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SOMMES EXPONENTIELLES DONT LA GEOMETRIE EST TRES BELLE: p-ADIC ESTIMATES

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In the present work we examine a family of multivariable exponential sums on a connected variety defined over a finite field.

0. Introduction. Let $K = \mathbb{F}_q$ be the field with q elements ($\operatorname{char} K = p \neq 2$, $q = p^{\ell}$), $\overline{x} \in K^{\times}$, $g_1, \dots, g_n$ positive integers relatively prime and prime to p ($n \geq 2$) and let $\mathscr{V}_{\overline{x}}$ be the variety defined over K by $\prod_{i=1}^{n} t_i^{g_i} = \overline{x}$. Let Ω be a complete algebraically closed field containing $\mathbb{Q}_p$, $\Theta: K \to \Omega^{\times}$ an additive character and for each $i \in \{1, \dots, n\}$ let $\chi_i: K^{\times} \to \Omega^{\times}$ be a multiplicative character. Let $\overline{c}_1, \dots, \overline{c}_n$ be non-zero elements of K, and let $\overline{f}(t) = \sum_{i=1}^{n} \overline{c}_i t_i^{k_i}$, where $k_1, \dots, k_n$ are positive integers prime to p. For each $m \in \mathbb{Z}_+$ let K_m be the extension of K of degree m. We consider the twisted exponential sums

$$S_m(\overline{f}, \mathscr{V}_{\overline{x}}) = \sum_{(\overline{t}_1, \dots, \overline{t}_n) \in \mathscr{V}_{\overline{x}}(K_n)} \prod_{i=1}^{n} \chi_i \circ N_{K_{m/K}}(\overline{t}_i) \times \Theta \circ \operatorname{Tr}_{K_{m/K}}(\overline{f}(\overline{t})) \tag{0.1}$$

and the associated L function:

$$L = L(\overline{f}, \mathscr{V}_{\overline{x}}, T) = \exp\left(- \sum_{m=1}^{\infty} S_m(\overline{f}, \mathscr{V}_{\overline{x}}) T^m / m \right). \tag{0.2}$$

Our main results are the following:

A. We show that $L^{(-1)^n}$ is a polynomial of degree

$$h = \left(\sum_{i=1}^{n} g_i / k_i \right) \prod_{i=1}^{n} k_i.$$

B. We compute explicitly a lower bound for the Newton polygon of $L^{(-1)^n}$; this lower bound is independent of the prime number p and its endpoints coincide with those of the Newton polygon (Theorem 5.1 and Corollary 5.1).

C. Provided p lies in certain congruence classes, we show that our lower bound is in fact the exact Newton polygon of $L^{(-1)^n}$ (Theorem 5.3).

D. As a consequence we obtain p-adic estimates for the sums (0.1), since they are related to the reciprocal roots $\{\gamma_i\}_{i=1}^h$ of (0.2) by the equation

$$(0.3) \qquad S_m(\overline{f}, \mathscr{V}_{\overline{x}}) = (-1)^{n+1}(\gamma_1^m + \cdots + \gamma_h^m).$$

We emphasize that our lower bound for the Newton polygon can be computed explicitly: To fix notations, we assume that the multiplicative characters χ_i are of the form $\chi_i(t) = \omega(t)^{-(q-1)\rho_i/r}$, where r and ρ_i are natural integers, $r|q-1$, $0 \leq \rho_i < r$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, let $\sigma(\alpha) = \operatorname{Inf}_i \alpha_i/g_i$ and $J(\alpha) = \frac{1}{r}\sum_{i=1}^n \alpha_i/k_i$. Let $\widetilde{\Delta}'_\rho$ be the finite subset of $\mathbb{Z}^n$ defined by

$$\alpha \in \widetilde{\Delta}'_\rho \Leftrightarrow \begin{cases} 0 \leq \sigma(\alpha) < r \\ \alpha_i \equiv \rho_i \pmod{r}, \qquad i = 1, \ldots, n \\ \sigma(\alpha) \leq \alpha_i/g_i \leq \sigma(\alpha) + rk_i/g_i, \qquad i = 1, \ldots, n. \end{cases}$$

Whenever two elements α and β of $\widetilde{\Delta}'_\rho$ satisfy $J(\alpha) = J(\beta)$ and $\alpha_i \equiv \beta_i \pmod{k_i}$ for all i, we only keep the first of these two elements for the lexicographic order and eliminate the other: let $\widetilde{\Delta}_\rho$ be the resulting set. $\widetilde{\Delta}_\rho$ contains $h = (\sum_{i=1}^n g_i/k_i)\prod_{i=1}^n k_i$ elements, and the slopes of our lower bound are the values on $\widetilde{\Delta}_\rho$ of the weight function $w(\alpha) = J(\alpha) - \frac{1}{r}\sigma(\alpha)\sum_{i=1}^n g_i/k_i$. For example, if $\mathscr{V}_{\overline{x}}$ is the variety $t_1t_2^2t_3^3 = 1$ and $\overline{f}(t) = t_1^3 + t_2^2 + t_3$, with trivial twisting characters χ_i, then L^{-1} is a polynomial of degree 26. When $p \equiv 1 \pmod{18}$ its reciprocal roots have p-adic ordinal 0, 1/3, 7/18, 4/9, 1/2, 2/3 (twice), 13/18, 7/9, 5/6, 8/9, 17/18, 1 (twice), 19/18, 10/9, 7/6, 11/9, 23/18, 4/3 (twice), 3/2, 14/9, 29/18, 5/3, 2. When $p \not\equiv 1 \pmod{18}$, the Newton polygon of L^{-1} lies above the Newton polygon whose sides have these slopes and their endpoints coincide.

If $n = 2$, $k_1 = k_2 = 1$, $g_1 = g_2 = 1$, and the twisting characters are trivial, the sum (0.1) is the Kloosterman sum, which was first investigated from a p-adic point of view by B. Dwork in **[9]**. More general situations have been studied by S. Sperber (**[13]**, **[14]**, **[15]**) and Adolphson-Sperber (**[1]**, **[2]**). We have made extensive use of the work of these authors, especially from **[15]**. On the other hand, using l-adic cohomology, P. Deligne **[6]** has shown, in the case $g_1 = \cdots = g_n = k_1 = \cdots = k_n = 1$, that the reciprocal roots $\{\gamma_i\}_{i=1}^h$ of $L^{(-1)^n}$ have complex absolute value $q^{n-1/2}$; this was later extended by N. Katz **[10]**—from whom we borrow the title of this article—to include the case $k_1 = \cdots = k_n$ and general $g_1, \ldots, g_n$. We complement here this result, by obtaining p-adic estimates for the γ_i's. Our approach departs from previous literature on the subject by the use of a new trace formula (Theorem 1.1) which provides a more balanced treatment and avoids the restriction $g_n = k_n = 1$ ([**4**], [**15**]).

Using Dwork's methods, we construct cohomology spaces $W_{x,\rho}$ on which a Frobenius map acts, $\overline{\mathscr{F}}_x\colon W_{x,\rho} \to W_{x^q,\rho}$. These spaces have dimension h, and if $x = x^q$ is a Teichmüller point, the eigenvalues of $\overline{\mathscr{F}}_x$ are the reciprocal zeros of (0.2). The choice of a good basis for the space $W_{x,\rho}$ is crucial in obtaining estimates for the Newton polygon of the L-function: its elements are those of the set $\{x^{-\sigma(\alpha)/r}t^{\alpha} | \alpha \in \widetilde{\Delta}_{\rho}\}$, chosen so as to minimize the weight function $w(\alpha)$.

Define $\rho^{(0)} = \rho, \rho^{(1)}, \dots, \rho^{(\ell)} = \rho$ by the conditions

$$\begin{cases} p\rho_i^{(j+1)} - \rho_i^{(j)} \equiv 0 & (\operatorname{mod} r) \\ 0 \le \rho_i^{(j)} < r & \forall i, j \end{cases}$$

For each $\alpha^{(j)} \in \widetilde{\Delta}_{\rho^{(j)}}$, there exist (Lemma 2.8) unique elements $\alpha^{(j+1)} \in \widetilde{\Delta}_{\rho^{(j+1)}}$ and $\delta^{(j)} \in \mathbb{Z}^n$ satisfying

$$\begin{cases} p\left(\dfrac{\alpha_i^{(j+1)}}{rk_i} - \sigma(\alpha^{(j+1)})\dfrac{g_i}{rk_i}\right) - \left(\dfrac{\alpha_i^{(j)}}{rk_i} - \sigma(\alpha^{(j)})\dfrac{g_i}{rk_i}\right) = \delta_i^{(j)} \\ 0 \le \delta_i^{(j)} < r \end{cases}$$

If $\alpha = \alpha^{(0)} \in \widetilde{\Delta}_{\rho}$, let $Z(\alpha) = \sum_{j=0}^{\ell-1} w(\alpha^{(j)})$. We show that the Newton polygon of $L^{(-1)^n}$ lies below that of $\mathscr{K}_{\rho}(T) = \prod_{\alpha\in\widetilde{\Delta}_{\rho}}(1 - p^{Z(\alpha)}T)$, and their endpoints coincide (Theorem 5.2 and Corollary 5.1). On the other hand, if $p \equiv 1 \ (\operatorname{mod} r)$, the Newton polygon of the L-function lies above that of $\mathscr{H}_{\rho}(T) = \prod_{\alpha\in\widetilde{\Delta}_{\rho}}(1 - q^{w(\alpha)}T)$ (Theorem 5.1). If furthermore $pg_i \equiv g_i \ \operatorname{mod}(k_i g_j)$ for all i, j, then $\mathscr{K}_{\rho}(T) = \mathscr{H}_{\rho}(T)$ and therefore their common Newton polygon is that of $L^{(-1)^n}$.

The precise determination of the Newton polygon in other congruence classes requires finer estimates for the Frobenius matrix. This question has been solved by Adolphson-Sperber ([**2**]) in the case $n = 2$, $g_1 = g_2 = 1$, $k_1 = k_2$. We expect to address this question more fully in a subsequent article.

In [**5**], we studied the deformation equation when $k_n = g_n = 1$. With only minor changes, this treatment can be reconciled with the point of view adopted here. Let us simply indicate that the deformation operator of [**5**, p. 9–04] should be replaced by

$$\eta_y = E_y + \pi M c_n \frac{d_n}{a_n} t_n^{d_n},$$

where

$$E_y(Y^\gamma t^\alpha) = \left(\gamma + M\frac{\alpha_n}{a_n}\right) Y^\gamma t^\alpha.$$

1. Trace formula. Let $g_1, \dots, g_n$ be positive integers ($n \geq 2$), $g = (g_1, \dots, g_n)$. We assume that g. c. d.$(g_1, \dots, g_n) = 1$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ we define:

$$(1.1) \qquad \begin{cases} \omega_{i,j}(\alpha) = \dfrac{\alpha_i}{g_i} - \dfrac{\alpha_j}{g_j}, & i, j = 1, \dots, n; \\ \sigma(\alpha) = \operatorname{Inf}\left\{\dfrac{\alpha_1}{g_1}, \dots, \dfrac{\alpha_n}{g_n}\right\}. \end{cases}$$

Let μ be a fixed positive integer; for any $\alpha \in \mathbb{Z}^n$ let $\phi_\alpha: \mathbb{Z}^n \to \mathbb{Z}/\mu\mathbb{Z}$ be the group homomorphism defined by $\phi_\alpha(\gamma_1, \dots, \gamma_n) = \sum_{i=1}^n \overline{\gamma_i \alpha_i}$.

LEMMA 1.1. *Let $\alpha \in \mathbb{Z}^N$; the following conditions are equivalent:*

(i) *There exists $\beta \in \mathbb{Z}^n$ such that $\omega_{i,j}(\alpha) = \mu\omega_{i,j}(\beta)$ for all $i, j = 1, \dots, n$.*

(ii) *There exist $\beta \in \mathbb{Z}^n$ and $l \in \{1, \dots, n\}$ such that $\omega_{i,l}(\alpha) = \mu\omega_{i,l}(\beta)$ for all $i = 1, \dots, n$.*

(iii) $\operatorname{Ker}(\phi_g) \subset \operatorname{Ker}(\phi_\alpha)$.

Proof. The equivalence of (i) and (ii) is obvious from the definitions. Suppose that α satisfies condition (ii) and let $\gamma = (\gamma_1, \dots, \gamma_n) \in \operatorname{Ker}(\phi_g)$. By assumption, $\alpha_i g_l = \alpha_l g_i + \mu(\beta_i g_l - \beta_l g_i)$ for all i, hence:

$$g_l \sum_{i=1}^n \gamma_i \alpha_i = \left(\sum_{i=1}^n \gamma_i g_i\right)(\alpha_l - \mu\beta_l) + \mu g_l \sum_{i=1}^n \gamma_i \beta_i.$$

Since $g_i(\alpha_l - \mu\beta_l) = g_l(\alpha_i - \mu\beta_i)$ for all i and g. c. d.$(g_1, \dots, g_n) = 1$, it follows that g_l divides $\alpha_l - \mu\beta_l$. Hence $\sum_{i=1}^n \gamma_i\alpha_i \equiv 0 \pmod{\mu}$ i.e. $\gamma \in \operatorname{Ker}(\phi_\alpha)$ and (ii)$\Rightarrow$(iii).

Suppose that $\operatorname{Ker}(\phi_g) \subset \operatorname{Ker}(\phi_\alpha)$ and, for $i = 1, \dots, n-1$, let $\tau_i =$ g. c. d.(g_i, g_n).

Since

$$\frac{g_n}{\tau_i} g_i - \frac{g_i}{\tau_i} g_n = 0,$$

our assumption implies the existence of integers $z_1, \dots, z_{n-1}$ satisfying

$$\frac{g_n}{\tau_i}\alpha_i - \frac{g_i}{\tau_i}\alpha_n = \mu z_i \quad \text{for all } i = 1, \dots, n-1.$$

Furthermore, for each such i, there are integers β_i and $\beta_n^{(i)}$ such that:

$$(1.2(\mathrm{i})) \qquad z_i = \beta_i \frac{g_n}{\tau_i} - \beta_n^{(i)} \frac{g_i}{\tau_i}.$$

Thus

$$\frac{\alpha_i}{g_i} - \frac{\alpha_n}{g_n} = \mu\left(\frac{\beta_i}{g_i} - \frac{\beta_n^{(i)}}{g_n}\right) \quad \text{for all } i = 1, \ldots, n-1.$$

Observe that, if $(\beta_i, \beta_n^{(i)})$ is a solution of equation (1.2(i)), then so is $(\beta_i + g_i/\tau_i, \beta_n^{(i)} + g_n/\tau_i)$. We must show the existence of solutions satisfying $\beta_n^{(1)} = \cdots = \beta_n^{(n-1)}$. Let $i, j \in \{1, \ldots, n-1\}$ with $i \neq j$:

$$\frac{\alpha_i}{g_i} - \frac{\alpha_j}{g_j} = \mu\left(\frac{\beta_n^{(j)} - \beta_n^{(i)}}{g_n} + \frac{\beta_i}{g_i} - \frac{\beta_j}{g_j}\right).$$

On the other hand, just as above, we can find integers ε_i and ε_j such that:

$$\frac{\alpha_i}{g_i} - \frac{\alpha_j}{g_j} = \mu\left(\frac{\varepsilon_i}{g_i} - \frac{\varepsilon_j}{g_j}\right).$$

Hence, letting $\delta_i = \beta_i - \varepsilon_i$, $\delta_j = \beta_j - \varepsilon_j$ and $\tau_{i,j} = \text{g. c. d.}(\tau_i, \tau_j)$ we can write:

$$(\beta_n^{(j)} - \beta_n^{(i)})\frac{g_i g_j \tau_{i,j}}{\tau_i \tau_j} = \frac{g_n \tau_{i,j}}{\tau_i \tau_j}(\delta_j g_i - \delta_i g_j).$$

Since $g_n\tau_{i,j}/\tau_i\tau_j$ and $g_i g_j \tau_{i,j}/\tau_i\tau_j$ are relatively prime, there exists $Z \in \mathbb{Z}$ such that

$$\beta_n^{(j)} - \beta_n^{(i)} = Z\frac{g_n \tau_{i,j}}{\tau_i \tau_j}.$$

In turn, there exist ξ, $\eta \in \mathbb{Z}$ such that $Z\tau_{i,j} = \xi\tau_i + \eta\tau_j$ and therefore

$$\beta_n^{(j)} - \beta_n^{(i)} = \xi\frac{g_n}{\tau_j} + \eta\frac{g_n}{\tau_i}.$$

If we let $r_k = g_n/\tau_k$ $(k = 1, \ldots, n-1)$, we have just proved that, for all $i, j \in \{1, \ldots, n-1\}$:

$$\beta_n^{(j)} - \beta_n^{(i)} \in r_i\mathbb{Z} + r_j\mathbb{Z}. \tag{1.3}$$

We now proceed by induction. Let $k < n-1$ and suppose that we have found solutions $(\widetilde{\beta}_i, \widetilde{\beta}_n^{(i)})$ of equations (1.2(i)) for all i, with the property that $\widetilde{\beta}_n^{(1)} = \cdots = \widetilde{\beta}_n^{(k)}$ $(= \widetilde{\beta}_n)$.

Let $m_k = \text{l. c. m.}(r_1, \ldots, r_k)$. By (1.3), $\widetilde{\beta}_n - \widetilde{\beta}_n^{(k+1)} \in m_k\mathbb{Z} + r_{k+1}\mathbb{Z}$ and therefore there are integers λ, ζ such that $\widetilde{\beta}_n + \lambda m_k = \widetilde{\beta}_n^{(k+1)} + \zeta r_{k+1}$.

Let:

$$\begin{cases} \beta_n^{(i)} = \widetilde{\beta}_n^{(i)} + \lambda m_k & 1 \le i \le k \\ \beta_i = \widetilde{\beta}_i + \lambda \dfrac{g_i}{g_n} m_k & 1 \le i \le k \\ \beta_n^{(k+1)} = \widetilde{\beta}_n^{(k+1)} + \zeta r_{k+1} & \\ \beta_{k+1} = \widetilde{\beta}_{k+1} + \zeta \dfrac{g_{k+1}}{\tau_{k+1}} & \\ \beta_n^{(j)} = \widetilde{\beta}_n^{(j)} & j > k+1 \\ \beta_j = \widetilde{\beta}_j & j > k+1 \end{cases}$$

For each $i = 1, \ldots, n-1$, $(\beta_i, \beta_n^{(i)})$ is a solution of (1.2(i)) and we have $\beta_n^{(1)} = \cdots = \beta_n^{(k+1)}$. Finally we obtain $\beta = (\beta_1, \ldots, \beta_n)$ with $\omega_{i,n}(\alpha) = \mu\omega_{i,n}(\beta)\ \forall i = 1, \ldots, n$.

Hence (iii)$\Rightarrow$(ii). □

Notation. If $\alpha, \beta \in \mathbb{Z}^n$ satisfy $\omega_{i,j}(\alpha) = \mu\omega_{i,j}(\beta)$ for all $i, j = 1, \ldots, n$ we shall write:

$$\omega(\alpha) = \mu\omega(\beta). \tag{1.4}$$

REMARK 1.1. Let $\alpha, \beta \in \mathbb{Z}^n$ satisfying (1.4) and let $l \in \{1, \ldots, n\}$, then

$$\sigma(\alpha) = \frac{\alpha_l}{g_l} \Leftrightarrow \sigma(\beta) = \frac{\beta_l}{g_l}. \tag{1.5}$$

Let:

$$S = \{\alpha \in \mathbb{Z}^n \mid 0 \le \sigma(\alpha) < 1\}. \tag{1.6}$$

LEMMA 1.2. *Let $\alpha, \beta \in S$; then $\alpha = \beta \Leftrightarrow \omega(\alpha) = \omega(\beta)$.*

Proof. The first implication is obvious. Conversely, suppose that $\omega(\alpha) = \omega(\beta)$ and let l be an index such that $\sigma(\alpha) = \alpha_l/g_l$. By the remark above, $\sigma(\beta) = \beta_l/g_l$.

By assumption, $g_i(\alpha_l - \beta_l) = g_l(\alpha_i - \beta_i)$ for all i. If $\gamma_1, \ldots, \gamma_n$ are integers satisfying $\sum_{i=1}^n \gamma_i g_i = 1$, then $\alpha_l - \beta_l = g_l \sum_{i=1}^n \gamma_i(\alpha_i - \beta_i)$ and therefore g_l divides $\alpha_l - \beta_l$.

Since α and β are elements of S, $-g_l < \alpha_l - \beta_l < g_l$, hence $\alpha_l = \beta_l$ and it follows that $\alpha_i = \beta_i$ for all i. □

We fix r, a positive integer, and for each $\alpha \in \mathbb{Z}^n$ we set

$$\mathscr{s}(\alpha) = \frac{1}{r}\sigma(\alpha). \tag{1.7}$$

Let:

$$E = \{\alpha \in \mathbb{Z}^n \mid 0 \leq \mathscr{s}(\alpha) < 1\} = \{\alpha \in \mathbb{Z}^n \mid 0 \leq \sigma(\alpha) < r\}. \tag{1.8}$$

If $\rho \in \mathbb{Z}^n$, with $0 \leq \rho_i < r$ we set

$$Z^{(\rho)} = \{\alpha \in \mathbb{Z}^n \mid \alpha_i \equiv \rho_i \pmod r \text{ for all } i\}, \tag{1.9}$$

$$E^{(\rho)} = Z^{(\rho)} \cap E. \tag{1.10}$$

LEMMA 1.3. *Let* $\alpha, \beta \in E^{(\rho)}$; *then* $\alpha = \beta \Leftrightarrow \omega(\alpha) = \omega(\beta)$.

