$. So $\varphi(\mathscr{L}_{2n}) = \varphi_1^{-1}(\mathscr{L}_{2n})$. Since $\varphi_1(\mathscr{L}_{2n}) = \mathscr{L}_{2n}$, $\varphi(\mathscr{L}_{2n}) = \varphi_1^{-1}(\mathscr{L}_{2n}) = \mathscr{L}_{2n}$. Let $i = 2k$ for some k. Let us consider $V_{2n-2k+1}$ in Example 3 and let $\varphi_2\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ be a surjective isometry in Example 3. Then $\varphi_2 \circ \varphi\colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ is a surjective isometry such that $\varphi_2 \circ \varphi(I) = I$ and $\varphi_2 \circ \varphi(E_{11}) = \varphi_2(E_{2k,2k}) = E_{2n,2n}$. By Corollary 22, $\varphi_2 \circ \varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^{\perp}$. So $\varphi(\mathscr{L}_{2n}) = \varphi_2^{-1}(\mathscr{L}_{2n}^{\perp})$. Since $\varphi_2(\mathscr{L}_{2n}) = \mathscr{L}_{2n}$, $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^{\perp}$.

If we summarize lemmas and corollaries, then we can get the following theorem.

THEOREM 25. *Let* $\varphi\colon \operatorname{Alg}\mathscr{L}_{2n} \to \operatorname{Alg}\mathscr{L}_{2n}$ *be a surjective isometry such that* $\varphi(I) = I$. *Let* $\varphi(E_{11}) = E_{ii}$. *If* i *is an odd number, then* $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}$. *If* i *is an even number, then* $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^{\perp}$.

Let $\varphi\colon \operatorname{Alg}\mathscr{L}_{2n} \to \operatorname{Alg}\mathscr{L}_{2n}$ be a surjective isometry such that $\varphi(I) = I$ and $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^{\perp}$. If J is the bijective conjugation which is defined below, then for all x, y in $\mathbf{C}^{2n}$ and all α in $\mathbf{C}$

(1) $J(x+y) = Jx + Jy$,
(2) $J(\alpha x) = \bar{\alpha} Jx$,
(3) $(Jx, Jy) = (y, x)$,
(4) $(Jx, y) = (Jy, x)$ and
(5) $J^2 = I$.

Define

$$J(x_1, x_2, \dots, x_{2n})^t = (\bar{x}_{2n}, \bar{x}_{2n-1}, \dots, \bar{x}_1)^t$$

for every $(x_1, x_2, \dots, x_{2n})^t$ in $\mathbf{C}^{2n}$.

If A is in $\operatorname{Alg}\mathscr{L}_{2n}$, then the map $A \to JA^*J$ is linear and "flips" A across the northeast-southwest diagonal (see Example 4).

Define $\varphi_1\colon \operatorname{Alg}\mathscr{L}_{2n} \to \operatorname{Alg}\mathscr{L}_{2n}$ by $\varphi_1(A) = JA^*J$ for every A in $\operatorname{Alg}\mathscr{L}_{2n}$. Then φ_1 is well-defined by the above statement, linear, φ_1 is a surjective isometry, and $\varphi_1(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^{\perp}$. If $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^{\perp}$, then define $\tilde{\varphi} = \varphi_1 \circ \varphi\colon \operatorname{Alg}\mathscr{L}_{2n} \to \operatorname{Alg}\mathscr{L}_{2n}$. Then $\tilde{\varphi}(\mathscr{L}_{2n}) = \varphi_1 \circ \varphi(\mathscr{L}_{2n}) = \varphi_1(\mathscr{L}_{2n}^{\perp}) = \mathscr{L}_{2n}$ by Lemma 23.

Since $(JAJ)^* = JA^*J$, $\varphi_1^{-1} = \varphi_1$ and we can get the following theorem.

THEOREM 26. *Let* $\varphi\colon \operatorname{Alg}\mathscr{L}_{2n} \to \operatorname{Alg}\mathscr{L}_{2n}$ *be a surjective isometry such that* $\varphi(I) = I$ *and* $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}^{\perp}$. *Then, there exist unitary operators* U *and* V *such that* $\tilde{\varphi}(A) = UAV$ *if and only if* $\varphi(A) = JV^*A^*U^*J$ *for every* A *in* $\operatorname{Alg}\mathscr{L}_{2n}$.