Proof. Suppose that $\omega(\alpha) = \omega(\beta)$ and assume that $\alpha_l \geq \beta_l$ for some index l. Then $\alpha_i \geq \beta_i$ for all i and, letting $\gamma_i = (\alpha_i - \beta_i)/r$, $\gamma = (\gamma_1, \ldots, \gamma_n)$ is an element of S, with $\omega(\gamma) = 0$. Lemma 1.2 implies that $\gamma = (0, \cdots, 0)$. □

We now fix p, a prime number, with $(p, r) = 1$. If $\rho \in \mathbb{Z}^n$, $0 \leq \rho_i < r$, we let $\rho' \in \mathbb{Z}^n$ be the unique element satisfying

$$\begin{cases} 0 \leq \rho'_i < r, \\ p\rho'_i - \rho_i \equiv 0 \pmod r). \end{cases} \tag{1.11}$$

LEMMA 1.4. *Let* $\alpha \in Z^{(\rho)}$ *satisfying the equivalent conditions of Lemma* 1.1 *with* $\mu = p$. *Then, in* (i) *and* (ii), β *can be chosen uniquely so that*

(1) $\beta \in E^{(\rho')}$;
(2) $\mathscr{s}(\alpha) - p\mathscr{s}(\beta) \in \mathbb{Z}$.

Proof. Suppose that $\omega(\alpha) = p\omega(\delta)$. Certainly, δ may be chosen (uniquely) so that $0 \leq \sigma(\delta) < 1$. By Remark 1.1, $g_i(\sigma(\alpha) - p\sigma(\delta)) = \alpha_i - p\delta_i \; \forall i$. Let $\gamma_1, \ldots, \gamma_n$ be integers satisfying $\sum_{i=1}^n \gamma_i g_i = 1$:

$$\sum_{i=1}^n g_i\gamma_i(\sigma(\alpha) - p\sigma(\beta)) = \sum_{i=1}^n \gamma_i(\alpha_i - p\delta_i),$$

hence $\sigma(\alpha) - p\sigma(\delta) \in \mathbb{Z}$. In particular, $p\delta - \alpha$ belongs to the cyclic subgroup of $\mathbb{Z}^n$ generated by g. Since g. c. d. $(p, r) = 1 =$ g. c. d.$(g_1, \ldots, g_n)$, there is a unique integer λ, $0 \leq \lambda < r$, such that $p(\delta + \lambda g) - \alpha \in r\mathbb{Z}^n$. Now set $\beta = \delta + \lambda g$. □

Let $\mathbb{Q}_p$ be the completion of the field of rational numbers for the p-adic valuation, and Ω an algebraically closed field containing $\mathbb{Q}_p$. We denote by "ord" the valuation on Ω normalized so that $\operatorname{ord} p = 1$. Let $\mathscr{f}$ be a positive integer such that $r \mid p^{\mathscr{f}} - 1$, let $q = p^{\mathscr{f}}$ and let

$x \in \Omega^\times$ be a Teichmüller point: $x^q = x$. Let K be an extension of $\mathbb{Q}_p$ in Ω containing x. Let $t_1, \dots, t_n$ be indeterminates. We shall use multi-index notation: if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $t^\alpha = t_1^{\alpha_1} \dots t_n^{\alpha_n}$.

Fix $k_1, \dots, k_n$ positive integers. Given $b, c \in \mathbb{R}$ with $b \geq 0$, let:

$$(1.12)\ \mathscr{L}(b,c) = \left\{ \xi = \sum_{\alpha \in \mathbb{N}^n} B_\alpha t^\alpha \mid B_\alpha \in K \text{ and } \operatorname{ord} B_\alpha \geq b \sum_{i=1}^n \frac{\alpha_i}{k_i} + c \right\};$$

$$(1.13) \qquad \mathscr{L}(b) = \bigcup_{c \in \mathbb{R}} \mathscr{L}(b,c).$$

For each $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{Z}^n$ with $0 \leq \rho_i < r$ we let

$$(1.14) \qquad \mathscr{L}_\rho(b,c) = \left\{ \xi = \sum B_\alpha t^\alpha \in \mathscr{L}(b,c) \mid B_\alpha = 0 \text{ if } \alpha \notin Z^{(\rho)} \right\};$$

$$(1.15) \qquad \mathscr{L}_\rho(b) = \bigcup_{c \in \mathbb{R}} \mathscr{L}_\rho(b,c).$$

$\mathscr{L}(b,c), \mathscr{L}(b), \mathscr{L}_\rho(b,c), \mathscr{L}_\rho(b)$ are p-adic Banach spaces with the norm

$$\|\xi\| = \operatorname{Sup}_\alpha p^{c_\alpha}, \qquad c_\alpha = b \sum_{i=1}^n \frac{\alpha_i}{k_i} - \operatorname{ord} B_\alpha.$$

Let $\mathscr{N} = \sum_{i=1}^n g_i/k_i$ and

$$(1.16)\ \ \overline{\mathscr{L}}(b,c) = \left\{ \eta = \sum_{\alpha \in E} C_\alpha t^\alpha \mid C_\alpha \in K \quad \text{and} \right.$$
$$\left. \operatorname{ord} C_\alpha \geq b \left(\sum_{i=1}^n \frac{\alpha_i}{k_i} - \mathscr{N} \sigma(\alpha) \right) + c \right\};$$

$$(1.17) \qquad \overline{\mathscr{L}}(b) = \bigcup_{c \in \mathbb{R}} \overline{\mathscr{L}}(b,c);$$

$$(1.18)\ \ \overline{\mathscr{L}}_\rho(b,c) = \left\{ \eta = \sum_{\alpha \in E} C_\alpha t^\alpha \in \overline{\mathscr{L}}(b,c) \mid C_\alpha = 0 \text{ if } \alpha \notin E^{(\rho)} \right\};$$

$$(1.19) \qquad \overline{\mathscr{L}}_\rho(b) = \bigcup_{c \in \mathbb{R}} \overline{\mathscr{L}}_\rho(b,c).$$

$\overline{\mathscr{L}}(b,c), \overline{\mathscr{L}}(b), \overline{\mathscr{L}}_\rho(b,c), \overline{\mathscr{L}}_\rho(b)$ are p-adic Banach spaces with the norm

$$\|\eta\| = \operatorname{Sup}_\alpha p^{c_\alpha}, \quad c_\alpha = b \left(\sum_{i=1}^n \frac{\alpha_i}{k_i} - \mathscr{N} \sigma(\alpha) \right) - \operatorname{ord} B_\alpha.$$

If $\alpha, \beta \in \mathbb{Z}^n$, there exist $\tau \in \mathbb{Z}$ and $\delta \in E$, uniquely defined, such that $\alpha + \beta = \delta + \tau r g$ and we set

$$t^{\alpha} * t^{\beta} = x^{\tau} t^{\delta}. \tag{1.20}$$

Since $\sigma(\alpha+\beta) \geq \sigma(\alpha) + \sigma(\beta)$ and $\sigma(\delta + \tau r g) = \sigma(\delta) + \tau r$, this operation makes $\overline{\mathscr{L}}(b)$ (respectively $\overline{\mathscr{L}}_{\rho}(b)$) into a K-algebra; if ζ is an element of $\overline{\mathscr{L}}(b, c')$, then $\eta \to \zeta * \eta$ maps $\overline{\mathscr{L}}(b,c)$ continuously into $\overline{\mathscr{L}}(b, c+c')$.

Let ϕ be the K-linear map whose action on monomials is given by

$$\phi(t^{\alpha}) = t_1^{\alpha_1} * t_2^{\alpha_2} * \cdots * t_n^{\alpha_n}. \tag{1.21}$$

For each ρ, ϕ is a continuous algebra homomorphism from $\mathscr{L}_{\rho}(b,c)$ into $\overline{\mathscr{L}}(b,c)$. If $\alpha \in Z^{(\rho)}$ we define

$$\psi(t^{\alpha}) = \begin{cases} x^{s(\alpha) - p s(\beta)} t^{\beta} & \text{if } \exists \beta \in E^{(\rho')} \text{ such that } \omega(\alpha) = p\omega(\beta), \\ 0 & \text{otherwise.} \end{cases} \tag{1.22}$$

Note that if $\alpha, \beta \in \mathbb{Z}^n$, then

$$\psi(t^{\alpha} * t^{\beta}) = \psi(t^{\alpha+\beta}). \tag{1.23}$$

It follows from Lemma 1.4 that ψ extends to a continuous linear map from $\overline{\mathscr{L}}_{\rho}(b,c)$ into $\overline{\mathscr{L}}_{\rho'}(pb,c)$. Since $r \mid q-1$, ψ^{f} maps $\overline{\mathscr{L}}_{\rho}(b,c)$ into $\overline{\mathscr{L}}_{\rho}(qb,c)$. If $b' > b$, then $\overline{\mathscr{L}}_{\rho}(b',c)$ is a subspace of $\overline{\mathscr{L}}_{\rho}(b,c)$ and the canonical injection $i: \overline{\mathscr{L}}_{\rho}(b',c) \to \overline{\mathscr{L}}_{\rho}(b,c)$ is completely continuous [**12**, §9].

We fix $F(t) = \sum_{\alpha \in \mathbb{N}^n} B_{\alpha} t^{\alpha}$ an element of $\mathscr{L}(rb)$ and we let $\overline{F}(t) = \phi(F(t^r)) \in \overline{\mathscr{L}}_0(b)$. We define $\mathscr{F}_{\rho}$ to be the composition:

$$\overline{\mathscr{L}}_{\rho}(qb) \xrightarrow{i} \overline{\mathscr{L}}_{\rho}(b) \xrightarrow{*\overline{F}(t)} \overline{\mathscr{L}}_{\rho}(b) \xrightarrow{\psi^{f}} \overline{\mathscr{L}}_{\rho}(qb).$$

By [**12**, §3], $\mathscr{F}_{\rho}$ is a completely continuous endomorphism of $\overline{\mathscr{L}}(qb)$. Its trace and Fredholm determinant are well defined and

$$\det(I - T\mathscr{F}_{\rho}) = \exp\left(-\sum_{m=1}^{\infty} \operatorname{tr}(\mathscr{F}_{\rho}^{m}) \frac{T^m}{m}\right) \text{ is a } p\text{-adic entire function.}$$

For $m \in \mathbb{N}^*$ we let

$$\mathscr{V}_m = \{(t_1, \ldots, t_n) \in K^n \mid t_i^{q^m - 1} = 1 \text{ and } t_1^{g_1} \times \cdots \times t_n^{g_n} = x\}. \tag{1.24}$$

THEOREM 1.1.

$$(q-1)^{n-1} \operatorname{tr}(\mathscr{F}_{\rho} \mid \overline{\mathscr{L}}_{\rho}(qb)) = \sum_{t \in \mathscr{V}_1} \left(\prod_{i=1}^{n} t_i^{-(q-1)\rho_i / r}\right) F(t).$$

Proof. Write $F(t) = \sum_{\alpha\in S}\sum_{\lambda\in\mathbb{N}} B_{\alpha+\lambda g} t^{\alpha+\lambda g}$. Let $G(t) = \sum_{\alpha\in S} C_\alpha t^\alpha$, with $C_\alpha = \sum_{\lambda\in\mathbb{N}} B_{\alpha+\lambda g} x^\lambda$. For each $i = 1, \dots, n$ let $\delta_i = -\rho_i(q-1)/r$ and set $X_\rho(t) = \prod_{i=1}^n t_i^{\delta_i}$. Then $\sum_{t\in\mathscr{Y}_1} X_\rho(t)F(t) = \sum_{t\in\mathscr{Y}_1} X_\rho(t)G(t)$.

On the other hand, $\overline{F}(t) = \phi(F(t^r)) = \sum_{\alpha\in S} C_\alpha t^{r\alpha} = G(t^r)$.

Note that for each $\beta \in \mathbb{Z}^n$ we can find $\gamma \in \mathbb{Z}^n$ such that $\omega(\gamma) = (q-1)\omega(\beta)$. Since $r \mid q-1$, we can choose γ so that $\gamma_i \equiv 0 \pmod r$ for all i. Furthermore, after adding or subtracting multiples of rg, we may assume that $\gamma \in E$. Accordingly, for each $\beta \in \mathbb{Z}^n$, we denote by $\widetilde{\beta}$ the unique (by Lemma 1.3) element of S satisfying $\omega(r\widetilde{\beta}) = (q-1)\omega(\beta)$.

For fixed $\beta \in E^{(\rho)}$,

$$\mathscr{F}_\rho(t^\beta) = \sum_{\alpha\in S} C_\alpha \psi^{\mathscr{I}}(t^{r\alpha} * t^\beta) = \sum C_\alpha x^{\mathscr{I}(r\alpha+\beta)-q\mathscr{I}(\gamma)} t^\gamma,$$

where the last sum is indexed by the set of all $\alpha \in S$ such that $\omega(r\alpha+\beta) = q\omega(\gamma)$, $\gamma \in E^{(\rho)}$. The coefficient of t^β in this sum is $C_{\widetilde{\beta}} x^{\mathscr{I}(r\widetilde{\beta})-(q-1)\mathscr{I}(\beta)}$, and therefore,

$$\operatorname{tr}(\mathscr{F}_\rho) = \sum_{\beta\in E^{(\rho)}} C_{\widetilde{\beta}} x^{\mathscr{I}(r\widetilde{\beta})-(q-1)\mathscr{I}(\beta)}. \tag{1.25}$$

There remains to show that $(q-1)^{n-1}\operatorname{tr}(\mathscr{F}_\rho) = \sum_{t\in\mathscr{Y}_1} X_\rho(t)G(t)$, and it is sufficient to check this when $G(t)$ is a single monomial, $G(t) = C_\alpha t^\alpha$. Let $G = (\mathbb{Z}/(q-1)\mathbb{Z})^n$; if $\overline{a} = (\overline{a}_1, \dots, \overline{a}_n)$ and $\overline{b} = (\overline{b}_1, \dots, \overline{b}_n)$ are two elements of G, we let $\overline{a}\bullet\overline{b} = \sum_{i=1}^n \overline{a}_i\overline{b}_i$. Fix ζ a primitive $(q-1)$-st root of unity. Since g.c.d.$(g_1, \dots, g_n) = 1$, we can find $\overline{\gamma} \in G$ such that $x = \zeta^{\overline{\gamma}\cdot\overline{g}}$. Let $H = \{\overline{\eta} \in G \mid \overline{\eta}\bullet\overline{g} = 0\}$:

$$\sum_{t\in\mathscr{Y}_1} X_\rho(t)t^\alpha = \zeta^{\overline{\gamma}\cdot(\overline{\delta}+\overline{\alpha})} \sum_{\eta\in H} \zeta^{\overline{\eta}\cdot(\overline{\delta}+\overline{\alpha})}.$$

The homomorphism from G into $\mathbb{Z}/(q-1)\mathbb{Z}$ sending $\overline{\eta} \in G$ into $\overline{\eta}\bullet\overline{g}$ is surjective, with kernel H; hence $|H| = (q-1)^{n-1}$. Furthermore, $\overline{\eta} \to \zeta^{\overline{\eta}\cdot(\overline{\delta}+\overline{\alpha})}$ is a character of H. Therefore

$$\sum_{\overline{\eta}\in H} \zeta^{\overline{\eta}\cdot(\overline{\delta}+\overline{\alpha})} = \begin{cases} (q-1)^{n-1} & \text{if } \overline{\eta}\bullet(\overline{\delta}+\overline{\alpha}) = \overline{0} \quad \forall\overline{\eta}\in H; \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 1.1, $\overline{\eta}\bullet(\overline{\delta}+\overline{\alpha}) = \overline{0}$ $\forall\overline{\eta} \in H$ if and only if there exists $\varepsilon \in \mathbb{Z}^n$ such that $\omega(\delta+\alpha) = (q-1)\omega(\varepsilon)$ or equivalently $\omega(r\alpha) = (q-1)\omega(r\varepsilon+\rho)$.

Thus $\overline{\eta} \bullet (\overline{\delta} + \overline{\alpha}) = \overline{0}\ \forall \overline{\eta} \in H$ if and only if there exists $\beta \in E^{(\rho)}$ (necessarily unique) such that $\omega(r\alpha) = (q-1)\omega(\beta)$. If so,

$$\alpha_i - \rho_i \frac{(q-1)}{r} \equiv g_i[\mathscr{s}(r\alpha) - (q-1)\mathscr{s}(\beta)] \ (\operatorname{mod} q-1) \quad \text{for all } i;$$

hence $\zeta^{\overline{\gamma}\cdot(\overline{\delta}+\overline{\alpha})} = x^{\mathscr{s}(r\alpha)-(q-1)\mathscr{s}(\beta)}$. □

LEMMA 1.5. *Let $F(t) \in \mathscr{L}(rb)$; then $\psi^{\ell} \circ (*\overline{F(t^q)}) = *\overline{F}(t) \circ \psi^{\ell}$.*

Proof. It is sufficient to check that, for a monomial t^β, $\beta \in \mathbb{Z}^n$:

$$\psi^{\ell}(t^{q\beta} * t^\alpha) = t^\beta * \psi^{\ell}(t^\alpha) \quad \text{for all } \alpha \in E.$$

$$\psi^{\ell}(t^{q\beta} * t^\alpha) = \begin{cases} x^{\mathscr{s}(q\beta+\alpha)-q\mathscr{s}(\delta)} t^\delta & \text{if } \omega(q\beta+\alpha) = q\omega(\delta); \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\omega(q\beta+\alpha) = q\omega(\delta)$. Then $\omega(\alpha) = q\omega(\delta-\beta)$; let $\lambda \in \mathbb{Z}$ be such that $\delta - \beta + \lambda rg = \gamma$ is an element of E:

$$\psi^{\ell}(t^\alpha) = x^{\mathscr{s}(\alpha)-q\mathscr{s}(\gamma)} t^\gamma; \quad \text{hence}$$
$$t^\beta * \psi^{\ell}(t^\alpha) = x^{\mathscr{s}(\alpha)-q\mathscr{s}(\gamma)+\lambda} t^\delta.$$

Suppose that $\sigma(\delta) = \delta_l/g_l$; Remark 1.1 shows that $\sigma(q\beta+\alpha) = (q\beta_l + \alpha_l)/g_l$. Thus,

$$\mathscr{s}(q\beta+\alpha) - q\mathscr{s}(\delta) = \frac{1}{rg_l}(q\beta_l + \alpha_l - q\delta_l) = \frac{1}{rg_l}(\alpha_l - q\gamma_l) + q\lambda.$$

Likewise, if $\sigma(\alpha) = \alpha_k/g_k$, then

$$\sigma(\gamma) = \frac{\gamma_k}{g_k} \quad \text{and} \quad \frac{1}{g_l}(\alpha_l - q\gamma_l) = \frac{1}{g_k}(\alpha_k - q\gamma_k).$$

Hence

$$\mathscr{s}(q\beta+\alpha) - q\mathscr{s}(\delta) \equiv \mathscr{s}(\alpha) - q\mathscr{s}(\gamma) + \lambda \qquad \operatorname{mod} q-1.$$ □

COROLLARY 1.1.

$$(q^m-1)^{n-1} \operatorname{tr}(\mathscr{F}_\rho^m \mid \overline{\mathscr{L}}_\rho(qb)) = \sum_{t \in \mathscr{T}_m} \left(\prod_{i=1}^n t_i^{-(q^m-1)\rho_i/r} \right) F(t)F(t^q)\cdots F(t^{q^{m-1}}).$$

2. Special subsets of $\mathbb{Z}^n$. Let $a = (a_1, \ldots, a_n)$ and $d = (d_1, \ldots, d_n)$ be two n-tuples of positive integers.

Let $M = \text{l.c.m.}(a_1, \ldots, a_n)$ and $D = \text{l.c.m.}(d_1, \ldots, d_n)$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ we let

$$s(\alpha) = \text{Inf}\left\{\frac{\alpha_1}{a_1}, \ldots, \frac{\alpha_n}{a_n}\right\}. \tag{2.1}$$

Let $J: \mathbb{Z}^n \to \frac{1}{D}\mathbb{Z}$ be the map defined by

$$J(\alpha) = \sum_{i=1}^{n} \frac{\alpha_i}{d_i}. \tag{2.2}$$

We define an equivalence relation on $\mathbb{Z}^n$ by setting:

(2.3) $\alpha \sim \alpha'$ if and only if $\alpha_i \equiv \alpha'_i \pmod{d_i}$ for all $i = 1, \ldots, n$.

There are $\prod_{i=1}^n d_i$ equivalence classes, which we call "congruence classes"; if $\alpha \in \mathbb{Z}^n$, we denote by $\overline{\alpha}$ its congruence class.