Let $\varphi\colon \operatorname{Alg}\mathscr{L}_\infty \to \operatorname{Alg}\mathscr{L}_\infty$ be a surjective isometry such that $\varphi(I) = I$, $\varphi(E_{ii}) = E_{ii}$ for each i; $i = 1, 2, \dots$ and $\varphi(\mathscr{L}_\infty) = \mathscr{L}_\infty$. Then by Corollary 20, there exists α_{ij} in $\mathbf{C}$ such that $\varphi(E_{ij}) = \alpha_{ij}E_{ij}$ for all E_{ij} in $\operatorname{Alg}\mathscr{L}_\infty(|i-j| = 1)$. Then we claim that there exists a diagonal unitary U such that $\varphi(E_{ij}) = UE_{ij}U^*$ for all E_{ij} in $\operatorname{Alg}\mathscr{L}_\infty(|i-j| = 1)$. Let U be a diagonal matrix whose (j,j)-component is $e^{i\theta_j}$ for all j $(j = 1, 2, \dots)$.

Then the equation $\varphi(E_{ij}) = UE_{ij}U^*$ holds for all E_{ij} in $\operatorname{Alg}\mathscr{L}_\infty$ provided the following system can be solved

$$\begin{aligned} e^{i(\theta_1-\theta_2)} &= \alpha_{12}. \\ e^{i(\theta_3-\theta_2)} &= \alpha_{32}. \\ e^{i(\theta_3-\theta_4)} &= \alpha_{34}. \\ &\vdots \end{aligned}$$

The equation can be solved recursively (θ_1 may be set equal to 0). From these facts, we can get the following theorem.

THEOREM 27. *If $\varphi\colon \operatorname{Alg}\mathscr{L}_\infty \to \operatorname{Alg}\mathscr{L}_\infty$ is a surjective isometry such that $\varphi(I) = I$, $\varphi(E_{ii}) = E_{ii}$ for all i $(i = 1, 2, \dots)$ and $\varphi(\mathscr{L}_\infty) = \mathscr{L}_\infty$, then there exists a diagonal unitary operator U whose (j, j)-component is $e^{i\theta_j}$ for all j $(j = 1, 2, \dots)$ such that $\varphi(A) = UAU^*$ for every A in* $\operatorname{Alg}\mathscr{L}_\infty$.

For the rest we will consider a surjective isometry such that $\varphi(\mathscr{L}_{2n}) = \mathscr{L}_{2n}$. As a special case, we first consider $n = 1$.

THEOREM 28. *Let $\varphi\colon \operatorname{Alg}\mathscr{L}_2 \to \operatorname{Alg}\mathscr{L}_2$ be a surjective isometry such that $\varphi(I) = I$ and $\varphi(E_{ii}) = E_{ii}$; $i = 1, 2$. Then there exists a unitary operator U such that $\varphi(A) = UAU^*$ for every A in* $\operatorname{Alg}\mathscr{L}_2$.

Proof. Let

$$U = \begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \quad \text{and} \quad \varphi(A) = \begin{bmatrix} a_{11} & b_{12} \\ 0 & a_{22} \end{bmatrix}.$$

Then there exists a complex number α such that $a_{12} = \alpha b_{12}$. This α depends only on φ (by linearity), not on the matrix entries. Note that $|\alpha| = 1$ because φ is an isometry. If we fix $e^{i\theta_1}$ and if we determine $e^{i\theta_2}$ such that $e^{i\theta_1}e^{-i\theta_2} = \alpha$, then $\varphi(A) = UAU^*$ for every A in $\operatorname{Alg}\mathscr{L}_2$.

LEMMA 29. *Let U be a unitary operator. Then $\|I + U\| = 2$ if and only if 1 is in $\sigma(U)$.*

PROPOSITION 30. *Let A be an $n \times n$ matrix $(n \geq 2)$ with 1 on the diagonal and just below it, 1 the $(1, n)$-component and 0 elsewhere. Then $\|A\| = 2$.*

Proof. Let U be an $n \times n$ matrix with 1 just below the diagonal, 1 the $(1, n)$-component and 0 elsewhere. Since $U(x_1, x_2, \dots, x_n)^t =$

$(x_n, x_1, \dots, x_{n-1})^t$ for every vector $(x_1, x_2, \dots, x_n)^t$ in $\mathbf{C}^n$, U is a unitary operator. Then $A = I + U$. Let X be a vector in $\mathbf{C}^n$ all of whose entries are 1. Then since $UX = X$, 1 is in $\sigma(U)$. So $\|A\| = 2$ by Lemma 29.