Let

$$\Delta' = \left\{\alpha \in \mathbb{Z}^n \mid s(\alpha) \leq \frac{\alpha_i}{a_i} \leq s(\alpha) + \frac{d_i}{a_i} \quad \forall i = 1, \ldots, n\right\}. \tag{2.4}$$

If α and β are two elements of Δ' we set

$$\begin{cases} \alpha \mathscr{R} \beta \text{ if and only if } \alpha \sim \beta \text{ and } J(\alpha) = J(\beta); \\ \Delta = \Delta'/\mathscr{R}. \end{cases} \tag{2.5}$$

We identify Δ with the subset of Δ' obtained by choosing, in each equivalence class for $\mathscr{R}$, the first element in lexicographic order.

LEMMA 2.1. *Let $\alpha \in \Delta$ and let $\beta \in \mathbb{Z}^n$ be such that $\beta \sim \alpha$ and $J(\beta) = J(\alpha)$; then*

$$s(\beta) \leq s(\alpha).$$

Proof. If $\beta \neq \alpha$, there is an index i such that $\beta_i < \alpha_i$. Since $\beta \sim \alpha$, we have in fact $\beta_i \leq \alpha_i - d_i$. Hence

$$\frac{\beta_i}{a_i} \leq \frac{\alpha_i}{a_i} - \frac{d_i}{a_i} \leq s(\alpha). \qquad \square$$

For each $i \in \{1, \ldots, n\}$ we denote by U_i the element of $\mathbb{Z}^n$ with 1 in the i-th position and 0 elsewhere.

LEMMA 2.2. *Let $K \in \frac{1}{D}\mathbb{Z}$ and let $\overline{\alpha}$ be a congruence class in $\mathbb{Z}^n$ such that $\overline{\alpha} \cap J^{-1}(K) \neq \emptyset$. Then there exists a unique element $\beta \in \Delta$ such that $\beta \in \overline{\alpha}$ and $J(\beta) = K$.*

Proof. Let $S(\overline{\alpha}, K) = \text{Max}\{s(\delta) \mid \delta \in \overline{\alpha} \text{ and } J(\delta) = K\}$.

Pick $\delta \in \overline{\alpha}$ with $J(\delta) = K$ and $s(\delta) = S(\overline{\alpha}, K)$.

If $\delta_i/a_i \leq s(\delta) + d_i/a_i$ for all i, then $\delta \in \Delta'$ so $\Delta' \cap J^{-1}(K) \neq \emptyset$ and we are done.

Suppose now that $\delta_i/a_i > s(\delta) + d_i/a_i$ for some index i and let k be the index such that δ_k/a_k is maximum among those satisfying the last inequality. Let also l be an index such that $s(\delta) = \delta_l/a_l$; note that necessarily $k \neq l$.

Let

$$\gamma = \delta - d_k U_k + d_l U_l: \quad \frac{\gamma_k}{a_k} > s(\delta) \quad \text{and} \quad \frac{\gamma_l}{a_l} > s(\delta).$$

Hence $s(\gamma) \geq s(\delta)$ and Lemma 2.1 implies $s(\gamma) = s(\delta)$.

Furthermore $\gamma_l/a_l = s(\gamma) + d_l/a_l$. Repeating the process if necessary, after a finite number of steps we obtain $\varepsilon \in \Delta' \cap \overline{\alpha}$ with $J(\varepsilon) = K$. □

Notation. If β satisfies the conditions of Lemma 2.2 we write

$$\beta = \tau(\overline{\alpha}, K). \tag{2.6}$$

Let

$$N = J(a) = \sum_{i=1}^{n} \frac{a_i}{d_i}. \tag{2.7}$$

Observe that $\alpha \in \Delta \Leftrightarrow \alpha + a \in \Delta$. Thus, if $\overline{\alpha} \cap J^{-1}(K) \neq \emptyset$:

$$\tau(\overline{\alpha}, K) + a = \tau(\overline{\alpha + a}, K + N). \tag{2.8}$$

LEMMA 2.3. *Let $K \in \frac{1}{D}\mathbb{Z}$ and let $\overline{\alpha}$ be a congruence class in $\mathbb{Z}^n$ such that $\overline{\alpha} \cap J^{-1}(K) \neq \emptyset$; let $\beta = \tau(\overline{\alpha}, K), \delta = \tau(\overline{\alpha}, K + 1)$; there exists an index $\lambda = \lambda(\overline{\alpha}, K) \in \{1, \ldots, n\}$ such that $\beta = \delta - d_\lambda U_\lambda$. Furthermore $s(\beta) = \beta_\lambda/a_\lambda$.*

Proof. Let

$$s = \max\left\{\frac{\delta_1 - d_1}{a_1}, \ldots, \frac{\delta_n - d_n}{a_n}\right\}$$

and let l be the smallest index such that $s = (\delta_l - d_l)/a_l$. Let $\gamma = \delta - d_l U_l$: for all $i \neq l$,

$$\frac{\delta_i}{a_i} \geq s(\delta) \geq \frac{\delta_l - d_l}{a_l} = \frac{\gamma_l}{a_l}, \quad \text{hence } s(\gamma) = \gamma_l/a_l = s.$$

Furthermore, for all $i \neq l$, $(\gamma_i - d_i)/a_i \leq s(\gamma)$ so $\gamma \in \Delta'$. Suppose that there exists $\varepsilon \in \Delta'$ such that $\varepsilon \mathscr{R} \gamma$ and ε precedes γ in the lexicographic ordering. Let j be the smallest index such that $\varepsilon_j \neq \gamma_j$; then $\varepsilon_j \leq \gamma_j - d_j$ and there exists $k > j$ such that $\varepsilon_k \geq \gamma_k + d_k$:

$$s(\varepsilon) \leq \frac{\varepsilon_j}{a_j} \leq \frac{\gamma_j - d_j}{a_j} \leq s(\gamma),$$
$$s(\gamma) \leq \frac{\gamma_k}{a_k} \leq \frac{\varepsilon_k - d_k}{a_k} \leq s(\varepsilon).$$

Hence $s(\gamma) = s(\varepsilon) = s$, $\varepsilon_j = \gamma_j - d_j$, $\varepsilon_k = \gamma_k + d_k$; in particular $s = (\gamma_j - d_j)/a_j$ so we must have $j \neq l$; hence $\varepsilon_j = \delta_j - d_j$ and therefore $j > l$. Let now $\delta' = \delta - d_j U_j + d_k U_k$:

$$s \leq \frac{\varepsilon_j}{a_j} = \frac{\delta_j - d_j}{a_j} \leq s(\delta)$$
$$s(\delta) \leq \frac{\delta_k}{a_k} = \frac{\gamma_k}{a_k} = \frac{\varepsilon_k - d_k}{a_k} = s.$$

Thus

$$s = s(\delta') = s(\delta) = \frac{\delta'_j}{a_j} = \frac{\delta_j - d_j}{a_j}.$$

Furthermore,

$$\frac{\delta'_i}{a_i} = \frac{\delta_i}{a_i} \leq s(\delta') + \frac{d_i}{a_i} \quad \text{if } i \neq j, k, \text{ and } \frac{\delta'_k}{a_k} = \frac{\delta_k + d_k}{a_k} = s(\delta') + \frac{d_k}{a_k}.$$

Hence $\delta' \in \Delta, \delta' \mathscr{R} \delta$ and δ' precedes δ in the lexicographic ordering. This contradicts the choice of δ. Hence $\gamma = \beta = \tau(\overline{\alpha}, K)$ and $l = \lambda(\overline{\alpha}, K)$. □

We now let

$$\widetilde{\Delta} = \{\alpha \in \Delta \mid 0 \leq s(\alpha) < 1\} \tag{2.9}$$

$$\overline{\Delta} = \{\alpha \in \Delta \mid 0 \leq J(\alpha) < N\} \tag{2.10}$$

Lemma 2.4. $|\widetilde{\Delta}| = |\overline{\Delta}|$.

Proof. We construct two maps:

$$\iota : \widetilde{\Delta} \to \overline{\Delta}$$
$$\iota^* : \overline{\Delta} \to \widetilde{\Delta}$$

Let $\alpha \in \widetilde{\Delta}$: we can find $\mu_\alpha \in \mathbb{N}$, $r_\alpha \in \frac{1}{D}\mathbb{N}$, unique such that $J(\alpha) = N\mu_\alpha + r_\alpha$ and we set:

$$\iota(\alpha) = \alpha - \mu_\alpha a. \tag{2.11}$$

Clearly, $\iota(\alpha) \in \Delta$ with $s(\iota(\alpha)) = s(\alpha) - \mu_\alpha$ and $0 \leq J(\iota(\alpha)) < N$; hence $\iota(\alpha) \in \overline{\Delta}$. If $\beta \in \overline{\Delta}$, there exist $\nu_\beta \in \mathbb{N}$ and $k_\beta < 1$ unique such that $s(\beta) = \nu_\beta + k_\beta$; we set:

$$\iota^*(\beta) = \beta - \nu_\beta a. \tag{2.12}$$

Clearly $\iota^*(\beta) \in \Delta$ with $0 \leq s(\iota^*(\beta)) < 1$, i.e. $\iota^*(\beta) \in \widetilde{\Delta}$.

It is now straightforward to check that ι and ι^* are inverse to each other. □

LEMMA 2.5. *Let* $\delta = \frac{1}{D}\prod_{i=1}^n d_i$. *If* $K \in \frac{1}{D}\mathbb{Z}$, *then* $J^{-1}(K)$ *meets exactly* δ *congruence classes in* $\mathbb{Z}^n$.

Proof. Let $G = \mathbb{Z}/d_i\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z}$ and let $H = \frac{1}{D}\mathbb{Z}/\mathbb{Z}$. $J: \mathbb{Z}^n \to \frac{1}{D}\mathbb{Z}$ induces a group homomorphism:

$$\overline{J}: G \to H. \tag{2.13}$$

It is sufficient to prove that $|\overline{J}^{-1}(h)| = \delta$ for any $h \in H$. Let

$$\delta_i = \prod_{\substack{1 \leq j \leq n \\ j \neq i}} d_i.$$

Observe that $\delta = \text{g.c.d.}\,(\delta_1, \ldots, \delta_n)$ and therefore there exist integers $\alpha_1, \ldots, \alpha_n$ such that $\delta = \sum_{i=1}^n \alpha_i \delta_i$. Dividing by $\prod_{i=1}^n d_i$ we obtain $\frac{1}{D} = \sum_{i=1}^n \alpha_i/d_i$, showing that $\overline{J}$ is surjective. Hence, for $h \in H$,

$$|J^{-1}(h)| = \frac{|G|}{|H|} = \frac{\prod_{i=1}^n d_i}{D} = \delta. \qquad \square$$

LEMMA 2.6. $|\widetilde{\Delta}| = N\prod_{i=1}^n d_i$.

Proof. By Lemma 2.5, $J^{-1}(K) \cap \Delta$ has exactly δ elements for each $K \in \frac{1}{D}\mathbb{Z}$. Hence, using the definition of $\overline{\Delta}$, $|\overline{\Delta}| = N\prod_{i=1}^n d_i$. The conclusion follows from Lemma 2.4. □

Let r be a fixed positive integer and let $g = (g_1, \ldots, g_n)$, $k = (k_1, \ldots, k_n)$ be n-tuples of positive integers, with $\text{g.c.d.}(g_1, \ldots, g_n) = 1$.

From now on we shall assume that $a_i = rg_i$ and $d_i = rk_i$ for all $i = 1, \ldots, n$. Thus, in (1.7) and (2.1):

$$s(\alpha) = \mathscr{s}(\alpha) \qquad \forall \alpha \in \mathbb{Z}^n. \tag{2.14}$$

If $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{Z}^n$, with $0 \le \rho_i < r$ we let

$$\Delta_\rho = \{\alpha \in \Delta \mid \alpha_i \equiv \rho_i \mod r\}; \tag{2.15}$$

$$\widetilde{\Delta}_\rho = \widetilde{\Delta} \cap \Delta_\rho; \tag{2.16}$$

$$\overline{\Delta}_\rho = \overline{\Delta} \cap \Delta_\rho. \tag{2.17}$$

LEMMA 2.7. $|\widetilde{\Delta}_\rho| = |\overline{\Delta}_\rho| = N \prod_{i=1}^n k_i$.

Proof. The map $\imath: \widetilde{\Delta} \to \overline{\Delta}$ of Lemma 2.4 restricts to a bijection between $\widetilde{\Delta}_\rho$ and $\overline{\Delta}_\rho$. Hence $|\widetilde{\Delta}_\rho| = |\overline{\Delta}_\rho|$. Let $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{Z}^n$, with $0 \le \eta_i < r$. If $\alpha \in \overline{\Delta}_\rho$ we let $\gamma = \alpha - \rho + \eta$. There is a unique integer λ_α such that $K_\alpha = J(\gamma) + \lambda_\alpha N$ satisfies $0 \le K_\alpha < N$, and we set $F_{\rho,\eta}(\alpha) = \overline{\tau(\gamma + \lambda_\alpha a, K_\alpha)}$. $F_{\rho,\eta}$ maps $\overline{\Delta}_\rho$ and $\overline{\Delta}_\eta$ and is easily seen to be injective. Hence, the r^n sets $\overline{\Delta}_\rho$, $0 \le \rho_i < r$, all have the same cardinality

$$|\overline{\Delta}_\rho| = \frac{1}{r^n}|\overline{\Delta}| = N \prod_{i=1}^n k_i. \qquad \square$$

LEMMA 2.8. *Let p be a prime number, with $(p, a_i) = (p, d_i) = 1$ for all i; let $\rho \in \mathbb{Z}^n$, with $0 \le \rho_i < r$ and let $\rho' \in \mathbb{Z}^n$ satisfying $0 \le \rho'_i < r$ and $p\rho'_i - \rho_i \equiv 0 \pmod r$ $\forall i$. If $\alpha' \in \widetilde{\Delta}_{\rho'}$, there exist $\alpha \in \widetilde{\Delta}_\rho$ and integers $\delta_1, \ldots, \delta_n$ uniquely determined by the conditions:*

$$\begin{cases} p\left(\dfrac{\alpha'_i}{d_i} - s(\alpha')\dfrac{a_i}{d_i}\right) - \left(\dfrac{\alpha_i}{d_i} - s(\alpha)\dfrac{a_i}{d_i}\right) = \delta_i, \\ 0 \le \delta_i < p - 1. \end{cases}$$

Furthermore:

(i) *Let $l \in \{1, \ldots n\}$, then*

$$s(\alpha) = \frac{\alpha_l}{a_l} \Leftrightarrow s(\alpha') = \frac{\alpha'_l}{a_l} \Leftrightarrow \delta_l = 0.$$

(ii) *$\alpha' \mapsto \alpha$ is a bijection between $\widetilde{\Delta}_{\rho'}$ and $\widetilde{\Delta}_\rho$.*

Proof. Certainly, using notation (1.4), there exists $\beta \in \mathbb{Z}^n$ such that $\omega(\beta) = p\omega(\alpha')$, and an argument similar to that of Lemma 1.4 shows that β can be chosen uniquely in $E^{(\rho)}$. Furthermore, if $s(\alpha') = \alpha'_l/a_l$, then $s(\beta) = \beta_l/a_l$. Since $\alpha' \in \widetilde{\Delta}$, we have

$$0 \le \frac{\alpha'_i}{a_i} - \frac{\alpha'_l}{a_l} \le \frac{d_i}{a_i},$$

hence

$$0 \le \frac{\beta_i}{a_i} - \frac{\beta_l}{a_l} \le p\frac{d_i}{a_i}$$

for all i.

If

$$\frac{\beta_i}{a_i} - \frac{\beta_l}{a_l} < p\frac{d_i}{a_i},$$

there is a unique integer δ_i, $0 \le \delta_i \le p-1$, such that

$$0 \le \frac{\beta_i - \delta_i d_i}{a_i} - \frac{\beta_l}{a_l} < \frac{d_i}{a_i}.$$

If

$$\frac{\beta_i}{a_i} - \frac{\beta_l}{a_l} = p\frac{d_i}{a_i}$$

we set $\delta_i = p-1$.

Now let $\alpha_i = \beta_i - \delta_i d_i$ for all i. It is straightforward to check that $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\delta = (\delta_1, \dots, \delta_n)$ have the required properties. □

LEMMA 2.9. *Let $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{N}^n$, with $0 \le \rho_i < r$. Then*

$$\sum_{\alpha \in \widetilde{\Delta}_\rho} w(\alpha) = N \prod_{i=1}^{n} k_i \frac{(n-1)}{2}.$$

Proof. Let $G = \prod_{i=1}^n \mathbb{Z}/d_i\mathbb{Z}$ and let $\not{p}: G \to (\mathbb{Z}/r\mathbb{Z})^n$ and $\not{q}: \mathbb{Z}^n \to G$ be the natural quotient maps. Let $\overline{\rho} = \not{p} \circ \not{q}(\rho)$ and $K_\rho = \not{p}^{-1}(\overline{\rho})$. Note that

$$|K_\rho| = \prod_{i=1}^{n} k_i, \quad \alpha \in \Delta_\rho \Leftrightarrow \alpha + a \in \Delta_\rho \quad \text{and} \quad \overline{\eta} \in K_\rho \Leftrightarrow \overline{\eta} + \not{q}(a) \in K_\rho.$$

Let H be the cyclic subgroup of G generated by $\not{q}(a)$ and let $\{G_l\}_{l=1}^{(G:H)}$ be the orbits of G under addition by elements of H: $G = \coprod_{l=1}^{(G:H)} G_l$. We have $K_\rho = \coprod_{K_\rho \cap G_l \neq \varnothing} G_l$ and $\overline{\Delta}_\rho = \coprod_{l=1}^{(G:H)} \overline{\Delta}_\rho(l)$, where $\overline{\Delta}_\rho(l) = \{\alpha \in \overline{\Delta} \mid \not{q}(\alpha) \in K_\rho \cap G_l\}$.

Let l be such that $K_\rho \cap G_l \neq \varnothing$ and let $\eta \in \overline{\Delta}_\rho(l)$ be such that $J(\eta)$ is minimum. Let $\varepsilon = |H|$; ε is the smallest integer such that

$\varepsilon a_i \equiv 0 \pmod{d_i}$ for all i. For any $\alpha \in \overline{\Delta}_\rho(l)$, there is a unique integer $\mu \in \mathbb{N}$ such that $0 \leq \mu < \varepsilon$ and $\alpha_i + \mu a_i \equiv \eta_i \pmod{d_i}$ for all i, and we have $J(\eta) \leq J(\alpha + \mu a) < J(\eta) + \varepsilon N$. Conversely, if $\beta \in \Delta$ satisfies $J(\eta) \leq J(\beta) < J(\eta) + \varepsilon N$ and $\beta_i \equiv \eta_i \pmod{d_i}$ for all i, there is a unique $\nu \in \mathbb{N}$, $0 \leq \nu < \varepsilon$ such that $J(\eta)+\nu N \leq J(\beta) < J(\eta)+(\nu+1)N$. Let $\gamma = \beta - \nu a$; then $J(\eta) \leq J(\gamma) < J(\eta) + N$. If $J(\gamma) \geq N$, then $J(\gamma - a) \geq 0$ and $J(\gamma - a) < J(\eta)$, contradicting the minimality of $J(\eta)$. Hence $\gamma \in \overline{\Delta}$.

Let $D_\rho(l) = \{\alpha \in \Delta | \alpha_i \equiv \eta_i \pmod{d_i}\ \forall i \text{ and } J(\eta) \leq J(\alpha) < J(\eta) + \varepsilon N\}$. Since $w(\alpha + a) = w(\alpha)$ for all $\alpha \in \mathbb{Z}^n$ we deduce that:

$$\sum_{\alpha \in \widetilde{\Delta}_\rho} w(\alpha) = \sum_{\alpha \in \overline{\Delta}_\rho} w(\alpha) = \sum_{l=1}^{(G:H)} \sum_{\alpha \in D_\rho(l)} w(\alpha).$$

It follows from Lemma 2.3 that $D_\rho(l) = \{\tau(\overline{\eta}, J(\eta) + k) \mid 0 \leq k \leq \varepsilon N - 1\}$. For each $k \in \mathbb{N}$, let $\alpha^{(k)} = \tau(\overline{\eta}, J(\eta) + k)$, $s_k = s(\alpha^{(k)})$, $J_k = J(\alpha^{(k)}) = J_0 + k$, $\lambda_k = \lambda(\overline{\eta}, J_k)$. By Lemma 2.3, $\alpha^{(k)} = \alpha^{(k-1)} + d_{\lambda_k} U_{\lambda_k}$ and $s_k = \alpha^{(k)}_{\lambda_{k+1}} / a_{\lambda_{k+1}}$. For each $i \in \{1, \dots n\}$ let μ_i be the integer satisfying $\varepsilon a_i = \mu_i d_i$. Since $\alpha^{(\varepsilon N)} = \eta + \varepsilon a$, it follows that $\varepsilon a = \sum_{k=1}^{\varepsilon N} d_{\lambda_k} U_{\lambda_k}$ and $\mu_i = \#\{k \mid 1 \leq k \leq \varepsilon N \text{ and } \lambda_k = i\}$.