PROPOSITION 31. *Let U be an $n \times n$ matrix with t_i the $(i+1, i)$-component and t_n the $(1, n)$-component $(i = 1, 2, \dots, n-1)$. If 1 is in $\sigma(U)$ and $|t_i| = 1$ for every i; $i = 1, 2, \dots, n$, then U is a unitary operator and $\prod_{i=1}^n t_i = 1$.*

Proof. Since $U(x_1, x_2, \dots, x_n)^t = (t_n x_n, t_1 x_1, t_2 x_2, \dots, t_{n-1} x_{n-1})^t$ for every vector $(x_1, x_2, \dots, x_n)^t$ in $\mathbf{C}^n$, U is a unitary operator. Since 1 is in $\sigma(U)$, there exists a non zero vector $(x_1, x_2, \dots, x_n)^t$ such that

$$\begin{aligned} U(x_1, x_2, \dots, x_n)^t &= (t_n x_n, t_1 x_1, t_2 x_2, \dots, t_{n-1} x_{n-1})^t \\ &= (x_1, x_2, \dots, x_n)^t. \end{aligned}$$

So $t_n x_n = x_1$, $t_1 x_1 = x_2$, $t_2 x_2 = x_3, \dots, t_{n-1} x_{n-1} = x_n$. If $x_i = 0$ for some i $(1 \le i \le n)$, then $x_1 = x_2 = \cdots = x_n = 0$. So $x_i \neq 0$ for every i $(i = 1, 2, \dots, n)$. Then $(\prod_{i=1}^n t_i) \prod_{i=1}^n x_i = \prod_{i=1}^n x_i$. Hence, $\prod_{i=1}^n t_i = 1$.

PROPOSITION 32. *Let A be an $n \times n$ matrix with a_i the (i, i)-component $(i = 1, 2, \dots, n)$, s_j the $(j+1, j)$-component $(j = 1, 2, \dots, n-1)$, s_n the $(1, n)$-component and 0 elsewhere. If $|a_i| = |s_i| = 1$ $(i = 1, 2, \dots, n)$ and $\|A\| = 2$, then $\prod_{i=1}^n a_i = \prod_{i=1}^n s_i$.*

Proof. Let U be an $n \times n$ diagonal matrix whose (i, i)-component is a_i^{-1} for all i $(i = 1, 2, \dots, n)$. Then UA is the $n \times n$ matrix with 1 on the diagonal, $a_{i+1}^{-1} s_i$ the $(i+1, i)$-component $(i = 1, 2, \dots, n-1)$, $a_1^{-1} s_n$ the $(1, n)$-component and 0 elsewhere. Let V be an $n \times n$ matrix with $a_{i+1}^{-1} s_i$ the $(i+1, 1)$-component $(i = 1, 2, \dots, n-1)$, $a_1^{-1} s_n$ the $(1, n)$-component and 0 elsewhere. Then V is a unitary operator and $UA = I + V$. Since U is a unitary operator, $\|UA\| = \|A\| = \|I + V\| = 2$. By Lemma 29, 1 is in $\sigma(V)$. Since

$$|a_1^{-1} s_n| = |a_2^{-2} s_1| = |a_3^{-1} s_2| = \cdots = |a_n^{-1} s_{n-1}| = 1,$$

by Proposition 31,

$$\left(\prod_{i=1}^{n} (a_{i+1})^{-1} s_i \right) a_1^{-1} s_n = \left(\prod_{i=1}^{n} a_i^{-1} \right) \left(\prod_{i=1}^{n} s_i \right) = 1.$$

Hence $\prod_{i=1}^n a_i = \prod_{i=1}^n s_i$.

LEMMA 33. *Let* φ: $\mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ *be a surjective isometry such that* $\varphi(E_{ii}) = E_{ii}$ *for each* i; $i = 1, 2, \ldots, 2n$ *and* $n \geq 2$. *Let* $\varphi(E_{ij}) = \alpha_{ij}E_{ij}$ *for all* E_{ij} *in* $\mathrm{Alg}\mathscr{L}_{2n}$, *where* $|\alpha_{ij}| = 1$ *for all* i, j. *Then*

$$\alpha_{12}\bar{\alpha}_{32}\alpha_{34}\bar{\alpha}_{54}\alpha_{56}\cdots\alpha_{2n-1,2n}\bar{\alpha}_{1,2n} = 1.$$

Proof. Let A be a $2n \times 2n$ matrix with 1 the $(2i-1, 2i)$-component $(i = 1, 2, \ldots, n)$ and the $(2j+1, 2j)$-component $(j = 1, 2, \ldots, n-1)$ and the $(1, 2n)$-component, and 0 elsewhere. Then, by hypothesis, $\varphi(A) = (\alpha_{ij})$. Let B be the $n \times n$ matrix with 1 on the diagonal and just below it, 1 the $(1, n)$-component and 0 elsewhere. Note that the $n \times n$ matrix B and the $2n \times 2n$ matrix A have the same norm. Let D be the $n \times n$ matrix with $\alpha_{2i-1,2i}$ the (i, i)-component $(i = 1, 2, \ldots, n)$, $\alpha_{1,2n}$ the $(1, n)$-component, $\alpha_{2j+1,2j}$ the $(j+1, j)$-component $(j = 1, 2, \ldots, n-1)$ and 0 elsewhere. Then $\|D\| = \|\varphi(A)\|$. Since φ preserves norm, $\|A\| = \|\varphi(A)\|$. So $\|B\| = \|D\|$. By Proposition 30 $\|B\| = 2$ and hence $\|D\| = 2$. Since $|\alpha_{2i-1,2i}| = |\alpha_{2i-1,2i-2}| = 1$ for each i; $i = 1, 2, \ldots, n$.

$$\alpha_{12}\bar{\alpha}_{32}\alpha_{34}\bar{\alpha}_{54}\alpha_{56}\cdots\bar{\alpha}_{2n-1,2n-2}\alpha_{2n-1,2n}\bar{\alpha}_{1,2n} = 1$$

by Proposition 32.

THEOREM 34. *Let* φ: $\mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ *be a surjective isometry such that* $\varphi(E_{ii}) = E_{ii}$ *for each* i; $i = 1, 2, \ldots, 2n$ *and* $n \geq 2$. *Then there exists a unitary operator* V *such that* $\varphi(A) = VAV^*$ *for every* A *in* $\mathrm{Alg}\mathscr{L}_{2n}$.

Proof. Let $A = (a_{ij})$ be in $\mathrm{Alg}\mathscr{L}_{2n}$ and let $\varphi(E_{ij}) = \alpha_{ij}E_{ij}$ for all E_{ij} in $\mathrm{Alg}\mathscr{L}_{2n}$, where $|\alpha_{ij}| = 1$ for all α_{ij}.

Let V be a $2n \times 2n$ diagonal matrix whose (j, j)-component is $e^{i\theta_j}$ for all j $(j = 1, 2, \ldots, 2n)$. Then VAV^* is the $2n \times 2n$ matrix with a_{rr} the (r, r)-component $(r = 1, 2, \ldots, 2n)$, $e^{i\theta_p}a_{p,p+1}e^{-\theta_{p+1}}$ the $(p, p+1)$-component $(p = 1, 3, \ldots, 2n-1)$, $e^{i\theta_q}a_{q,q-1}e^{-i\theta_{q-1}}$ the $(q, q-1)$-component $(q = 3, 5, \ldots, 2n-1)$, $e^{i\theta_1}a_{1,2n}e^{-i\theta_{2n-1}}$ the $(1, 2n)$-component and 0 elsewhere.

So the theorem will be proved if we can determine $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_{2n}}$ satisfying the following relations;

$$\begin{aligned}
e^{i\theta_1}e^{-i\theta_2} &= \alpha_{12}.\\
e^{i\theta_3}e^{-i\theta_2} &= \alpha_{32}.\\
e^{i\theta_3}e^{-i\theta_4} &= \alpha_{34}.\\
&\vdots\\
e^{i\theta_{2n-1}}e^{-i\theta_{2n}} &= \alpha_{2n-1,2n}.\\
e^{i\theta_1}e^{-i\theta_{2n}} &= \alpha_{1,2n}.
\end{aligned}$$

Let $\alpha_{ij} = e^{i\theta}$ for all i, j such that E_{ij} is in $\operatorname{Alg}\mathscr{L}_{2n}$. Then $\theta_{12}, \theta_{32}, \theta_{34}, \dots, \theta_{2n-1,2n}$ and $\theta_{1,2n}$ are known by $\alpha_{12}, \alpha_{32}, \alpha_{34}, \dots, \alpha_{2n-1,2n}$ and $\alpha_{1,2n}$ respectively. It will suffice to solve the linear system; $(*) \dots, \theta_1 - \theta_2 = \theta_{12}$, $\theta_3 - \theta_2 = \theta_{32}, \dots, \theta_{2n-1} - \theta_{2n} = \theta_{2n-1,2n}$ and $\theta_1 - \theta_{2n} = \theta_{1,2n}$.