We have

$$\begin{aligned}
\sum_{k=0}^{\varepsilon N-1} s_i &= \sum_{j=1}^{n} \sum_{\lambda_k = j} \alpha^{(k)}_{\lambda_{k+1}} / a_j = \sum_{j=1}^{n} \frac{1}{a_j} \left(\sum_{\nu=0}^{\mu_j - 1} \eta_j + \nu d_j \right) \\
&= \sum_{j=1}^{n} \left[\frac{\mu_j}{a_j} \left(\eta_j + \frac{(\mu_j - 1)}{2} d_j \right) \right] \\
&= \varepsilon \sum_{j=1}^{n} \left(\frac{\mu_j}{d_j} + \frac{\mu_j - 1}{2} \right) = \varepsilon \left(J_0 + \frac{\varepsilon N - n}{2} \right).
\end{aligned}$$

On the other hand:

$$\sum_{k=0}^{\varepsilon N-1} J_k = \varepsilon N J_0 + \frac{N(\varepsilon N - 1)}{2}.$$

Thus

$$\begin{aligned}
\sum_{\alpha \in D_\rho(l)} w(\alpha) &= \sum_{k=0}^{\varepsilon N-1} (J_k - N s_k) \\
&= \varepsilon N \frac{(n-1)}{2} = |K_\rho \cap G_l| N \frac{(n-1)}{2}.
\end{aligned}$$

Hence

$$\sum_{\alpha \in \widetilde{\Delta}_p} w(\alpha) = |K_p| N \frac{(n-1)}{2}. \qquad \square$$

3. Cohomology: The generic case.

a. *Definitions.* Let K_r be the unramified extension of $\mathbb{Q}_p$ in Ω of degree r, $\zeta_p \in \Omega$ a primitive p-th root of unity, $\Omega_0 = K_r(\zeta_p)$ and let $\tau \in \mathrm{Gal}(\Omega_0 \mid \mathbb{Q}_p(\zeta_p))$ denote the Frobenius automorphism. Let $\mathscr{O}_0$ be the ring of integers of Ω_0.

Let $M = \text{l.c.m.}(a_1, \dots, a_n)$ and, for $m \in \mathbb{N}^*$:

(3.1) $S_m = \{(\alpha; \gamma) \in \mathbb{N}^n \times \mathbb{Z} \mid \gamma \geq -mMs(\alpha)\}$;

(3.2) $E_m = \{(\alpha; \gamma) \in E \times \mathbb{Z} \mid \gamma \geq -mMs(\alpha)\}$;

(3.3) $A_m = \Omega_0$-algebra generated by $\{t^\alpha Y^\gamma \mid (\alpha; \gamma) \in S_m\}$;

(3.4) $P^{(m)} = t^a Y^{-mM} - 1$;

(3.5) $\overline{A}_m = A_m / (P^{(m)})$;

(3.6) $\mathscr{R}_m = \Omega_0$-span of $\{t^\alpha Y^\gamma \mid (\alpha; \gamma) \in E_m\}$.

If $\alpha \in \mathbb{Z}^n$, $\gamma \in \mathbb{Z}$, we set:

(3.7) $$w_m(\alpha; \gamma) = J(\alpha) + \frac{N\gamma}{mM}.$$

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(3.8) $w_m(\alpha; \gamma) \geq 0 \quad$ for all $(\alpha; \gamma) \in S_m$

(3.9) If $W \in \mathbb{Q}$, the set $\{(\alpha; \gamma) \in E_m \mid w_m(\alpha; \gamma) = W\}$ is finite.

If $\alpha, \beta \in \mathbb{Z}^n$, there exist $\delta = \delta(\alpha, \beta) \in E$, $\lambda = \lambda(\alpha, \beta) \in \mathbb{Z}$ unique, such that $\alpha + \beta = \delta + \lambda a$ and we set:

(3.10) $$t^\alpha *_m t^\beta = Y^{\lambda m M} t^\delta.$$

If $(\alpha; \gamma)$ and $(\beta; \varepsilon)$ are two elements of S_m, $\delta = \delta(\alpha, \beta)$, $\lambda = \lambda(\alpha, \beta)$ as above, then $(\delta, \gamma + \varepsilon + \lambda) \in E_m$. In particular, the operation $*_m$ makes $\mathscr{R}_m$ into an $\Omega_0[Y]$ algebra and, if we set

(3.11) $$\Phi_m(t^\alpha) = t_1^{\alpha_1} *_m t_2^{\alpha_2} *_m \cdots *_m t_n^{\alpha_n} \qquad (\alpha \in \mathbb{Z}^n),$$

then Φ_m extends to an $\Omega_0[Y]$-algebra homomorphism $\Phi_m \colon A_m \to \mathscr{R}_m$.

Furthermore, Φ_m induces an $\Omega_0[Y]$-algebra isomorphism.

(3.12) $$\overline{\Phi}_m \colon \overline{A}_m \xrightarrow{\sim} \mathscr{R}_m.$$

$A_m, \overline{A}_m, \mathscr{R}_m$ are graded algebras with

$$w_m(Y^\gamma t^\alpha) = w_m(\alpha;\gamma). \tag{3.13}$$

Both Φ_m and $\overline{\phi}_m$ are homogeneous of degree 0.

Note. When no confusion can arise, we shall omit the subscript "m" and write $*$ instead of $*_m$.

For $b, c \in \mathbb{R}$, $b \geq 0$, let

$$L(b,c) = \{\eta = \sum A(\alpha)t^\alpha \mid \alpha \in \mathbb{N}^n,\ A(\alpha) \in \Omega_0,\ \operatorname{ord} A(\alpha) \geq bJ(\alpha) + c\}; \tag{3.14}$$

$$L(b) = \bigcup_{c\in\mathbb{R}} L(b,c). \tag{3.15}$$

$L(b)$ and $L(b,c)$ are p-adic Banach spaces with the norm

$$||\eta|| = \operatorname*{Sup}_\alpha p^{-c_\alpha}, \qquad c_\alpha = \operatorname{ord} A(\alpha) - bJ(\alpha). \tag{3.16}$$

Let

$$L_m(b,c) = \left\{\xi = \sum B(\alpha;\gamma)t^\alpha Y^\gamma \mid (\alpha;\gamma) \in E_m,\ B(\alpha;\gamma) \in \Omega_0,\ \operatorname{ord} B(\alpha;\gamma) \geq bw_m(\alpha;\gamma) + c\right\}; \tag{3.17}$$

$$L_m(b) = \bigcup_{c\in\mathbb{R}} L_m(b,c). \tag{3.18}$$

$L_m(b)$ and $L_m(b,c)$ are p-adic Banach spaces with the norm

$$||\xi||_m = \operatorname{Sup}_{(\alpha;\gamma)} p^{-c_{\alpha,\gamma}}, \qquad c_{\alpha,\gamma} = \operatorname{ord} B(\alpha;\gamma) - bw_m(\alpha;\gamma). \tag{3.19}$$

Let

$$R_m(b,c) = \Omega_0[[Y]] \cap L_m(b,c), \tag{3.20}$$

$$R_m(b) = \Omega_0[[Y]] \cap L_m(b) = \bigcup_{c\in\mathbb{R}} R_m(b,c). \tag{3.21}$$

The operation $*_m$ described in (3.10) makes $L_m(b)$ into an $R_m(b)$-algebra. (3.9) ensures that this is well defined. Furthermore, if $\eta \in L_m(b)$, the mapping $\xi \mapsto \eta *_m \xi$ is a continuous endomoprhism of $L_m(b)$. Note that $L_m(b)$ is the completion of $\mathscr{R}_m$ for the norm $||\ \ ||_m$.

For each $c \in \mathbb{R}$, there is a continuous Ω_0-linear map from $L(b,c)$ into $L_m(b,c)$ whose action on monomials is given by (3.11). This map will again be denoted Φ_m.

Let $\bar{c}_1, \dots, \bar{c}_n$ be non-zero elements of $\mathbb{F}_q$ and, for each i let c_i be the Teichmüller representative of $\bar{c}_i$ in Ω_0 (so $c_i^q = c_i$).

Let:

$$(3.22)\qquad f(t) = \sum_{i=1}^{n} c_i t_i^{k_i}.$$

Let $\{\gamma_j\}_{j=0}^{\infty}$ be a sequence of elements of $\mathbb{Q}_p(\zeta_p)$ such that

$$(3.23)\qquad \begin{cases} \operatorname{ord} \gamma_0 = \dfrac{1}{p-1}, \\ \operatorname{ord} \gamma_j \geq \dfrac{p^{j+1}}{p-1} - (j+1), \qquad j \geq 1. \end{cases}$$

If $t^\alpha Y^\gamma$ is a monomial, we set

$$(3.24)\qquad E_i(t^\alpha Y^\gamma) = \left(\frac{\alpha_i}{a_i} - \frac{\alpha_n}{a_n}\right) t^\alpha Y^\gamma, \qquad i = 1, \dots, n-1.$$

Note that $E_i(t^\alpha * t^\beta) = E_i(t^\alpha) * t^\beta + t^\alpha * E_i(t^\beta)$ so that E_i acts as a derivation on all the rings and Banach spaces which have been defined so far.

Let

$$(3.25)\qquad \overline{H}(t) = \gamma \circ f(t^r).$$

$$(3.26)\qquad H(t) = \sum_{l=0}^{\infty} \gamma_l f^{\tau^l}(t^{rp^l}) = \sum_{l=0}^{\infty} \gamma_l \left(\sum_{i=1}^{n} c_i^{p^l} t_i^{p^l d_i} \right);$$

$$(3.27)\qquad \overline{H}_i = E_i \overline{H}(t) = \gamma_0 \left(c_i \frac{d_i}{a_i} t_i^{d_i} - c_n \frac{d_n}{a_n} t_n^{d_n} \right), \qquad i = 1, \dots, n-1;$$

$$(3.28)\qquad H_i = E_i H(t), \qquad i = 1, \dots, n-1;$$

$$(3.29)\qquad D_i = E_i + H_i, \qquad i = 1, \dots, n-1;$$

From now on we assume:

$$(3.30)\qquad \text{g.c.d.}(p, M) = \text{g.d.c.}(p, D) = 1,$$

and we let

$$(3.31)\qquad \varepsilon_i = c_i \frac{d_i}{a_i}, \qquad i = 1, \dots, n.$$

Each ε_i is therefore a unit in $\mathscr{O}_0$.

Let $e = b - 1/(p-1)$: we have $\overline{H}_i \in L(b, -e)$ and $\overline{H}_i \in L_m(b, -e)\ \forall m$. Also, if $b \leq p/(p-1)$, $H_i \in L(b, -e)$ and $H_i \in L_m(b, -e)\ \forall m$.

b. *Reduction.*

LEMMA 3.1. *Let* $\alpha \in \mathbb{N}^n$, $K = J(\alpha)$, $\beta = \tau(\overline{\alpha}, K)$; *then* $t^\alpha = u(\alpha)t^\beta + \gamma_0^{-1}\sum_{i=1}^{n-1}\overline{H}_i p_{i,\alpha}$, *where* $u(\alpha) \in \mathscr{O}_0$ *is a unit and, for each* i, $p_{i,\alpha} \in \mathscr{O}_0[t_1, \ldots, t_n]$.

Furthermore, $p_{i,\alpha}$ *has unit coefficients and, if* t^δ *is any monomial of* $p_{i,\alpha}$ *having non-zero coefficient, then*

(i) $J(\delta) = J(\alpha) - 1$

(ii) $s(\delta) \geq s(\alpha)$.

Proof. If $\delta \in \mathbb{Z}^n$, we can write

$$t^\delta = \varepsilon_j\varepsilon_i^{-1}t^{\alpha - d_iU_i + d_jU_j} + \gamma_0^{-1}\varepsilon_i^{-1}(\overline{H}_i - \overline{H}_j)t^{\alpha - d_iU_i}, \qquad i, j = 1, \ldots, n-1;$$

$$t^\delta = \varepsilon_n\varepsilon_i^{-1}t^{\alpha - d_iU_i + d_nU_n} + \gamma_0^{-1}\varepsilon_i^{-1}\overline{H}_i t^{\alpha - d_iU_i}, \qquad i = 1, \ldots, n-1.$$

By assumption, there are integers $\lambda_1, \ldots, \lambda_n$ such that $\alpha = \beta + \sum_{i=1}^n \lambda_i d_i U_i$, with $\sum_{i=1}^n \lambda_i = 0$. The result follows immediately, except maybe for (ii): if $\alpha \neq \beta$, there is an index i such that $\lambda_i > 0$; hence $\alpha_i \geq \beta_i + d_i$. Thus $(\alpha_i - d_i)/a_i \geq \beta_i/a_i \geq s(\beta)$ and $s(\beta) \geq s(\alpha)$ since $\beta \in \Delta$. □

LEMMA 3.2. *Let* $Y^\gamma t^\alpha$ *be a monomial in* $\mathscr{R}_m$ *and let* $\widetilde{\alpha} \in \widetilde{\Delta}$, $\tau \in \mathbb{N}$, *satisfying* $\alpha \sim \widetilde{\alpha} + \tau a$ *and* $J(\alpha) = J(\widetilde{\alpha}) + \tau N$. *Then*

$$Y^\gamma t^\alpha = u(\alpha)Y^{\gamma + \tau m M}t^{\widetilde{\alpha}} + \gamma_0^{-1}\sum_{i=1}^{n-1}\overline{H}_i *_m q_{i,\alpha,\gamma},$$

where $u(\alpha) \in \mathscr{O}_0$ *is a unit and, for each* i, $q_{i,\alpha,\gamma} \in \mathscr{R}_m$. *Furthermore, each* $q_{i,\alpha,\gamma}$ *has unit coefficients and, if* $Y^\delta t^\varepsilon$ *is a monomial of* $q_{i,\alpha,\gamma}$ *with non-zero coefficient, then* $w_m(\varepsilon; \delta) = w_m(\alpha; \gamma) - 1$.

Proof. Using Lemma 3.1 we can write:

$$Y^\gamma t^\alpha = u(\alpha)Y^\gamma t^\beta + \gamma_0^{-1}\sum_{i=1}^{n-1}\overline{H}_i p_{i,\alpha,\gamma}, \tag{3.32}$$

where β is the unique element of Δ such that $\beta\mathscr{R}\alpha$, and $p_{i,\alpha,\gamma} = Y^\gamma p_{i,\alpha}$. Let t^δ be a monomial of $p_{i,\alpha}$ with non-zero coefficient:

Lemma 3.2 (ii)$\Rightarrow$ $\gamma \geq -mMs(\delta)$ so that $p_{i,\alpha,\gamma} \in A_m$ and equation (3.32) is valid in A_m.

Applying the map $\Phi_m : A_m \to \mathscr{R}_m$ to equation (3.32) we obtain the desired result with $q_{i,\alpha,\gamma} = \Phi_m(p_{i,\alpha,\gamma})$. □

Let $V_m(b)$ be the $R_m(b)$-vector space generated by

$$\{Y^{-mMs(\alpha)}t^\alpha \mid \alpha \in \widetilde{\Delta}\},$$

and let $V_m(b, c) = V_m(b) \cap L_m(b, c)$.

PROPOSITION 3.1.

$$L_m(b,c) = V_m(b,c) + \sum_{i=1}^{n-1} \overline{H}_i * L_m(b,c+e).$$

Proof. Let $\xi = \sum_{(\alpha;\gamma)\in E_m} A(\alpha;\gamma)t^\alpha Y^\gamma \in L_m(b,c)$. We apply Lemma 3.2 to all the monomials in ξ.

If $\widetilde{\alpha} \in \widetilde{\Delta}$ and $\nu \geq -mMs(\widetilde{\alpha})$ we let

$$B_{\widetilde{\alpha}}(\nu) = A(\alpha;\gamma)u(\alpha), \tag{3.33}$$

where $u(\alpha)$ has been defined in Lemma 3.2 and the sum is taken over the set

$$E(\widetilde{\alpha},\nu) = \{(\alpha;\gamma) \in E_m \mid \nu = \mu m M + \gamma,\ \alpha \sim \widetilde{\alpha} + \mu a,\ J(\alpha) = J(\widetilde{\alpha}) + \mu N\}.$$

If $(\alpha,\gamma) \in E(\widetilde{\alpha},\nu)$, then $w_m(\alpha;\gamma) = w_m(\widetilde{\alpha};\nu)$; hence by (3.9) the sum (3.33) is finite and $\operatorname{ord} B_{\widetilde{\alpha}}(\nu) \geq bw_m(\widetilde{\alpha};\nu) + c$.

Thus, for each $\widetilde{\alpha} \in \widetilde{\Delta}$, $B_{\widetilde{\alpha}}(Y)t^{\widetilde{\alpha}} = \sum_{\nu \geq -mMs(\widetilde{\alpha})} B_{\widetilde{\alpha}}(\nu)Y^\nu t^{\widetilde{\alpha}}$ is an element of $V_m(b,c)$. On the other hand, let $\zeta_i = \gamma_0^{-1} \sum_{(\alpha;\gamma)\in E_m} A(\alpha;\gamma)q_{i,\alpha,\gamma}$ and write

$$\zeta_i = \sum_{(\beta,\nu)\in E_m} C_i(\beta;\nu)t^\beta Y^\nu, \qquad i = 1,\dots,n-1. \tag{3.34}$$

If $(\alpha;\gamma) \in E_m$ we can write $q_{i,\alpha,\gamma} = \sum D_{i,\alpha,\gamma}(\varepsilon;\delta)t^\varepsilon Y^\delta$, the sum being taken over all $(\varepsilon;\delta) \in E_m$ such that $w_m(\varepsilon;\delta) = w_m(\alpha;\gamma) - 1$. Thus

$$C_i(\beta,\nu) = \gamma_0^{-1} \sum D_{i,\alpha,\gamma}(\beta,\nu)A(\alpha;\gamma), \tag{3.35}$$

the sum being over the set $\{(\alpha;\gamma) \in E_m \mid w_m(\alpha;\gamma) = w_m(\beta;\nu) + 1\}$. This set is finite and

$$\operatorname{ord} C_i(\beta;\nu) \geq b[w_m(\beta;\nu) + 1] + c - \frac{1}{p-1} = bw_m(\beta;\nu) + c + e.$$

Hence the sum (3.34) is meaningful, $\zeta_i \in L_m(b,c+e)$, and we can write

$$\xi = \sum_{\alpha\in\widetilde{\Delta}} B_{\widetilde{\alpha}}(Y)t^{\widetilde{\alpha}} + \sum_{i=1}^{n=1} \overline{H}_i * \zeta_i. \tag{3.36}$$

c.

PROPOSITION 3.2. $V_m(b) \cap \sum_{i=1}^{n-1} \overline{H}_i * L_m(b) = (0)$.

Proof. Let $v \in V_m(b)$. For $W \in \mathbb{Q}$ we let $v^{(W)}$ be the component of v which is of homogeneous weight W: we can write $v^{(W)} = \sum_{\alpha\in\widetilde{\Delta}} P_\alpha(Y)t^\alpha$, where each $P_\alpha(Y)$ is a Laurent polynomial in Y.

Let $\iota:\widetilde{\Delta} \to \overline{\Delta}$ be the map described in the proof of Lemma 2.4. Let $Z = Y^{mM}$ and, for $\alpha \in \widetilde{\Delta}$ let $\beta = \iota(\alpha) = \alpha - \tau a$ ($\tau \in \mathbb{N}$):

$$t^\alpha = Z^\tau t^\beta + (t^a - Z)(t^{\alpha-a} + Zt^{\alpha-2a} + \cdots + Z^{\tau-1}t^{\alpha-\tau a}).$$

Hence we can write:

$$v^{(W)} = \sum_{\beta\in\overline{\Delta}} Q_\beta(Y)t^\beta + (t^a - Z)\sum_{\beta\in\overline{\Delta}} R_\beta(t, Y),$$

where for each β, $Q_\beta(Y)$ is a Laurent polynomial in Y and $R_\beta(t, Y)$ is a Laurent polynomial in $Y, t_1, \ldots, t_n$. Furthermore:

(i) if $y \in \Omega^\times$ and $\alpha \in \widetilde{\Delta}$, then $P_\alpha(y) = 0 \Leftrightarrow Q_{\iota(\alpha)}(y) = 0$;
(ii) if $Y^\gamma t^\delta$ is any monomial in $R_\beta(t, Y)$ with non-zero coefficient, then $J(\delta) \geq 0$.

Suppose $v \in \sum_{i=1}^{n-1} \overline{H}_i * L_m(b)$: we can write

$$v^{(W)} = \sum_{i=1}^{n-1} \overline{H}_i * \zeta_i,$$

where, for each i, $\zeta_i \in \Omega_0[Y, \frac{1}{Y}, t_1, \ldots, t_n]$ and is of homogeneous weight $W - 1$.

Let $\alpha, \beta \in E$ and suppose $\alpha + \beta = \delta + \tau a$, with $\delta \in E$ and $\tau \in \mathbb{N}$: $t^\alpha *_m t^\beta = t^{\alpha+\beta} - (t^{\alpha+\beta-a} + Zt^{\alpha+\beta-2a} + \cdots + Z^{\tau-1}t^{\alpha+\beta-\tau a})(t^a - Z)$. Hence we can write

$$\overline{H}_i * \zeta_i = \overline{H}_i\zeta_i + \eta_i(t^a - Z), \qquad \text{with } \eta_i \in \Omega_0\left[Y, \frac{1}{Y}, t_1, \ldots, t_n\right].$$

For each $i = 1, \ldots, n$, fix $\xi_i \in \Omega$ with $\xi_i^{d_i} = \varepsilon_n\varepsilon_i^{-1}$ and let μ_{d_i} be the group of d_i-th roots of unity in Ω.

Let $s_i = \prod_{j\neq i} d_j$, $s = \prod_{j=1}^n d_j$. Let $\hat{v}(Y, t) = \sum_{\beta\in\overline{\Delta}} Q_\beta(Y)t^\beta$ and suppose $v^{(W)} \neq 0$: there exists $\alpha \in \widetilde{\Delta}$ such that $P_\alpha(Y) \neq 0$; hence there exists $\beta = \iota(\alpha) \in \overline{\Delta}$ such that $Q_\beta(Y) \neq 0$. For such a fixed β let $\overline{\Delta}(\beta) = \{\gamma \in \overline{\Delta} \mid J(\gamma) = J(\beta)\}$ and let $y \in \Omega^\times$ such that $Q_\beta(y) \neq 0$.

We claim that there exists $(\zeta_1, \ldots, \zeta_n) \in \prod_{i=1}^n \mu_{d_i}$ such that

$$\hat{v}(y, u_1, \ldots, u_n) \neq 0, \tag{3.37}$$

where $u_i = \xi_i\zeta_i t_n^{s_i}$, $i = 1, \ldots, n$.

Indeed, the coefficient of $t_n^{sJ(\beta)}$ in (3.37) is

$$\sum_{\gamma\in\overline{\Delta}(\beta)} Q_\gamma(y)\xi_1^{\gamma_1}\dots\xi_n^{\gamma_n}\zeta_1^{\gamma_1}\dots\zeta_n^{\gamma_n}.$$

For each $\gamma=(\gamma_1,\dots,\gamma_n)\in\overline{\Delta}(\beta)$, $\chi_\gamma\colon(\zeta_1,\dots,\zeta_n)\mapsto\zeta_1^{\gamma_1}\dots\zeta_n^{\gamma_n}$ is a character of $\prod_{i=1}^n\mu_{d_i}$.

The elements of $\overline{\Delta}(\beta)$ all belong to distinct congruence classes, so these characters are all distinct, and therefore linearly independent. Our claim follows since $Q_\beta(y)\neq 0$.

Let now

$$S(Y;t)=\sum_{i=1}^n\eta_i-\sum_{\delta\in\overline{\Delta}}R_\delta(Y;t),$$

$$u=\prod_{i=1}^n(\xi_i\zeta_i)^{a_i}\qquad\text{and}\quad A=\sum_{i=1}^n a_ir_i=N\prod_{i=1}^n d_i.$$

We have:

$$\hat{v}(y;u_1,\dots,u_n)=(ut_n^A-y^{mM})S(y;u_1,\dots,u_n). \tag{3.38}$$

The left-hand side of (3.38) is a non-zero polynomial in t_n, of degree less than A, while the right-hand side vanishes for any choice of t_n satisfying $t_n^A=u^{-1}y^{mM}$, a contradiction. Hence $v^{(W)}=0$. □

LEMMA 3.3. *Let K be a field of arbitrary characteristic, $u_1,\dots,u_n$ elements of $K^\times,\nu_1,\dots,\nu_n,\lambda$ positive integers; let*

$$B=K[t_1,\dots,t_n,Y,Y^{-1}t^a],\qquad f=(Y^{-1}t^a)^\lambda-1,$$

$\overline{B}=B/(f)$, $h_i=u_it_i^{\nu_i}-u_nt_n^{\nu_n}$ $(i=1,\dots,n-1)$; then the family $\{h_i\}_{i=1}^{n-1}$ in any order forms a regular sequence on $\overline{B}$.

Proof. Let $I\subsetneq\{1,\dots,n-1\}$ and let $\mathfrak{A}_I$ be the ideal of $\overline{B}$ generated by $\{h_i\}_{i\in I}$. We must show that $(\mathfrak{A}_I\colon h_k)=\mathfrak{A}_I$ for any $k\notin I$. By relabelling we may assume that $I=\{1,\dots,j\}$, with $j<n-1$, and that $k=j+1$. Accordingly, we write $\mathfrak{A}_j$ instead of $\mathfrak{A}_I$. Let $B_1=K[t_1,\dots,t_n,Y,Z]$ and $\overline{B}_1=B_1/(Z^\lambda-1,YZ-t^a)$.

The mapping $Z\mapsto Y^{-1}t^a$ induces a ring isomorphism from $\overline{B}_1$ into $\overline{B}$. Thus, if $\mathfrak{B}_j$ is the ideal of B_1 generated by $\{h_1,\dots,h_j,Z-1, YZ-t^a\}$, we must show that $(\mathfrak{B}_j\colon h_{j+1})=\mathfrak{B}_j$, or equivalently that h_{j+1} does not belong to any associated prime of $\mathfrak{B}_j$. Since $\mathfrak{B}_j$ has $j+2$ generators, its dimension is at least $n-j$. On the other hand, the ring $B_1/\mathfrak{B}_j$ is integral over $K[t_{j+1},\dots,t_n]$ (note that $Y^\lambda - t^{\lambda a} = 0$ in $B_1/\mathfrak{B}_j$). Hence $\dim \mathfrak{B}_j = n-j$. By Macaulay's theorem [**16**, Ch. VII, §8], $\mathfrak{B}_j$ is unmixed. Likewise, $\mathfrak{B}_{j+1} = (\mathfrak{B}_j, h_{j+1})$ is unmixed, of dimension $n-j-1$. Let $\mathfrak{p}$ be an associated prime of $\mathfrak{B}_j$ and suppose that $h_{j+1} \in \mathfrak{p}\colon \mathfrak{p} \supset (\mathfrak{B}_j, h_{j+1}) = \mathfrak{B}_{j+1}$; hence $\dim \mathfrak{p} \le n-j-1$, a contradiction since $\dim \mathfrak{p} = n-j$. □

Let

$$R = \Omega_0[t_1,\dots,t_n, Y, Y^{-1}t^a] \tag{3.39}$$

$$f^{(m)} = (Y^{-1}t^a)^{mM} - 1 \tag{3.40}$$

$$\overline{R}^{(m)} = R/f(^{(m)}) \tag{3.41}$$

$$h_i^{(m)} = \varepsilon_i t_i^{mMd_i} - \varepsilon_n t_n^{mMd_n}, \qquad i = 1,\dots,n-1. \tag{3.42}$$

For any monomial $t^\alpha Y^\gamma$ we set:

$$\widetilde{w}_m(\alpha;\gamma) = \widetilde{w}_m(t^\alpha Y^\gamma) = \frac{1}{mM}(J(\alpha) + N\gamma). \tag{3.43}$$

$\widetilde{w}_m$ makes $\overline{R}^{(m)}$ into a graded ring, and each $h_i^{(m)}$ is homogeneous of weight 1.

LEMMA 3.4. *Let I be a non-empty subset of $\{1,\dots,n-1\}$ and let $\{P_i\}_{i\in I}$ be a family of elements of $\overline{R}^{(m)}$ such that $\sum_{i\in I} P_i h_i^{(m)} = 0$. Then there exists a skew-symmetric set $\{\eta_{i,j}\}_{i,j\in I}$ such that $P_i = \sum_{j\in I} \eta_{i,j} h_j^{(m)}$ for each $i \in I$. Furthermore, if each P_i is of homogeneous weight $\widetilde{w}_m(P_i) = W$ independent of i:*

(a) *if $W \ge 1$, each $\eta_{i,j}$ may be chosen of homogeneous weight $\widetilde{w}_m(\eta_{i,j}) = W - 1$ with $\mathrm{Min}_{j\in I}\{\mathrm{ord}\,\eta_{i,j}\} \ge \mathrm{ord}\,P_i$ for all $i \in I$;*

(b) *if $W < 1$ then $P_i = 0$ for all $i \in I$ (i.e. each $\eta_{i,j}$ may be chosen to be zero).*

Proof. To simplify notation, we write h_i instead of $h_i^{(m)}$. We proceed by induction on the number of elements in I. By relabelling, we may assume that $I = \{1,\dots,r+1\}$, $r \ge 0$. If $r = 0$, then $P_i = 0$ and hence we can assume $r \ge 1$. Let $\mathfrak{A}_r$ be the ideal of $\overline{R}^{(m)}$ generated by $\{h_i\}_{i=1}^r$; by Lemma 3.3, $(\mathfrak{A}_r\colon h_{r+1}) = \mathfrak{A}_r$; hence $P_{r+1} \in \mathfrak{A}_r$. Thus there exist $y_1,\dots,y_r \in \overline{R}^{(m)}$ such that

$$P_{r+1} = \sum_{i=1}^r y_i h_i. \tag{3.44}$$

Now

$$\sum_{i=1}^{r}(P_i + y_i h_{r+1})h_i = \sum_{i=1}^{r} P_i h_i + \left(\sum_{i=1}^{r} y_i h_i\right) h_{r+1}$$
$$= \sum_{i=1}^{r+1} P_i h_i = 0.$$

By induction hypothesis, there exists a skew-symmetric set $\{\eta_{i,j}\}_{i,j=1}^{r}$ such that $P_i + y_i h_{r+1} = \sum_{i=1}^{r} \eta_{i,j} h_j$ for $i = 1, \ldots, r$.

We can now set $\eta_{r+1,i} = y_i$ and $\eta_{i,r+1} = -y_i$, $i = 1, \ldots, r$ and the first assertion follows.

If each P_i is of homogeneous weight $W \geq 1$, in (3.44) we can choose each y_i to be of homogeneous weight $W - 1$. If $W < 1$, since $\widetilde{w}_m(h_i) = 1$ both sides of equation (3.44) must be zero and the induction hypothesis shows that each $P_i = 0$, $i = 1, \ldots, r + 1$.

For the estimate on $\operatorname{ord} \eta_{i,j}$ we refer the reader to [**7**, Lemma 3.1] where a similar result is proved. □

The argument of Lemmas 3.5 and 3.6 is due to S. Sperber and can be used to close a gap in the proof of directness of sum in [**15**, Theorem 3.9].

LEMMA 3.5. *Let $T_m = \{(\alpha; \gamma) \in (mM\mathbb{Z})^n \times \mathbb{Z} \mid t^\alpha Y^\gamma \in R\}$; then the mapping $(\alpha; \gamma) \mapsto (mM\alpha; \gamma)$ establishes a bijection between S_m and T_m. In particular, $t_i \mapsto t_i^{mM}$ $(i = 1, \ldots, n)$ maps A_m into a subring of R and $\overline{A}_m$ into a subring of $\overline{R}^{(m)}$.*

Proof. Let $(\alpha; \gamma) \in S_m$ and let $\beta = mM\alpha$:

$$t^\beta Y^\gamma = (Y^{-1} t^a)^{s(\beta)} Y^{\gamma + s(\beta)} t^{\beta - s(\beta)a}.$$

$s(\beta) = mMs(\alpha)$ is an integer and, by assumption, $\gamma \geq -mMs(\alpha)$ and $\alpha_i \geq s(\alpha)a_i$ for all i. Hence $\gamma + s(\beta) \geq 0$, $\beta_i - s(\beta)a_i \geq 0$ $\forall i$ and $t^\beta Y^\gamma \in R$.

Conversely, if $t^\delta Y^\gamma$ is a monomial in R, then $\gamma \geq -s(\delta)$: this is clearly true of the generators of R and, for any $\delta, \varepsilon \in \mathbb{Z}^n$, $s(\delta + \varepsilon) \geq s(\delta) + s(\varepsilon)$. Thus, if $(\beta; \gamma) \in T_m$, with $\beta = mM\alpha$, then $(\alpha; \gamma) \in S_m$. □

LEMMA 3.6. *Let I be a non-empty subset of $\{1, \ldots, n-1\}$; then the family $\{\overline{H}_i\}_{i \in I}$ in any order forms a regular sequence in $\mathscr{R}_m$. More precisely, if $\{P_i(t, Y)\}_{i \in I}$ is a set of non-zero elements of $\mathscr{R}_m$, of homogeneous weight $w_m(P_i) = W$ independent of i, and such that*

$\sum_{i\in I} \overline{H}_i * P_i = 0$, *then there exists a skew-symmetric set* $\{\xi_{i,j}\}_{i,j\in I}$ *of elements of* $\mathscr{R}_m$ *such that*

(i) $P_i(t, Y) = \sum_{j\in I} \overline{H}_j * \xi_{i,j}$;

(ii) *each* $\xi_{i,j}$ *has homogeneous weight* $w_m(\xi_{i,j}) = W - 1$ *for all* $(i, j) \in I \times I$;

(iii) $\mathrm{Min}_{j\in I}\{\mathrm{ord}\, \xi_{i,j}\} \geq \mathrm{ord}\, P_i - 1/(p-1)$ *for all* $i \in I$.

Proof. Assume that

$$\sum_{i\in I} \overline{H}_i * P_i(t, Y) = 0. \tag{3.45}$$

Applying $\overline{\Phi}_m^{-1}$ to equation (3.45) we obtain the following equation in $\overline{A}_m$:

$$\sum_{i\in I} \overline{H}_i P_i(t, Y) = 0. \tag{3.46}$$

Replacing t_i by t_i^{mM} $(i = 1, \dots, n)$, and multiplying by γ_0^{-1}, we get

$$\sum_{i\in I} h_i^{(m)} P_i(t^{mM}, Y) = 0. \tag{3.47}$$

Let $Q_i(t, Y) = P_i(t^{mM}, Y)$; by Lemma 3.5, $Q_i(t, Y) \in \overline{R}_m$ and, if $t^\alpha Y^\gamma$ is any monomial in $Q_i(t, Y)$ with non-zero coefficient, then $\widetilde{w}_m(\alpha; \gamma) = W$. Lemma 3.4 implies the existence of a skew-symmetric set $\{\eta_{i,j}\}_{i,j\in I}$ of elements of $\overline{R}_m$ such that $Q_i(t, Y) = \sum_{j\in I} \eta_{i,j} h_j^{(m)}$ for each $i \in I$, with $\widetilde{w}_m(\eta_{i,j}) = W - 1$ and $\mathrm{ord}\, \eta_{i,j} \geq \mathrm{ord}\, P_i$ for all i, j.

If $t^\alpha Y^\gamma$ is any monomial in $Q_i(t, Y)$ with non-zero coefficient then $(\alpha; \gamma) \in T_m$. The same is true of each $h_i^{(m)}$. Hence we may choose the elements $\eta_{i,j}$ so that $\eta_{i,j} = \xi'_{i,j}(t^{mM}, Y)$:

$$P_i(t^{mM}, Y) = \sum_{j\in I} \xi'_{i,j}(t^{mM}, Y) h_j^{(m)}. \tag{3.48}$$

Therefore, letting $\xi_{i,j}(t, Y) = \gamma_0^{-1} \xi'_{i,j}(t, Y)$:

$$P_i(t, Y) = \sum_{j\in I} \xi_{i,j}(t, Y) \overline{H}_j. \tag{3.49}$$

Equation (3.49) is now valid in $\overline{A}_m$ and, for any monomial $t^\alpha Y^\gamma$ in $\xi_{i,j}(t, Y)$ with non-zero coefficient, $w_m(\alpha; \gamma) = \widetilde{w}_m(mM\alpha; \gamma) = W - 1$. Applying $\overline{\Phi}_m$ to equation (3.49) yields the result. □

Using the results already attained in this section, Lemmas 3.7 and 3.8 and Theorems 3.1, 3.2, and 3.3 can be obtained with a slight reworking of the arguments in [7, §3]. We shall therefore omit the proofs.

LEMMA 3.7 (*see* [7, *Lemma* 3.4]). *If* $b \le p/(p-1)$, *then*

$$L_m(b,c) = V_m(b,c) + \sum_{i=1}^{n-1} H_i * L_m(b, c+e).$$

LEMMA 3.8 (*see* [7, *Lemma* 3.5]). *If* $b \le p/(p-1)$, *then*

$$V_m(b) \cap \sum_{i=1}^{n-1} H_i * L_m(b) = (0).$$

THEOREM 3.1 (*see* [7, *Lemma* 3.6]). *If* $1/(p-1) \le b \le p/(p-1)$, *then*

$$L_m(b,c) = V_m(b,c) + \sum_{i=1}^{n-1} D_i * L_m(b, c+e).$$

THEOREM 3.2 (*see* [7, *Lemma* 3.10]). *Let* I *be a non-empty subset of* $\{1,\dots,n-1\}$ *and assume that* $1/(p-1) < b \le p/(p-1)$; *if* $\{\xi_i\}_{i\in I}$ *is a set of elements of* $L_m(b,c)$ *such that* $\sum_{i\in I} D_i * \xi_i = 0$, *then there exists a skew-symmetric set* $\{\eta_{i,j}\}_{i,j\in I}$ *in* $L_m(b,c+e)$ *such that* $\xi_i = \sum_{j\in I} D_j * \eta_{i,j}$ *for all* $i \in I$. *In particular, the family* $\{D_i\}_{i=1}^{n-1}$ *in any order forms a regular sequence on the* $R_m(b)$-*module* $L_m(b,c)$.

THEOREM 3.3 (*see* [7, *Lemma* 3.11]). *If* $1/(p-1) < b \le p/(p-1)$, *then*

$$V_m(b) \cap \sum_{i=1}^{n-1} D_i * L_m(b) = (0).$$

d. *A Comparison Theorem.*

We now undertake to compare reduction modulo

$$\sum_{i=1}^{n-1} H_i * L_m(b, c+e) \qquad \left(\text{respectively } \sum_{i=1}^{n-1} D_i * L_m(b, c+e)\right)$$

with reduction modulo $\sum_{i=1}^{n-1} \overline{H}_i * L_m(b, c+e)$ studied in §2.

Fix $\xi \in L_m(b,c)$. Using Theorem 3.1, Lemma 3.8, and Proposition 3.1 we write:

$$(3.50)\quad \xi = v + \sum_{i=1}^{n-1} D_i * \zeta_i, \qquad v \in V_m(b,c),\ \zeta_i \in L_m(b,c+e);$$

$$(3.51)\quad \xi = \widetilde{v} + \sum_{i=1}^{n-1} H_i * \widetilde{\zeta}_i, \qquad \widetilde{v} \in V_m(b,c),\ \widetilde{\zeta}_i \in L_m(b,c+e);$$

$$(3.52)\quad \xi = \overline{v} + \sum_{i=1}^{n-1} \overline{H}_i * \overline{\zeta}_i, \qquad \overline{v} \in V_m(b,c),\ \overline{\zeta}_i \in L_m(b,c+e).$$

LEMMA 3.9. *Let $\xi, v, \zeta_1, \ldots, \zeta_{n-1}$ be as in* (3.50); *then in* (3.51) $\widetilde{v}$ *satisfies $v - \widetilde{v} \in V_m(b,c+e)$ and each $\widetilde{\zeta}_i$ can be chosen so that $\zeta_i - \widetilde{\zeta}_i \in L_m(b,c+2e)$.*

Proof.

$$\sum_{i=1}^{n-1} D_i * \zeta_i - \sum_{i=1}^{n-1} H_i * \zeta_i = \sum_{i=1}^{n-1} E_i \zeta_i \in L_m(b,c+e).$$

By Lemma 3.8, there exist $v' \in V_m(b,c+e)$ and $\zeta_i' \in L_m(b,c+2e)$, $i = 1, \ldots, n-1$, such that

$$\sum_{i=1}^{n-1} E_i \zeta_i = v' + \sum_{i=1}^{n-1} H_i * \zeta_i'.$$

Hence

$$\xi = v + v' + \sum_{i=1}^{n-1} H_i * (\zeta_i + \zeta_i')$$

and we may set $\widetilde{v} = v + v'$, $\widetilde{\zeta}_i = \zeta_i + \zeta_i'$, $i = 1, \ldots, n-1$. □

In the rest of this section we fix $b = 1/(p-1)$ (so $e = 1$).

LEMMA 3.10. *For each $i \in \{1, \ldots, n-1\}$ there exist*

$$\Gamma_i \in L_m(p/(p-1), 0) \quad \textit{and} \quad G_i \in L_m(p/(p-1), 0)$$

such that $H_i = \overline{H}_i * G_i + \Gamma_i$. Furthermore, G_i is invertible and $G_i^{-1} \in L_m(p/(p-1), 0)$.

Proof. By definition,

$$H_i = \sum_{l=0}^{\infty} p^l \gamma_l \left(c_i^{p^l} \frac{d_i}{a_i} t_i^{p^l d_i} - c_n^{p^l} \frac{d_n}{a_n} t_n^{p^l d_n} \right)$$

(recall that $c_i^q = c_i$, and therefore $c_i^\tau = c_i^p$).

Let

$$\Gamma_i = \sum_{l=0}^{\infty} p^l \gamma_l \left[\frac{d_i}{a_i} - \left(\frac{d_i}{a_i}\right) p^l\right] c_i^{p^l} t_i^{p^l d_i} - \sum_{l=0}^{\infty} p^l \gamma_l \left[\frac{d_n}{a_n} - \left(\frac{d_n}{a_n}\right)^{p^l}\right] c_n^{p^l} t_n^{p^l d_n}.$$

Then

$$H_i = \sum_{l=0}^{\infty} p^l \gamma_l \left[(\varepsilon_i t_i^{d_i})^{p^l} - (\varepsilon_n t_n^{d_n})^{p^l}\right] + \Gamma_i.$$

If we set

$$G_i = 1 + \sum_{l=1}^{\infty} \gamma_0^{-1} \gamma_l p^l \sum_{j=0}^{p^l - 1} (\varepsilon_i t_i^{d_i})^j (\varepsilon_n t_n^{d_n})^{p^l - j - 1},$$

then formally: $H_i = \overline{H}_i G_i + \Gamma_i$.