Let A be the matrix of coefficients of $(*)$ and let $A^1, A^2, \dots, A^{2n}$ be the column vectors of A. Let $B = (\theta_{12}, \theta_{32}, \theta_{34}, \dots, \theta_{2n-1,2n}, \theta_{1,2n})^t$. Then the system $(*)$ has solutions if and only if $\operatorname{rank} A = \operatorname{rank}(A^1, A^2, A^3, \dots, A^{2n}, B)$.

It is easy to check that the left hand side is $n-1$. Thus, the rank of the right hand side must be $n-1$ and the ranks will be equal if

$$\theta_{12} - \theta_{32} + \theta_{34} - \cdots + \theta_{2n-1,2n} - \theta_{1,2n} = 0.$$

But the last equation is the same as $\alpha_{12}\bar{\alpha}_{32}\alpha_{34}\bar{\alpha}_{54}\cdots\alpha_{2n-1,2n}\bar{\alpha}_{1,2n} = 1$, which we know to be true by Lemma 33. So $(*)$ has solutions. Hence $\varphi(A) = VAV^*$ for every A in $\operatorname{Alg}\mathscr{L}_{2n}$.

THEOREM 35. *If $\varphi\colon \operatorname{Alg}\mathscr{L}_{2n} \to \operatorname{Alg}\mathscr{L}_{2n}$ is a surjective isometry such that $\varphi(I) = I$, $\varphi(E_{11}) = E_{2i+1,2i+1}$, $\varphi(E_{22}) = E_{2i,2i}$, $\varphi(E_{33}) = E_{2i-1,2i-1}, \dots, \varphi(E_{2i-1,2i-1}) = E_{22}$, $\varphi(E_{2i,2i}) = E_{11}$, $\varphi(E_{2i+1,2i+1}) = E_{2n,2n}, \dots, \varphi(E_{2n,2n}) = E_{2i+2,2i+2}$. Then there exists a unitary operator W such that $\varphi(A) = WAW^*$ for all A in $\operatorname{Alg}\mathscr{L}_{2n}$.*

Proof. Let $U_{2i+1} = D_{2i+1} \oplus D_{2n-2i-1}$.

Define $\varphi_1\colon \operatorname{Alg}\mathscr{L}_{2n} \to \operatorname{Alg}\mathscr{L}_{2n}$ by $\varphi_1(A) = U_{2i+1}AU_{2i+1}^*$ for every A in $\operatorname{Alg}\mathscr{L}_{2n}$. where $U_{2i+1} = U_{2i+1}^*$. Then φ_1 is a surjective isometry because $U_{2i+1}AU_{2i+1}$ is in $\operatorname{Alg}\mathscr{L}_{2n}$ for every A in $\operatorname{Alg}\mathscr{L}_{2n}$. See Example 2. Define $\tilde{\varphi} = \varphi_1 \circ \varphi$. Then $\tilde{\varphi}(E_{ii}) = \varphi_1 \circ \varphi(E_{ii}) = E_{ii}$ for each i, $i = 1, 2, 3, \dots, 2n$. So there exists a unitary operator V such that $\tilde{\varphi}(A) = VAV^*$ for every A in $\operatorname{Alg}\mathscr{L}_{2n}$ by Theorem 34. Since $\tilde{\varphi}(A) = \varphi_1 \circ \varphi(A) = U_{2i+1}\varphi(A)U_{2i+1}^* = VAV^*$ for every A in $\operatorname{Alg}\mathscr{L}_{2n}$, $\varphi(A) = U_{2i+1}^*VAV^*U_{2i+1}$. Set $U_{2i+1}^*V = W$. Then $\varphi(A) = WAW^*$ for every A in $\operatorname{Alg}\mathscr{L}_{2n}$.