Since $d_k/a_k \in \mathbb{Q}$ and $(p, M) = 1$ we have

$$\operatorname{ord}\left[\frac{d_k}{a_k} - \left(\frac{d_k}{a_k}\right)^{p^l}\right] \geq 1 \quad \text{for all } k = 1, \ldots, n.$$

Hence both Γ_i and G_i are elements of $L(p/(p-1), 0)$. G_i is of the form $G_i = 1 - \sum_{\alpha_i \geq 0} C_\alpha t^\alpha$; such a series is invertible in $L(p/(p-1), 0)$, with inverse $G_i^{-1} = 1 + \sum_{j=0}^{\infty} (\sum_{\alpha_i \geq 0} C_\alpha t^\alpha)^j$.

Now apply $\Phi_m \colon L(p/(p-1)) \to L_m(p/(p-1))$. □

LEMMA 3.11. *Let $\xi, \widetilde{v}, \widetilde{\zeta}_1, \ldots, \widetilde{\zeta}_{n-1}$ be as in (3.51); then in (3.52) $\overline{v}$ satisfies $\widetilde{v} - \overline{v} \in V_m(p/(p-1), c+1)$ and each $\overline{\zeta}_i$ can be chosen so that*

$$\widetilde{\zeta}_i - G_i * \overline{\zeta}_i \in L_m\left(\frac{p}{p-1}, c+2\right).$$

Proof. We construct a sequence $(\xi^{(\nu)}, v^{(\nu)}, \zeta_1^{(\nu)}, \ldots, \zeta_{n-1}^{(\nu)})_{\nu \in \mathbb{N}}$ with

$$\xi^{(\nu)} \in L_m\left(\frac{p}{p-1}, c+\nu\right), \quad v^{(\nu)} \in V_m\left(\frac{p}{p-1}, c+\nu\right),$$
$$\zeta_i^{(\nu)} \in L_m\left(\frac{p}{p-1}, c+\nu+1\right)$$

by letting $\xi^{(0)} = \xi$, $v^{(0)} = \widetilde{v}$, $\zeta_i^{(0)} = \widetilde{\zeta}_i$ and the following recursion. Given $\xi^{(\nu)} \in L_m(p/(p-1), c+\nu)$ we can write, using Lemma 3.8:

$$\xi^{(\nu)} = v^{(\nu)} + \sum_{i=1}^{n-1} H_i * \zeta_i^{(\nu)}, \quad v^{(\nu)} \in L_m\left(\frac{p}{p-1}, c+\nu\right),$$
$$\zeta_i^{(\nu)} \in L_m\left(\frac{p}{p-1}, c+\nu+1\right).$$

By Lemma 3.10,

$$\xi^{(\nu)} = v^{(\nu)} + \sum_{i=1}^{n-1} \overline{H}_i * G_i * \zeta_i^{(\nu)} + \xi^{(\nu+1)}, \quad \text{with} \tag{3.53}$$

$$\xi^{(\nu+1)} = \Gamma_i * \zeta_i^{(\nu)} \in L_m\left(\frac{p}{p-1}, c+\nu+1\right).$$

Let $s \in \mathbb{N}$. Writing equation (3.53) for $0 \leq \nu \leq s$ and adding yields, after cancellations:

$$\xi = \sum_{\nu=0}^{s} v^{(\nu)} + \sum_{i=1}^{n-1} \overline{H}_i * \sum_{\nu=0}^{s} G_i * \zeta_i^{(\nu)} + \xi^{(s+1)}.$$

Letting $s \to \infty$, $\sum_{\nu=0}^{s} v^{(\nu)}$ converges to $\overline{v} \in V_m(p/(p-1), c)$, $\sum_{\nu=0}^{s} \zeta_i^{(\nu)}$ converges to $\overline{\zeta}_i \in L_m(p/(p-1), c+1)$ and $\xi^{(s+1)}$ converges to zero. □

THEOREM 3.4. *Let $\xi \in L_m(p/(p-1), c)$; if we express ξ in the form $\xi = \overline{v} + \sum_{i=1}^{n-1} \overline{H}_i * \overline{\zeta}_i$ on the one hand, with $\overline{v} \in V_m(p/(p-1), c)$, $\overline{\zeta}_i \in L_m(p/(p-1), c+1)$ and if we express ξ in the form $\xi = v + \sum_{i=1}^{n-1} D_i * \zeta_i$ on the other hand, with $v \in V_m(p/(p-1), c)$, $\zeta_i \in L_m(p/(p-1), c+1)$, then $v - \overline{v} \in V_m(p/(p-1), c+1)$ and ζ_i and $\overline{\zeta}_i$ may be chosen so that $\zeta_i - G_i * \overline{\zeta}_i \in L_m(p/(p-1), c+2)$ for all i.*

Proof. This is a consequence of Lemmas 3.9 and 3.11. □

4. Specialization. In order to obtain estimates for the exponential sum (0.4), we need to specialize the spaces $L_m(b, c)$ by setting $Y = y$ for some $y \in \Omega^{\times}$. We first observe that elements of $L_m(b, c)$ are convergent for $\operatorname{ord} t_i > -b/d_i$ and $\operatorname{ord} Y > -Nb/mM$. Furthermore, if we fix $Y = y$ with $\operatorname{ord} y > -Nb/mM$, the resulting series in $t_1, \ldots, t_n$ are convergent for t_i satisfying $\operatorname{ord} t_i \geq (mM/d_iN) \operatorname{ord} y$.

Throughout this section, we assume that $(p, M) = 1 = (p, D)$ and $1/(p-1) < b \leq p/(p-1)$.

For $\alpha \in \mathbb{Z}^n$ we let

$$w(\alpha) = J(\alpha) - Ns(\alpha). \tag{4.1}$$

For $x \in \Omega_0^{\times}$, let

$$L(x; b, c) = \left\{ \xi = \sum_{\alpha \in E} A(\alpha) t^{\alpha} \mid A(\alpha) \in \Omega_0, \right. \tag{4.2}$$
$$\left. \operatorname{ord} A(\alpha) \geq bw(\alpha) - s(\alpha) \cdot \operatorname{ord} x + c \right\};$$

(4.3) $$L(x;b) = \bigcup_{c\in\mathbb{R}} L(x,b,c);$$

(4.4) $$V = \Omega_0\text{-span of } \{t^\alpha \mid \alpha \in \widetilde{\Delta}\};$$

(4.5) $$V(x;b,c) = V \cap L(x,b,c).$$

$L(x;b)$ is a Banach space with the norm

(4.6) $$||\xi||_x = \operatorname*{Sup}_{\alpha\in E} p^{-c_\alpha}, \quad c_\alpha = \operatorname{ord} A(\alpha) - bw(\alpha) + s(\alpha) \operatorname{ord} x.$$

We equip $L(x;b,c)$ with an Ω_0-algebra structure in the following way: if $\alpha, \beta \in E$, there exist $\delta \in E$, $\lambda \in \mathbb{N}$ unique such that $\alpha + \beta = \delta + \lambda a$ and we set:

(4.7) $$t^\alpha * t^\beta = x^\lambda t^\delta.$$

If $\eta = \sum_{\alpha\in E} B(\alpha) t^\alpha$ is an element of $L(x;b,c')$, then $\xi \mapsto \eta * \xi$ is a continuous mapping from $L(x;b,c)$ into $L(x;b,c+c')$. Note that $\overline{H}_i$ and H_i (as defined in (3.27) and (3.28) respectively) can be viewed as elements of $L(x;b,0)$ and that $\overline{H}_i$, H_i, and D_i act continuously on $L(x;b,c)$ for any $c \in \mathbb{R}$. Given $x \in \Omega_0^\times$, $\operatorname{ord} x^m > -Nb$, we fix $y \in \Omega^\times$ with $y^M = x$. Let $L_m(b,c)'$, $L_m(b)'$, $V_m(b,c)'$, $L(x;b,c)'$, $L(x;b)'$, V' be defined as their unprimed counterparts, with the difference that the coefficients are allowed to lie in $\Omega_0' = \Omega_0(y)$. We can define an Ω_0'-linear specialization map

$$S_y: L_m(b)' \to L(x^m;b)'$$

by sending Y into y. S_y is continuous of norm 1 and is surjective, sending $V_m(b)'$ onto V' and $D_1 * L_m(b)'$ onto $D_i * L(x^m,b)'$ for all i. Indeed, there is an Ω_0'-linear section

(4.8) $$T_y: \sum_{\alpha\in E} A(\alpha) t^\alpha \to \sum_{\alpha\in E} x^{ms(\alpha)} Y^{-mMs(\alpha)} t^\alpha.$$

Proposition 4.1. $\operatorname{Ker}(S_y \mid L_m(b,c)') = (Y-y) L_m(b, c - \operatorname{ord} y)$.

In particular, $L_m(b)'/(Y-y)L_m(b)' \xrightarrow{\sim} L(x^m;b)'$.

Proof. Let $\xi = \sum_{(\alpha;\gamma)\in E_m} A(\alpha;\gamma) t^\alpha Y^\gamma \in L_m(b,c)'$ and assume that $S_y(\xi) = 0$.

For each $\alpha \in E$ we must have $\sum_{\gamma \geq -mMs(\alpha)} A(\alpha;\gamma)y^\gamma = 0$. Multiplying by $y^{mMs(\alpha)}$ we obtain $\sum_{\gamma \geq 0} A(\alpha;\gamma - mMs(\alpha))t^\gamma = 0$. Thus

$$\xi = \sum_{\alpha \in E} \left[\sum_{\gamma \geq 0} A(\alpha;\gamma - mMs(\alpha))(Y^\gamma - y^\gamma) \right] Y^{mMs(\alpha)} t^\alpha = (Y-y)\xi', \text{ with}$$

$$\xi' = \sum_{\alpha \in E} \left[\sum_{\gamma \geq 0} A(\alpha;\gamma - mMs(\alpha)) \sum_{\lambda=0}^{\gamma-1} Y^\lambda y^{\gamma-\lambda-1} \right] Y^{mMs(\alpha)} t^\alpha.$$

$\xi' \in L_m(b, c - \operatorname{ord} y)'$ since $\operatorname{ord} y > -Nb/mM$. □

It follows from Theorem 3.2 that the operators D_i, $i = 1, \ldots, n-1$, acting on the $R_m(b)$-module $L_m(b)$ (respectively the $R_m(b)'$-module $L_m(b)'$) form a completely secant family ([**3**, §9, n° 5, Proposition 5]). In other words, the associated Koszul complexes are acyclic: if

$$\mathbb{H}_\mu(\{D_i\}_{i=1}^{n-1}, L_m(b)) \quad [\text{respectively } \mathbb{H}_\mu(\{D_i\}_{i=1}^{n}, L_m(b)')]$$

is the μ-th homology group of the corresponding complex, then:

$$\mathbb{H}_\mu(\{D_i\}_{i=1}^{n-1}, L_m(b)) = 0, \qquad \mu \geq 1; \tag{4.9}$$

$$\mathbb{H}_\mu(\{D_i\}_{i=1}^{n-1}, L_m(b)') = 0, \qquad \mu \geq 1. \tag{4.10}$$

LEMMA 4.1. *$(Y-y)$ is not a zero divisor in $L_m(b)'/\sum_{i=1}^{n-1} D_i * L_m(b)'$.*

Proof. Let $\xi \in L_m(b)'$ and assume that

$$(Y - y)\xi = \sum_{i=1}^{n-1} D_i * \zeta_i, \qquad \zeta_i \in L_m(b)'. \tag{4.11}$$

By Theorem 3.1, we can write

$$\xi = v + \sum_{i=1}^{n-1} D_i * \eta_i, \qquad v \in V_m(b)',\ \eta_i \in L_m(b)'. \tag{4.12}$$

Thus (4.11), (4.12), and Theorem 3.3 imply $(Y-y)v = 0$; hence $v = 0$. □

THEOREM 4.1.

(i) $\mathbb{H}_\mu(\{D_i\}_{i=1}^{n-1}, L(x^m;b)') = 0$ *for all* $\mu \geq 1$;

(ii) $\mathbb{H}_0(\{D_i\}_{i=1}^{n-1}, L(x^m,b)') \xrightarrow{\sim} V'$.

Proof. (i) Let $D_m = Y - y$. As a consequence of Lemma 4.1, the family $\{D_i\}_{i=1}^n$ forms a regular sequence on the $R_m(b)'$-module $L_m(b)'$.

In particular,

$$(4.13)\qquad \mathbb{H}_\mu(\{D_i\}_{i=1}^n, L_m(b)') = 0 \quad \text{for all } \mu \geq 1.$$

Using [11, Ch. 8, Theorem 4] and Proposition 4.1, for all $\mu \geq 0$ there is an Ω_0'-linear isomorphism.

$$(4.14)\qquad \mathbb{H}_\mu(\{D_i\}_{i=1}^n, L_m(b)') \xrightarrow{\sim} \mathbb{H}_\mu(\{D_i\}_{i=1}^{n-1}, L(x^m; b)').$$

(ii) S_y maps $V_m(b,c)'$ onto $V(x^m; b, c)'$ and $D_i * L_m(b, c+e)'$ onto $D_i * L(x^m; b, c+e)'$ for all $i = 1, \dots, n-1$.

Hence using Theorems 3.1 and 3.3:

$$(4.15)\qquad L(x^m; b, c)' = V(x^m; b, c)' + \sum_{i=1}^{n-1} D_i * L(x^m; b, c+e)'.$$

Now

$$\mathbb{H}_0(\{D_i\}_{i=1}^{n-1}, L(x^m; b)') = L(x^m; b)' / \sum_{i=1}^{n-1} D_i * L(x^m; b)'. \qquad \square$$

PROPOSITION 4.2. $L(x; b, c) = V(x; b, c) + \sum_{i=1}^{n-1} D_i * L(x; b, c+e)$.

Proof. Let $\eta = \sum_{\alpha \in E} A(\alpha) t^\alpha$ be an element of $L(x; b, c)$. Assume that, for any $\alpha \in E$ such that $A(\alpha) \neq 0$, $s(\alpha)$ is equal to some value s independent of α, and let $\xi = y^{-Ms} T_y(\eta)$.

Let $c_s = s \cdot \operatorname{ord} x$; $\xi = \sum_{\alpha \in E} A(\alpha) t^\alpha Y^{-Ms}$ is an element of $L_1(b, c+c_s)$ and, by Theorem 3.1, there exist $v = \sum_{\beta \in \widetilde{\Delta}} P_\beta(Y) t^\beta \in V_1(b, c+c_s)$ and $\zeta_i \in L_1(b, c+c_s+e)$ such that $\xi = v + \sum_{i=1}^{n-1} D_i * \zeta_i$. For each $\beta \in \widetilde{\Delta}$, write $P_\beta(Y) = \sum_\gamma P_{\beta,\gamma} Y^\gamma$ and, for each $i = 1, \dots, n-1$, $\zeta_i = \sum_{(\alpha;\gamma)} \zeta_{i,\alpha,\gamma} t^\alpha Y^\gamma$.

For $l \in \mathbb{N}$, $0 \leq l < M$ we let:

$$P_{\beta,l}(Y) = \sum_{\gamma + Ms \equiv l \ (\operatorname{mod} M)} P_{\beta,\gamma} Y^\gamma,$$

$$\zeta_{i,l} = \sum_{\gamma + Ms \equiv l \ (\operatorname{mod} M)} \zeta_{i,\alpha,\gamma} t^\alpha Y^\gamma, \qquad i = 1, \dots, n-1.$$

Note that if $t^\alpha Y^\gamma$ is any monomial in $D_i * \zeta_{i,l}$ with non-zero coefficient, then again $\gamma + Ms \equiv l \ (\operatorname{mod} M)$. Thus, if $l \neq 0$:

$$\sum_{\beta \in \widetilde{\Delta}} P_{\beta,l}(Y) + \sum_{i=1}^{n-1} D_i * \zeta_{i,l} = 0.$$

Applying Theorem 3.3, $P_{\beta,l}(Y) = 0$ for all $\beta \in \widetilde{\Delta}$ and we may choose each $\zeta_{i,l}$ to be zero. Therefore:

$$\xi = \sum_{\beta \in \widetilde{\Delta}} P_{\beta,0}(Y) t^{\beta} + \sum_{i=1}^{n-1} D_i * \zeta_{i,0}.$$

Certainly $y^{Ms} P_{\beta,0}(Y) \in \Omega_0$ for all $\beta \in \widetilde{\Delta}$ and $y^{Ms} S_y(\zeta_{i,0})$ has its coefficients in Ω_0 for all $i = 1, \dots, n-1$. Hence

$$\eta \in V(x; b, c) + \sum_{i=1}^{n-1} D_i * L(x; b, c+e).$$

Now observe that if $\alpha \in E$, $s(\alpha)$ can assume only a finite set of values. Finally, directness of sum follows from (4.15). □

COROLLARY 4.1.

(i) $\mathbb{H}_{\mu}(\{D_i\}_{i=1}^{n-1}, L(x^m; b)) = 0$ *for all* $\mu \geq 1$.
(ii) $\mathbb{H}_0(\{D_i\}_{i=1}^{n-1}, L(x^m; b)) \xrightarrow{\sim} V$.

Proof. (i) follows from Theorem 4.1 and the fact that

$$\mathbb{H}_{\mu}(\{D_i\}_{i=1}^{n-1}, L(x^m; b)') = \mathbb{H}_{\mu}(\{D_i\}_{i=1}^{n-1}, L(x^m; b)) \otimes_{\Omega_0} \Omega_0'$$

(ii) follows from Proposition 4.2 and the fact that

$$\mathbb{H}_0(\{D_i\}_{i=1}^{n-1}, L(x^m; b)) = L(x^m; b) / \sum_{i=1}^{n-1} D_i * L(x^m; b). \qquad \square$$

5. The Frobenius map. We first review some of the definitions and results in [7, §4] concerning the lifting of characters. Let

$$E(z) = \exp\left(\sum_{j=0}^{\infty} \frac{z^{p^j}}{p^j} \right)$$

be the Artin-Hasse exponential series. For $s \in \mathbb{N}^* \cup \{\infty\}$, fix $\gamma_{s,0} \in \mathbb{Q}_p(\zeta_p)$ satisfying

$$\operatorname{ord} \gamma_{s,0} = \frac{1}{p-1} \quad \text{and} \quad \sum_{j=0}^{s} \frac{\gamma_{s,0}^{p^j}}{p^j} = 0,$$

and let θ_s be the splitting function

$$\theta_s(z) = E(\gamma_{s,0} z). \tag{5.1}$$

Let

$$(5.2)\qquad a_s = \begin{cases} \dfrac{1}{p-1} - \dfrac{1}{p^s}\left(s + \dfrac{1}{p-1}\right) & \text{if } s \in \mathbb{N}^*, \\ \dfrac{1}{p-1} & \text{if } s = \infty. \end{cases}$$

As a power series in z:

$$(5.3)\qquad \theta_s(z) = \sum_{l=0}^{\infty} B_l^{(s)} z^l,$$

with

$$(5.4)\qquad \begin{cases} \operatorname{ord} B_l^{(s)} \geq l a_{s+1} & \text{for all } l \geq 0. \\ B_l^{(s)} = \dfrac{\gamma_{s,0}^l}{l!} & \text{for } 0 \leq l \leq p-1. \end{cases}$$

In particular:

$$(5.5)\qquad \operatorname{ord} B_l^{(s)} = \frac{l}{p-1} \quad \text{for } 0 \leq l \leq p-1.$$

For a fixed choice of s, we can choose $\gamma_{s,0}$ so that

$$(5.6)\qquad \theta_s(t) = \theta(\bar{t}) \quad \text{whenever } t^p = t,$$

where θ is the additive character of $\mathbb{F}_p$ chosen in (0.5). Let

$$(5.7)\qquad \begin{cases} F(t) = \prod_{i=1}^{n} \theta_s(c_i t_i^{k_i}); \\ G(t) = \prod_{j=0}^{\ell-1} F^{\tau^j}(t^{p^j}). \end{cases}$$

As a consequence of [7, §4], for all $m \geq 0$:

$$(5.8)\quad S_m(\bar{f}, \mathscr{V}_{\bar{X}}, \Theta, \rho) = \sum_{t \in \mathscr{V}_m} \left(\prod_{i=1}^{n} t_i^{-(q^m-1)\rho_i/r} \right) G(t) G(t^q) \cdots G(t^{q^{m-1}}).$$

Clearly, $F(t) \in L(r a_{s+1}, 0)$ and $G(t) \in L\left(\frac{p}{q} r a_{s+1}, 0\right)$.

Let $\rho \in \mathbb{N}^n$, $0 \leq \rho_i < r$. We define elements $\rho^{(0)} = \rho$, $\rho' = \rho^{(1)}, \ldots, \rho^{(\ell)} = \rho$ satisfying:

$$(5.9)\qquad \begin{cases} p\rho_i^{(j+1)} - \rho_i^{(j)} \equiv 0 \pmod{r}, \\ 0 \leq \rho_i^{(j)} < r, \end{cases} \qquad i = 1, \ldots, n;\ j = 0, \ldots, \ell.$$

For each of the Banach spaces which have been defined, we indicate by the subscript "ρ" the subspace where all monomials t^α have zero coefficient unless $\alpha \in Z^{(\rho)}$. Thus, for example,

$$L_{m,\rho}(b,c) = \left\{ \xi = \sum B(\alpha;\gamma) t^\alpha Y^\gamma \in L_m(b,c) \mid B(\alpha;\gamma) = 0 \text{ if } \alpha \notin E^{(\rho)} \right\}.$$

Let $X = Y^M$. If $\alpha \in Z^{(\rho)}$ we set

$$\psi(t^\alpha) = \begin{cases} t^{\alpha/p}, & \text{if } p \mid \alpha_i,\ 1 \le i \le n; \\ 0, & \text{otherwise.} \end{cases} \tag{5.10}$$

$$\psi_X(t^\alpha) = \begin{cases} X^{s(\alpha)-ps(\beta)} t^\beta, & \text{if } \exists \beta \in E^{(\rho')} \text{ such that } \omega(\alpha) = p\omega(\beta); \\ 0, & \text{otherwise.} \end{cases} \tag{5.11}$$

$$\psi_x(t^\alpha) = S_y \circ \psi_X(t^\alpha). \tag{5.12}$$

ψ defines a continuous Ω_0-linear map $\psi: L_\rho(b/p, c) \to L_{\rho'}(b, c)$; ψ_X defines a continuous $R_1(b)$-linear map $\psi_X: L_{1,\rho}(b/p, c) \to L_{p,\rho'}(b, c)$; ψ_x defines a continuous Ω_0-linear map $\psi_x: L_\rho(x; b/p, c) \to L_{\rho'}(x^p; b, c)$. For all $m \ge 0$ the following diagram is commutative:

$$\begin{array}{ccccc} L_\rho(b/p) & \xrightarrow{\phi_m} & L_{m,\rho}(b/p) & \xrightarrow{S_y} & L_\rho(x^m; b/p) \otimes_{\Omega_0} \Omega_0' \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi_{X^m}} & & \downarrow{\scriptstyle \psi_{x^m} \otimes \mathrm{id}} \\ L_{\rho'}(b) & \xrightarrow{\phi_m} & L_{pm,\rho'}(b) & \xrightarrow{S_y} & L_{\rho'}(x^{pm}; b) \otimes_{\Omega_0} \Omega_0' \end{array} \tag{5.13}$$

Let:

$$\begin{cases} \psi_X^{\ell} = \psi_{X^{q/p}} \circ \psi_{X^{q/p^2}} \circ \cdots \circ \psi_X; \\ \psi_x^{\ell} = \psi_{x^{q/p}} \circ \psi_{x^{q/p^2}} \circ \cdots \circ \psi_x. \end{cases} \tag{5.14}$$

$$\begin{cases} F_j(t, X) = [\phi_{p^j}(F(t^r))]^{\tau^j} \in L_{p^j}(a_{s+1}, 0), \quad 0 \le j \le \ell - 1; \\ G_0(t, X) = \phi_1(G(t^r)). \end{cases} \tag{5.15}$$

If $b \le pa_{s+1}$ we define maps

$$\begin{cases} \mathscr{F}: L_\rho(b, c) \to L_\rho(b/q, c) \xrightarrow{\times G(t^r)} L_\rho(b/q, c) \xrightarrow{\psi^{\ell}} L_\rho(b, c); \\ \mathscr{F}_X: L_{1,\rho}(b, c) \to L_{1,\rho}(b/q, c) \xrightarrow{* G_0(t,X)} L_{1,\rho}(b/q, c) \xrightarrow{\psi_X^{\ell}} L_{q,\rho}(b, c); \\ \mathscr{F}_x: L_\rho(x; b, c) \to L_\rho(x; b/q, c) \xrightarrow{* G_0(t,x)} L_\rho(x; b/q, c) \xrightarrow{\psi_x^{\ell}} L_\rho(x^q; b, c). \end{cases} \tag{5.16}$$

By **[12**, §9], $\mathscr{F}$ (respectively $\mathscr{F}_X$, respectively $\mathscr{F}_x$) is a completely continuous Ω_0-linear map (respectively $R_1(b)$-linear, respectively Ω_0-linear).

Let δ be the operator defined on $1 + T\Omega[[T]]$ by

$$g(T)^\delta = \frac{g(T)}{g(qT)}. \tag{5.17}$$

If $x \in \Omega_0^\times$ is the Teichmüller lifting of $\overline{x} \in \mathbb{F}_q$, it follows from Corollary 1.1 that

$$L(\overline{f}, \mathscr{V}_{\overline{x}}, \Theta, \rho, T)^{(-1)^n} = \det(I - T\mathscr{F}_x)^{\delta^{n-1}}. \tag{5.18}$$

We now fix the choice of constants in (3.23) by setting

$$\gamma_j = \begin{cases} \sum_{l=0}^{j} \frac{\gamma_{0,s}^{p^l}}{p^l}, & \text{if } j \leq s-1, \\ 0, & \text{if } j \geq s. \end{cases} \tag{5.19}$$

Let $\hat{F}(t^r) = \exp H(t)$ ($H(t)$ has been defined in (3.26)).
We recall ([7, (4.22)]) that

$$\begin{cases} F(t) = \dfrac{\hat{F}(t)}{\hat{F}^\tau(t^p)}, \\ G(t) = \dfrac{\hat{F}(t)}{\hat{F}(t^q)}. \end{cases} \tag{5.20}$$

As operators on $L(0)$:

$$D_i = \frac{1}{\hat{F}(t^r)} \circ E_i \circ \hat{F}(t^r), \qquad i = 1, \dots, n-1. \tag{5.21}$$

On the other hand, $\mathscr{F} = \psi^{\mathscr{f}} \circ G(t^r)$ maps $L(0)$ into itself, and it follows from (5.20) that

$$\mathscr{F} = \frac{1}{\hat{F}(t^r)} \circ \psi^{\mathscr{f}} \circ \hat{F}(t^r). \tag{5.22}$$

Since $\psi^{\mathscr{f}} \circ E_i = qE_i \circ \psi^{\mathscr{f}}$ for all i, we deduce:

$$\mathscr{F} \circ D_i = qD_i \circ \mathscr{F}, \qquad i = 1, \dots, n-1, \tag{5.23}$$

and this last equation is now valid in $L(b) \subset L(0)$. Using (5.13) and the definition of ϕ_m we deduce:

$$\begin{cases} \mathscr{F}_X \circ D_i = qD_i \circ \mathscr{F}_X, \\ \mathscr{F}_x \circ D_i = qD_i \circ \mathscr{F}_x. \end{cases} \tag{5.24}$$

Let

$$\begin{cases} W_{X^m,\rho} = L_{m,\rho}(b) / \sum_{i=1}^{n-1} D_i * L_{m,\rho}(b); \\ W_{x,\rho} = L_\rho(x; b) / \sum_{i=1}^{n-1} D_i * L_\rho(x; b). \end{cases} \tag{5.25}$$

As a consequence of (5.24), $\mathscr{F}_x$ acts on the Koszul complex

$K(\{D_i\}_{i=1}^{n-1}, L_\rho(x;b))$. Specifically, there is a commutative diagram:

$$(5.26)\quad \begin{array}{ccccccc} 0 \to L_\rho(x;b) & \to \cdots \to & L_\rho(x;b)^{\binom{n-1}{i}} & \to \cdots \to & L_\rho(x;b) & \to W_{x,\rho} \to 0 \\ \downarrow q^{n-1}\mathscr{F}_x & & \downarrow (q^i\mathscr{F}_x)^{\binom{n-1}{i}} & & \downarrow \mathscr{F}_x & \downarrow \overline{\mathscr{F}}_x \\ 0 \to L_\rho(x^q;b) & \to \cdots \to & L_\rho(x^q;b)^{\binom{n-1}{i}} & \to \cdots \to & L_\rho(x^q;b) & \to W_{x^q,\rho} \to 0 \end{array}$$

Corollary 4.1 implies that both rows of diagram (5.26) are exact. Therefore, taking the alternating product of the Fredholm determinants, we obtain

$$(5.27)\qquad \det(I - T\mathscr{F}_x)^{\delta^{n-1}} = \det(I - T\overline{\mathscr{F}}_x).$$

For $j \geq 0$ let

$$(5.28)\qquad \begin{cases} \mathscr{F}^{(j)} = \psi \circ F^{\tau^j}(t^r); \\ \mathscr{F}_X^{(j)} = \psi_{X^{p^j}} \circ [*F_j(t,X)]; \\ \mathscr{F}_x^{(j)} = \psi_{x^{p^j}} \circ [*F_j(t,x)]. \end{cases}$$

$\mathscr{F}_X^{(j)}$ maps $L_{p^j,\rho^{(j)}}(b,c)$ into $L_{p^{j+1},\rho^{(j+1)}}(b,c)$, while $\mathscr{F}_x^{(j)}$ maps $L_{\rho^{(j)}}(x^{p^j};b,c)$ into $L_{\rho^{(j+1)}}(x^{p^{j+1}};b,c)$. If we set:

$$(5.29)\qquad D_i^{(j)} = E_i + H_i^{\tau^j}, \qquad i = 1,\ldots,n-1;\ j = 0,\ldots,\ell,$$

then, as above,

$$(5.30)\qquad \mathscr{F}^{(j)} \circ D_i^{(j)} = pD_i^{(j+1)} \circ \mathscr{F}^{(j)}.$$

Hence:

$$(5.31)\qquad \begin{cases} \mathscr{F}_X^{(j)} \circ D_i^{(j)} = pD_i^{(j+1)} \circ \mathscr{F}_X^{(j)}; \\ \mathscr{F}_x^{(j)} \circ D_i^{(j)} = pD_i^{(j+1)}. \end{cases}$$

Let

$$(5.32)\qquad \begin{cases} W_{X,\rho}^{(j)} = L_{p^j,\rho^{(j)}}(b) / \sum_{i=1}^{n-1} D_i^{(j)} * L_{p^j,\rho^{(j)}}(b), \\ W_{x,\rho}^{(j)} = L_{\rho^{(j)}}(x^{p^j};b) / \sum_{i=1}^{n-1} D_i^{(j)} * L_{\rho^{(j)}}(x^{p^j};b) \end{cases}$$

$\mathscr{F}_X^{(j)}$ and $\mathscr{F}_x^{(j)}$ define quotient maps:

$$(5.33)\qquad \begin{cases} \overline{\mathscr{F}}_X^{(j)} : W_{X,\rho}^{(j)} \to W_{X,\rho}^{(j+1)}; \\ \overline{\mathscr{F}}_x^{(j)} : W_{x,\rho}^{(j)} \to W_{x,\rho}^{(j+1)}. \end{cases}$$

With these notations, $W_{X,\rho}^{(\ell)} = W_{X^q,\rho}$, $W_{x,\rho}^{(\ell)} = W_{x^q,\rho}$ and the following factorizations hold:

$$(5.34)\qquad \begin{cases} \overline{\mathscr{F}}_X = \overline{\mathscr{F}}_X^{(\ell-1)} \circ \cdots \circ \overline{\mathscr{F}}_X^{(1)} \circ \overline{\mathscr{F}}_X^{(0)}; \\ \overline{\mathscr{F}}_x = \overline{\mathscr{F}}_x^{(\ell-1)} \circ \cdots \circ \overline{\mathscr{F}}_x^{(1)} \circ \overline{\mathscr{F}}_x^{(0)}. \end{cases}$$

We now fix:

$$(5.35)\qquad s = \infty; \quad b = \frac{p}{p-1}.$$

Proposition 5.1. (i) *Let* $C^{(j)}(Y) = (C^{(j)}_{\beta,\alpha}(Y))$ *be the matrix of* $\overline{\mathscr{F}}^{(j)}_X: W^{(j)}_{X,p} \to W^{(j+1)}_{X,p}$ *with respect to the bases* $\{Y^{-Mp^j s(\alpha)} t^\alpha \mid \alpha \in \widetilde{\Delta}_{p^{(j)}}\}$ *of* $W^{(j)}_{X,p}$ *and* $\{Y^{-Mp^{j+1} s(\alpha)} t^\alpha \mid \alpha \in \widetilde{\Delta}_{p^{(j+1)}}\}$ *of* $W^{(j+1)}_{X,p}$ *respectively; then for any* $\alpha \in \widetilde{\Delta}_{p^{(j)}}$ *and* $\beta \in \widetilde{\Delta}_{p^{(j+1)}}$, $C^{(j)}_{\beta,\alpha}(Y)$ *is analytic in the disk* $\{y \mid \operatorname{ord} y > -N/Mp^j(p-1)\}$.

(ii) *Let* $x \in \Omega^\times$ *with* $\operatorname{ord} x = 0$ *and let* $A^{(j)} = (A^{(j)}_{\beta,\alpha}(x))$ *be the matrix of* $\overline{\mathscr{F}}^{(j)}_x: W^{(j)}_{x,p} \to W^{(j+1)}_{x,p}$ *with respect to the bases* $\{t^\alpha \mid \alpha \in \widetilde{\Delta}_{p^{(j)}}\}$ *of* $W^{(j)}_{x,p}$ *and* $\{t^\alpha \mid \alpha \in \widetilde{\Delta}_{p^{(j+1)}}\}$ *of* $W^{(j+1)}_{x,p}$ *respectively; then for any* $\alpha \in \widetilde{\Delta}_{p^{(j)}}$ *and* $\beta \in \widetilde{\Delta}_{p^{(j+1)}}$, $\operatorname{ord} A^{(j)}_{\beta,\alpha}(x) \geq (pw(\beta) - w(\alpha))/(p-1)$.

Proof. (i) If $\alpha \in \widetilde{\Delta}_{p^{(j+1)}}$, then

$$Y^{-p^j M s(\alpha)} t^\alpha \in L_{p^j}\left(\frac{1}{p-1}, \frac{-w(\alpha)}{p-1}\right)$$

so that

$$\mathscr{F}^{(j)}_X(Y^{-p^j M s(\alpha)} t^\alpha) \in L_{p^{j+1}}\left(\frac{p}{p-1}, \frac{-w(\alpha)}{p-1}\right).$$

Using Theorem 3.1, we may write

$$\begin{aligned}(5.36)\quad & \mathscr{F}^{(j)}_X(Y^{-p^j M s(\alpha)} t^\alpha) \\ &= \sum_{\beta \in \widetilde{\Delta}_{p^{(j+1)}}} C^{(j)}_{\beta,\alpha}(Y) Y^{-p^{j+1} M s(\beta)} t^\beta + \sum_{i=1}^{n-1} D^{(j+1)}_i * \zeta_i(t, Y).\end{aligned}$$

with

$$\begin{aligned} C^{(j)}_{\beta,\alpha}(Y) &\in R_{p^{j+1}}\left(\frac{p}{p-1}, \frac{pw(\beta) - w(\alpha)}{p-1}\right) \quad \text{and} \\ \zeta_i(t, Y) &\in L_{p^{j+1}}\left(\frac{p}{p-1}, \frac{-w(\alpha)}{p-1} + 1\right).\end{aligned}$$

(ii) Applying the map S_y to equation (5.36) and multiplying by $x^{p^j s(\alpha)}$ we obtain:

$$\begin{aligned}(5.37)\qquad \mathscr{F}^{(j)}_x(t^\alpha) = & \sum_{\beta \in \widetilde{\Delta}_{p^{(j+1)}}} C^{(j)}_{\beta,\alpha}(y) x^{p^j s(\alpha) - p^{j+1} s(\beta)} t^\beta \\ & + \sum_{i=1}^{n-1} D^{(j+1)}_i * \zeta_i(t, y).\end{aligned}$$

Since $\mathscr{F}_x^{(j)}$ is defined over Ω_0, Proposition 4.2 shows that in fact $C_{\beta,\alpha}^{(j)}(y)x^{p^j s(\alpha)-p^{j+1}s(\beta)} \in \Omega_0$ and we may write:

$$A_{\beta,\alpha}^{(j)}(x) = C_{\beta,\alpha}^{(j)}(y)x^{p^j s(\alpha)-p^{j+1}s(\beta)}. \tag{5.38}$$

The estimates now follow from the fact that

$$C_{\beta,\alpha}(y) \in L\left(x^{p^{j+1}}; \frac{p}{p-1}, \frac{pw(\beta)-w(\alpha)}{p-1}\right)' \cap \Omega_0'.$$

THEOREM 5.1. *Let $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{Z}^n$, $0 \le \rho_i < r$ and suppose that $\rho = \mathbf{0}$ or $p \equiv 1 \pmod r$; let $\mathscr{H}_\rho(T) = \prod_{\alpha\in\widetilde{\Delta}_\rho}(1-q^{w(\alpha)}T)$. Then the Newton polygon of $L(\overline{f}, \Theta, \rho, T)$ lies over the Newton polygon of $\mathscr{H}_\rho(T)$.*

Proof. Let $\mathscr{T}$ be the completion of the maximal unramified extension of $\mathbb{Q}_p$ in Ω. For $x \in \mathscr{T}(\zeta_p)$ satisfying $\operatorname{ord} x \ge 0$ and $\tau(x) = x^p$ we can define

$$\tau^{-1}: W_{x,\rho}^{(1)} \to W_{x,\rho}^{(0)} = W_{x,\rho}, \tag{5.39}$$

by sending $\xi = \sum_{\alpha\in E^{(\rho)}} A(\alpha)t^\alpha \in L_\rho(x^p; b, c)$ into

$$\tau^{-1}(\xi) = \sum_{\alpha\in E^{(\rho)}} \tau^{-1}(A(\alpha))t^\alpha \in L_\rho(x; b, c).$$

Certainly,

$$\tau^{-1}(D_i^{(1)} *_p L(x^p; b)) \subset D_i *_1 L(x; b) \quad \text{for all } i,$$

so that τ^{-1} is defined on the quotient. Let $x \in \Omega_0^\times$ with $x^q = x$ and let

$$\mathscr{F}_x' = \tau^{-1} \circ \mathscr{F}_x^{(0)}. \tag{5.40}$$

If $p \equiv 1 \pmod r$, then $\rho^{(j)} = \rho$ for all $j \in \mathbb{N}$ and $\mathscr{F}_x'$ is a τ^{-1}-semilinear map and a completely continuous endomorphism of $L_\rho(x; b)$ over $\Omega_1 = \mathbb{Q}_p(\zeta_p)$. If we let

$$\overline{\mathscr{F}}_x' = \tau^{-1} \circ \overline{\mathscr{F}}_x^{(0)}, \tag{5.41}$$

then:

$$\overline{\mathscr{F}}_x = (\overline{\mathscr{F}}_x')^{f}. \tag{5.42}$$

It follows from [**8**, Lemma 7.1] that the Newton polygon of $\det_{\Omega_0}(I - T\overline{\mathscr{F}}_x)$ can be obtained from that of $\det_{\Omega_1}(I - T\overline{\mathscr{F}}_x')$ by reducing both ordinates and abscissae by the factor $1/\ell$ and interpreting the ordinates as normalized so that $\operatorname{ord} q = 1$. If $x \in \Omega_0^\times$ is the Teichmüller representative of $\overline{x} \in \mathbb{F}_q$, we let $\mathscr{A}(x) = (\mathscr{A}_{\beta,\alpha}(x))$ be the matrix of $\mathscr{F}_x' \colon W_{x,\rho} \to W_{x,\rho}$ over Ω_0 with respect to the basis $\{t^\alpha \mid \alpha \in \widetilde{\Delta}_\rho\}$. By Proposition 5.1:

$$\operatorname{ord} \mathscr{A}_{\beta,\alpha}(x) \geq \frac{pw(\beta) - w(\alpha)}{p-1} \quad \text{for all } \alpha, \beta \in \widetilde{\Delta}_\rho. \tag{5.43}$$

We fix an integral basis $\{\eta_i\}_{i=1}^{\ell}$ of Ω_0 over Ω_1 with the property that $\{\overline{\eta}_i\}_{i=1}^{\ell}$ is a basis of $\mathbb{F}_q$ over $\mathbb{F}_p$. In particular, if $\omega \in \Omega_0$, $\omega = \sum_{i=1}^{\ell} \omega_i \eta_i$, $\omega_i \in \Omega_1$, then $\operatorname{ord} \omega = \operatorname{Inf}_{1 \leq i \leq \ell}\{\operatorname{ord} \omega_i\}$. Write:

$$\overline{\mathscr{F}}'_x(\eta_i t^\alpha) = \sum_{\beta \in \widetilde{\Delta}_\rho} \sum_{1 \leq j \leq \ell} \mathscr{A}((\beta, j), (\alpha, i)) \eta_j t^\beta. \tag{5.44}$$

$\overline{\mathscr{F}}'_x$ is an Ω_1-linear endomorphism of $W_{x,\rho}$ with matrix

$$\mathscr{A}' = [\mathscr{A}((\beta, j), (\alpha, i))]$$

with respect to the basis $\{\eta_i t^\alpha \mid \alpha \in \widetilde{\Delta}_\rho,\ 1 \leq i \leq \ell\}$. Furthermore:

$$\operatorname{ord} \mathscr{A}((\beta, j), (\alpha, i)) \geq \frac{pw(\beta) - w(\alpha)}{p-1} \quad \text{for all } i, j.$$

We now proceed as in **[8**, §7]:

$$\det_{\Omega_1}(I - T\overline{\mathscr{F}}'_x) = 1 + \sum_{j=1}^{Q} m_j T^j,$$

where $Q = \ell N \prod_{i=1}^n k_i$ and m_j is (up to sign) the sum of the $j \times j$ principal minors of the matrix $\mathscr{A}'$. Thus, $\operatorname{ord} m_j$ is greater than or equal to the minimum of all j-fold sums $\sum_{l=1}^{j} w(\beta_{(l)})$, in which $\{(\beta_{(l)}, i_l)\}_{l=1}^{j}$ is a set of j distinct elements in $\{(\beta, i) \mid \beta \in \widetilde{\Delta}_\rho,\ 1 \leq i \leq \ell\}$. □

Proposition 5.2. *For each $\alpha \in \widetilde{\Delta}_{\rho^{(j)}}$, let $\alpha' \in \widetilde{\Delta}_{\rho^{(j+1)}}$ and $\delta \in \mathbb{Z}^n$ be the unique elements such that $0 \leq \delta_i \leq p-1$ and*

$$p\left(\frac{\alpha_i'}{d_i} - s(\alpha')\frac{a_i}{d_i}\right) - \left(\frac{\alpha_i}{d_i} - s(\alpha)\frac{a_i}{d_i}\right) = \delta_i \quad \textit{for all } i;$$

Let $C^{(j)} = (C^{(j)}_{\beta,\alpha}(Y))$ be the matrix of $\overline{\mathscr{F}}^{(j)}_X \colon W^{(j)}_{X,\rho} \to W^{(j+1)}_{X,\rho}$.
Then:

(i) $\operatorname{ord} C^{(j)}_{\alpha',\alpha}(0) = \frac{pw(\alpha') - w(\alpha)}{p-1} = \sum_{i=1}^n \delta_i$.