THEOREM 36. *If $\varphi\colon \operatorname{Alg}\mathscr{L}_{2n} \to \operatorname{Alg}\mathscr{L}_{2n}$ is a surjective isometry such that $\varphi(I) = I$, $\varphi(E_{11}) = E_{2i+1,2i+1}$,*

$$\begin{aligned}\varphi(E_{22}) &= E_{2i+2,2i+2}, \dots, \varphi(E_{2n-2i,2n-2i}) \\ &= E_{2n,2n}, \varphi(E_{2n-2i+1,2n-2i+1}) \\ &= E_{11}, \dots, \varphi(E_{2n,2n}) = E_{2i,2i},\end{aligned}$$

then there exists a unitary operator W such that $\varphi(A) = WAW^*$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$.

Proof. Let

$$V_{2n-2i+1} = \begin{bmatrix} 0 & I_{2n-2i} \\ I_{2i} & 0 \end{bmatrix}.$$

Define $\varphi_1 \colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ by $\varphi_1(A) = V_{2n-2i+1}AV^*_{2n-2i+1}$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$. Then since $V_{2n-2i+1}AV^*_{2n-2i+1}$ and $V^*_{2n-2i+1}AV_{2n-2i+1}$ are in $\mathrm{Alg}\mathscr{L}_{2n}$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$, φ_1 is a surjective isometry. See Example 3. Define $\tilde{\varphi} = \varphi_1 \circ \varphi$. Then $\tilde{\varphi}(E_{ii}) = E_{ii}$ for each i, $i = 1, 2, \ldots, 2n$. So there exists a unitary operator U such that $\tilde{\varphi}(A) = UAU^*$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$ by Theorem 34. Since $\tilde{\varphi}(A) = \varphi_1 \circ \varphi(A) = V_{2n-2i+1}\varphi(A)V^*_{2n-2i+1} = UAU^*$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$, $\varphi(A) = V^*_{2n-2i+1}UAU^*V_{2n-2i+1}$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$. Set $V^*_{2n-2i+1}U = W$. Then $\varphi(A) = WAW^*$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$.

THEOREM 37. *If $\varphi \colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ is a surjective isometry such that $\varphi(I) = I$, $\varphi(E_{11}) = E_{11}$, $\varphi(E_{22}) = E_{2n,2n}$, $\varphi(E_{33}) = E_{2n-1,2n-1}, \ldots$, $\varphi(E_{2i-1,2i-1}) = E_{2n-(2i-1-2),2n-(2i-1-2)}, \ldots, \varphi(E_{2n,2n}) = E_{22}$, then there exists a unitary operator W such that $\varphi(A) = WAW^*$ for every A in* $\mathrm{Alg}\mathscr{L}_{2n}$.

Proof. Let $U = D_1 \oplus D_{2n-1}$. Define $\varphi_1 \colon \mathrm{Alg}\mathscr{L}_{2n} \to \mathrm{Alg}\mathscr{L}_{2n}$ by $\varphi_1(A) = UAU^*$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$, where $U = U^*$. Then φ_1 is a surjective isometry because UAU is in $\mathrm{Alg}\mathscr{L}_{2n}$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$. Define $\tilde{\varphi} = \varphi_1 \circ \varphi$. Then $\tilde{\varphi}(E_{ii}) = \varphi_1 \circ \varphi(E_{ii}) = E_{ii}$ for each i, $i = 1, 2, \ldots, 2n$. So there exists a unitary operator V such that $\tilde{\varphi}(A) = VAV^*$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$ by Theorem 34. Since $\tilde{\varphi}(A) = \varphi_1 \circ \varphi(A) = U\varphi(A)U^* = VAV^*$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$, $\varphi(A) = U^*VAV^*U$. Set $U^*V = W$. Then $\varphi(A) = WAW^*$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$.

The last three theorems exhaust all possible cases where $\varphi(E_{11}) = E_{kk}$ and k is an odd number. Then the last three theorems show that there exists a diagonal unitary operator U such that $\varphi(A) = UAU^*$ for every A in $\mathrm{Alg}\mathscr{L}_{2n}$. If k is an even number, then Theorem 26 and the last three theorems show that there exists a unitary operator W and a conjugation J such that $\varphi(A) = JWA^*W^*J$ for each A in $\mathrm{Alg}\mathscr{L}_{2n}$. If $\varphi(I) = U \neq I$, then the reduction following Lemma 8 shows that there exists a unitary U so that the isometry $\hat{\varphi}(A) = U^*\varphi(A)$ has one of the above two forms. Thus the main theorem has been proved.

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