(ii) *If* $\beta \neq \alpha'$ *then*

$$\operatorname{ord} C^{(j)}_{\beta,\alpha}(0) > \frac{pw(\beta) - w(\alpha)}{p-1},$$

provided one of the following conditions holds:

(a) β *and* α' *lie in distinct congruence classes*;

(b) $\beta \sim \alpha'$ *and* $s(\beta) \neq s(\alpha')$;

(c) $\beta \sim \alpha'$, $s(\beta) = s(\alpha')$, $w(\beta) < w(\alpha')$.

Proof. To simplify notation, we shall assume that $j = 0$. For each $l \in \mathbb{N}$ we write B_l instead of $B_l^{(\infty)}$ in (5.3). For $\alpha \in \mathbb{N}^n$ let

$$B(\alpha) = \begin{cases} \prod_{i=1}^n c_i^{\alpha_i/d_i} B_{\alpha_i/d_i}, & \text{if } d_i \mid \alpha_i \text{ for all } i; \\ 0, & \text{otherwise.} \end{cases} \tag{5.45}$$

By (5.4), $\operatorname{ord} B(\alpha) \geq J(\alpha)/(p-1)$, and by (5.5), $\operatorname{ord} B(\alpha) = J(\alpha)/(p-1)$, if $\alpha_i/d_i \leq p-1$ for all i.

With these notations:

$$\begin{cases} F(t^r) = \sum_{\alpha\in\mathbb{N}^n} B(\alpha) t^\alpha, \\ F_0(t, X) = \sum_{\alpha\in E} \sum_{\lambda\in\mathbb{N}} B(\alpha + \lambda a) t^\alpha Y^{\lambda M}. \end{cases} \tag{5.46}$$

Let $\alpha \in \widetilde{\Delta}_p$:

$$\begin{aligned} &\mathscr{F}_X^{(0)}(Y^{-Ms(\alpha)} t^\alpha) \\ &\quad = \sum_{\lambda\in\mathbb{N}} \sum B(\eta + \lambda a) Y^{Ms(\alpha+\eta) - pMs(\sigma) - Ms(\alpha) + \lambda M} t^\sigma, \end{aligned} \tag{5.47}$$

where the inner sum is indexed by the set

$$\{(\eta, \sigma) \in E^{(\mathbf{0})} \times E^{(p')} \mid \eta_i + \lambda a_i \equiv 0 \mod d_i,\ \omega(\alpha + \mu) = p\omega(\sigma)\}.$$

Let

$$\xi \in L_p\left(\frac{p}{p-1}, c\right), \quad \xi = \sum_{(\alpha,\gamma)\in E_p} A(\alpha;\gamma) t^\alpha Y^\gamma.$$

If we write

$$\xi = \sum_{\beta\in\widetilde{\Delta}} E_\beta(Y) t^\beta + \sum_{i=1}^{n-1} \overline{H}_i^\tau * \zeta_i,$$

we saw in the proof of Proposition 3.1 that the coefficient of $Y^{-pMs(\beta)}$ in $E_\beta(Y)$ is $\sum u(\widehat{\alpha}) A(\widehat{\alpha};\gamma)$, where the sum is indexed by the set

$$\begin{aligned} \{(\widehat{\alpha};\gamma) \in E \times \mathbb{N} \mid -pMs(\beta) = \mu pM + \gamma,\ &\widehat{\alpha} \sim \beta + \mu a, \\ &J(\widehat{\alpha}) = J(\beta) + \mu a,\ \mu \in \mathbb{N}\}, \end{aligned}$$

and where each $u(\widehat{\alpha})$ is a unit in $\mathscr{O}_0$. Thus, if we write

$$(5.48)\qquad \mathscr{F}_X^{(0)}(Y^{-Ms(\alpha)}t^{\alpha}) = \sum_{\beta\in\widetilde{\Delta}_{\rho'}} \overline{C}_{\beta,\alpha}(Y)Y^{-pMs(\beta)}t^{\beta} + \sum_{i=1}^{n-1}\overline{H}_i^{\tau} * \zeta_i,$$

then the constant coefficient of $\overline{C}_{\beta,\alpha}(Y)$ is

$$(5.49)\qquad \overline{C}_{\beta,\alpha}(0) = \sum u(\sigma)B(\mu+\lambda a),$$

where the sum is indexed by the set $S(\beta,\alpha)$ of all $(\eta,\sigma,\lambda) \in E^{(\mathbf{0})} \times E^{(\rho')} \times \mathbb{N}$ satisfying:

$$(5.50)\qquad \begin{cases} ps(\beta) - s(\alpha) + s(\alpha+\eta) - ps(\sigma) + \lambda + p\mu = 0 \\ \sigma \sim \beta + \mu a, \qquad \mu \in \mathbb{N} \\ J(\sigma) = J(\beta) + \mu a \\ \omega_{i,j}(\alpha+\eta) = p\omega_{i,j}(\sigma) \qquad i,j = 1,\dots,n. \\ \eta_i + \lambda a_i \equiv 0 \mod d_i \qquad i = 1,\dots,n. \end{cases}$$

Let $(\eta,\sigma,\lambda) \in S(\beta,\alpha)$. If $\sigma \sim \beta + \mu a$ and $J(\sigma) = J(\beta) + \mu a$ for some $\mu \in \mathbb{N}$, then necessarily $s(\sigma) \le s(\beta) + \mu$. On the other hand, $s(\alpha+\eta) \ge s(\alpha) + s(\eta)$. Hence:

$$\begin{aligned} 0 &= ps(\beta) - s(\alpha) + s(\alpha+\eta) - ps(\sigma) + \lambda + p\mu \\ &\ge s(\alpha+\eta) - s(\alpha) + \lambda \ge s(\eta) + \lambda \ge 0. \end{aligned}$$

We conclude that $s(\alpha+\eta) = s(\alpha)$, $s(\sigma) = s(\beta) + \mu$, $\lambda = 0$, $s(\eta) = 0$. Furthermore, since σ and β are elements of E, $s(\sigma) < 1$ and $s(\beta) < 1$; hence $\mu = 0$. Thus

$$(5.51)\qquad \overline{C}_{\beta,\alpha}(0) = \sum u(\sigma)B(\eta),$$

where the sum is indexed by the set $T(\beta,\alpha)$ of all $(\eta,\sigma) \in E^{(\mathbf{0})} \times E^{(\rho')}$ which satisfy

$$(5.52)\qquad \begin{cases} s(\alpha+\eta) = s(\alpha) \\ s(\eta) = 0 \\ s(\sigma) = s(\beta) \\ \sigma \sim \beta, \\ J(\sigma) = J(\beta) \\ \omega_{i,j}(\alpha+\eta) = p\omega_{i,j}(\sigma) \quad \text{for all } i,j \\ \eta_i \equiv 0 \mod d_i \quad \text{for all } i. \end{cases}$$

Let $(\eta, \sigma) \in T(\beta, \alpha)$: there is an index l such that $\eta_l = 0$ and $s(\alpha) = s(\alpha + \eta) = \alpha_l/a_l$ and, by Remark 1.1, $s(\sigma) = \sigma_l/a_l$. Hence:

$$(5.53) \quad p\left(\frac{\sigma_i}{d_i} - s(\sigma)\frac{a_i}{d_i}\right) - \left(\frac{\alpha_i}{d_i} - s(\alpha)\frac{a_i}{d_i}\right) - \frac{\eta_i}{d_i} = \nu_i \in \mathbb{N} \quad \text{for all } i.$$

By assumption:

$$(5.54) \quad p\left(\frac{\alpha'_i}{d_i} - s(\alpha')\frac{a_i}{d_i}\right) - \left(\frac{\alpha_i}{d_i} - s(\alpha)\frac{a_i}{d_i}\right) = \delta_i \in \mathbb{N} \quad \text{for all } i.$$

by Lemma 2.8, $s(\alpha') = \alpha'_l/a_l$ and we deduce from (5.53) and (5.54) that

$$pg_i\frac{(\sigma_l - \alpha'_l)}{g_l} \in \mathbb{Z} \quad \text{for all } i = 1, \dots, n.$$

Since $\text{g.c.d.}(g_1, \dots, g_n) = 1$ and $(p, M) = 1$, this implies $\sigma_l \equiv \alpha'_l \mod g_l$; but σ and α' are elements of $E^{(p')}$: $\sigma_l/g_l < r$, $\alpha'_l/g_l < r$ and $\sigma_l \equiv \alpha'_l \mod r$. Hence $\sigma_l = \alpha'_l$ and $s(\sigma) = s(\alpha')$. (5.53) and (5.54) now imply $p(\sigma_i - \alpha'_i) \equiv 0 \mod d_i$ for all i; since $(p, D) = 1$ we deduce $\alpha' \sim \sigma \sim \beta$. In particular, $T(\beta, \alpha) = \varnothing$ if β and α' lie in distinct congruence classes, or if $s(\beta) \neq s(\alpha')$. Furthermore, since $s(\sigma) = s(\beta)$, (5.53) yields

$$(5.55) \quad p\left(\frac{\beta_i}{d_i} - s(\beta)\frac{a_i}{d_i}\right) - \left(\frac{\alpha_i}{d_i} - s(\alpha)\frac{a_i}{d_i}\right) = \varepsilon_i \in \mathbb{Z} \quad \text{for all } i.$$

Suppose $\beta \neq \alpha'$: by Lemma 2.8 there exists an index j such that $\varepsilon_j < 0$ or alternatively an index k such that $\varepsilon_k > p - 1$.

If $\varepsilon_j < 0$, (5.53) and (5.54) imply $p(\sigma_j/d_j - \beta_j/d_j) = \nu_j - \varepsilon_j > 0$, hence $\sigma_j > \beta_j$ and therefore $\sigma_j \geq \beta_j + d_j$; but $J(\sigma) = J(\beta)$, hence there exists an index m such that $\beta_m \geq \sigma_m + d_m$. Subtracting (5.53) from (5.54) then yields $\varepsilon_m - \nu_m \geq p$; hence $\varepsilon_m > p - 1$. Now subtracting (5.54) from (5.55) we obtain

$$p\left(\frac{\beta_m}{d_m} - \frac{\alpha'_m}{d_m}\right) = \varepsilon_m - \delta_m > 0,$$

hence $\beta_m > \alpha'_m$. If $\beta \sim \alpha'$, this last inequality implies that $\beta_i \geq \alpha'_i$ for all i (Lemma 2.3) and therefore $w(\beta) > w(\alpha')$ since $s(\beta) = s(\alpha')$. Thus, if $\beta \sim \alpha'$, $\beta \neq \alpha'$, $s(\beta) = s(\alpha')$, and $w(\beta) \leq w(\alpha')$ the set $T(\beta, \alpha)$ is empty and $\overline{C}_{\beta,\alpha}(0) = 0$.

Suppose finally that $\beta = \alpha'$. Since $J(\sigma) = J(\alpha')$, if $\sigma \neq \alpha'$ there is an index i such that $\alpha'_i \geq \sigma_i + d_i$; but this implies $\delta_i - \nu_i \geq p$ in (5.53) and (5.54); hence $\delta_i \geq p$, a contradiction. Hence $\sigma = \alpha'$ and the set $T(\alpha', \alpha)$ contains the single element (η, α') with $\eta = (\delta_1 d_1, \dots, \delta_n d_n)$. In particular, $\operatorname{ord} \overline{C}_{\alpha',\alpha}(0) = \sum_{i=1}^n \delta_i$.

Summarizing:

(i) $\operatorname{ord} \overline{C}_{\alpha',\alpha}(0) = (pw(\alpha') - w(\alpha))/(p-1)$;

(ii) if $\beta \neq \alpha'$ then $\overline{C}_{\beta,\alpha}(0) = 0$ whenever one of the following holds:

(a) β and α' lie in distinct congruence classes;

(b) $\beta \sim \alpha'$ and $s(\beta) \neq s(\alpha')$;

(c) $\beta \sim \alpha'$, $s(\beta) = s(\alpha')$, and $w(\beta) \leq w(\alpha')$.

The proposition now follows from the fact that, by (5.36) and Theorem 3.4:

$$(5.56) \quad C_{\beta,\alpha}(Y) - \overline{C}_{\beta,\alpha}(Y) \in R_p\left(\frac{p}{p-1}, \frac{pw(\beta) - w(\alpha)}{p-1} + 1\right) \quad \forall \alpha, \beta \in \Delta. \quad \square$$

Let π be a uniformizer of $\mathbb{Q}_p(\zeta_p)$ and let π' be a root of $Z^{MD} - \pi$ in Ω. If $\mathscr{T}$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$ in Ω, we let $\mathscr{T} = \mathscr{T}(\pi')$ and we extend τ to $\mathscr{T}'$ by setting $\tau(\pi') = \pi'$.

Let $\mathscr{C}^{(j)}(Y)$ be the matrix of $\overline{\mathscr{F}}_X^{(j)}: W_{X,p}^{(j)} \to W_{X,p}^{(j+1)}$ with respect to the bases $\{\pi^{w(\alpha)} Y^{-p^j s(\alpha)} t^\alpha \mid \alpha \in \widetilde{\Delta}_{p^{(j)}}\}$ of $W_{X,p}^{(j)}$ and $\{\pi^{w(\beta)} Y^{-p^j s(\beta)} t^\beta \mid \beta \in \widetilde{\Delta}_{p^{(j+1)}}\}$ of $W_{X,p}^{(j+1)}$.

For $x \in \Omega_0^\times$, with $\operatorname{ord} x = 0$, let also $\mathscr{A}^{(j)}(x)$ be the matrix of $\overline{\mathscr{F}}_x^{(j)}: W_{x,p}^{(j)} \to W_{x,p}^{(j+1)}$ with respect to the bases $\{\pi^{w(\alpha)} t^\alpha \mid \alpha \in \widetilde{\Delta}_{p^{(j)}}\}$ of $W_{x,p}^{(j)}$ and $\{\pi^{w(\beta)} t^\beta \mid \beta \in \widetilde{\Delta}_{p^{(j+1)}}\}$ of $W_{x,p}^{(j+1)}$.

By Proposition 5.2, the following estimates hold:

$$(5.57) \quad \begin{cases} \operatorname{ord} \mathscr{C}_{\beta,\alpha}^{(j)}(0) \geq w(\beta) & \text{for all } (\alpha, \beta) \in \widetilde{\Delta}_{p^{(j)}} \times \widetilde{\Delta}_{p^{(j+1)}}; \\ \operatorname{ord} \mathscr{C}_{\alpha',\alpha}^{(j)}(0) = w(\alpha') & \text{for all } \alpha \in \widetilde{\Delta}_{p^{(j)}}; \\ \mathscr{C}_{\beta,\alpha}^{(j)}(0) = 0 & \text{if } \beta \text{ and } \alpha \text{ satisfy condition (a),} \\ & \text{(b), or (c) of Proposition 5.2 (ii).} \end{cases}$$

$$(5.58) \quad \begin{cases} \operatorname{ord} \mathscr{A}_{\beta,\alpha}^{(j)}(x) \geq w(\beta) & \text{for all } (\alpha, \beta) \in \widetilde{\Delta}_{p^{(j)}} \times \widetilde{\Delta}_{p^{(j+1)}}; \\ \operatorname{ord} \mathscr{A}_{\alpha',\alpha}^{(j)}(x) = w(\alpha') & \text{for all } \alpha \in \widetilde{\Delta}_{p^{(j)}}; \\ \operatorname{ord} \mathscr{A}_{\beta,\alpha}^{(j)}(x) > w(\beta) & \text{if } \beta \text{ and } \alpha \text{ satisfy condition (a),} \\ & \text{(b), or (c) of Proposition 5.2 (ii).} \end{cases}$$

If $\alpha \in \widetilde{\Delta}$, we let $Z(\alpha) = w(\alpha) + w(\alpha') + \cdots + w(\alpha^{(\ell-1)})$ and, for fixed ρ, we let

$$\mathscr{K}_\rho(T) = \prod_{\alpha \in \widetilde{\Delta}_\rho} (1 - p^{Z(\alpha)} T) \in \Omega_1[T].$$

Let$Q = \ell N \prod_{i=1}^n k_i$.

THEOREM 5.2. *The Newton polygon of $L(\overline{f}, \Theta, \rho, T)$ lies below the Newton polygon of $\mathscr{H}_\rho(T)$ and their endpoints coincide at* $(0,0)$ *and* $(Q, Q(n-1)/2)$.

Proof. Let $R = N\prod_{i=1}^n k_i = \dim_{\Omega_0}(W_{X,\rho})$. We can write

$$\det_{\Omega_0}(I - T\overline{\mathscr{F}}_X \mid W_{X,\rho}) = 1 + \sum_{i=1}^R m_i(Y)T^i,$$

and by Proposition 5.1 each $m_i(Y)$ is analytic in the disk $\{y \mid \operatorname{ord} y > -Np/Mq(p-1)\}$. If y satisfies $\operatorname{ord} y = 0$, by the maximum modulus theorem, $\operatorname{ord}(m_i(y)) \leq \operatorname{ord}(m_i(0))$. Observe that if $\alpha, \beta \in \widetilde{\Delta}$ satisfy $\alpha \sim \beta$, $s(\alpha) = s(\beta)$ and $w(\alpha) \leq w(\beta)$, then $w(\alpha') \leq w(\beta')$. Thus, using (5.57), we can order the elements of $\widetilde{\Delta}_{\rho^{(j)}}$ for each j, $0 \leq j \leq \not{f} - 1$, so that the matrices $\mathscr{C}^{(j)}(0)$ are simultaneously upper triangular, with diagonal entries $\{\mathscr{C}^{(j)}_{\alpha^{(j+1)},\alpha^{(j)}}(0) \mid \alpha \in \widetilde{\Delta}_\rho\}$ and $\operatorname{ord}\mathscr{C}^{(j)}_{\alpha^{(j+1)},\alpha^{(j)}}(0) = w(\alpha^{(j+1)})$. Hence for each i, $1 \leq i \leq R$, $\operatorname{ord}(m_i(0))$ is the infimum of all the i-fold sums $\sum Z(\alpha)$, where α runs over a subset of i distinct elements of $\widetilde{\Delta}_\rho$. This establishes the first assertion. By Lemma 2.9, $\sum_{\alpha\in\widetilde{\Delta}_\rho} w(\alpha) = R(n-1)/2$ for any ρ. Hence $\operatorname{ord} m_Q(0) = \not{f}R(n-1)/2$.

On the other hand, estimates (5.58) imply that, for all j, $0 \leq j \leq \not{f} - 1$,

$$\operatorname{ord}(\det \mathscr{A}^{(j)}(x)) = \sum_{\alpha\in\widetilde{\Delta}_\rho(j)} w(\alpha).$$

The second assertion follows. □

COROLLARY 5.1. *If* $p \equiv 1 \pmod r$, *the endpoints of the Newton polygons of* $L(\overline{\not{f}}, \Theta, \rho, T)$ *and of* $\mathscr{H}_\rho(T)$ *coincide.*

THEOREM 5.3. *If* $p \equiv 1 \pmod r$, *(or* $\rho = (0, \ldots, 0)$*), and* $pg_i \equiv g_i \pmod{k_i g_j}$ *for all* $i, j \in \{1, \ldots, n\}$, *the Newton polygons of* $L(\overline{\not{f}}, \Theta, \rho, T)$ *and of* $\mathscr{H}_\rho(T)$ *coincide.*

Proof. Under our assumptions, the permutation $\alpha \mapsto \alpha'$ of Lemma 2.8 is the identity on $\widetilde{\Delta}_\rho$. Using the estimates (5.58), the remainder of the proof is identical to that of **[15**, Theorem 5.46]. □

REMARK. Theorem 5.3 holds in particular when $p \equiv 1 \pmod{MD}$.

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