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BLASCHKE COCYCLES AND GENERATORS

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Dedicated to Professor Shōzo Koshi on his 60th birthday

Using a local product decomposition, we establish a certain class of Blaschke cocycles with the property that a simply invariant subspace has a single generator if and only if its cocycle is cohomologous to one of this class. Some applications are also obtained. We show, among other things, every simply invariant subspace is approximated by a singly generated one as near as desired.

1. Preliminaries. Let Γ be a dense subgroup of the real line R, endowed with the discrete topology, and let K be the dual group of Γ. For each t in R, e_t denotes the element of K defined by $e_t(\lambda) = e^{i\lambda t}$ for any λ in Γ. Then the mapping from t to e_t embeds R continuously onto a dense subgroup of K. Choose and fix a positive γ in Γ, and let K_γ be the subgroup consisting of all x in K such that $x(\gamma) = 1$. Then K may be identified measure theoretically, and almost topologically, with $K_\gamma \times [0, 2\pi/\gamma)$ via the mapping $y + e_s$ to (y, s). We assume, for simplicity, that 2π lies in Γ throughout the paper. Thus K may be regarded as $K_{2\pi} \times [0, 1)$. This local product decomposition is very useful for understanding the group K. We denote by σ and σ_1 the normalized Haar measures on K and $K_{2\pi}$, respectively. Then $d\sigma$ is carried by the above mapping to the restriction of $d\sigma_1 \times dt$ to $K_{2\pi} \times [0, 1)$.

A Borel function V on $K_{2\pi} \times R$ is *automorphic* if $V(y, t+1) = V(y + e_1, t)$ for $d\sigma_1 \times dt$-a.e. (y, t) in $K_{2\pi} \times R$. Every Borel function φ on K has the *automorphic extension* $\varphi^\#$ to $K_{2\pi} \times R$ by

$$\varphi^\#(y, t) = \varphi(y + e_{[\![t]\!]}, t - [\![t]\!])$$

for each (y, t) in $K_{2\pi} \times R$, where $[\![t]\!]$ denotes the largest integer not exceeding t. Conversely, if V is automorphic on $K_{2\pi} \times R$, then there is a function φ on K of which the automorphic extension is V, since V is determined by its values on $K_{2\pi} \times [0, 1)$.

A function φ in $L^1(\sigma)$ is *analytic* if its Fourier coefficients

$$a_\lambda(\varphi) = \int_K \overline{\chi}_\lambda(x)\varphi(x)\, d\sigma(x)$$

vanish for all negative λ in Γ, where χ_λ denotes the character on K defined by $\chi_\lambda(x) = x(\lambda)$. The *Hardy space* $H^p(\sigma)$, $1 \leq p \leq \infty$, is defined to be the space of all analytic functions in $L^p(\sigma)$, and $H^p_0(\sigma)$ denotes the space of all functions φ in $H^p(\sigma)$ for which $a_0(\varphi) = 0$. Recall that a complex-valued function of modulus one is said to be a *unitary function*. An analytic unitary function is called *inner*. A function φ in $H^p(\sigma)$ is *outer* if φ satisfies

$$\log|a_0(\varphi)| = \int_K \log|\varphi(x)|\,d\sigma(x) > -\infty.$$

A closed subspace $\mathfrak{M}$ of $L^2(\sigma)$ is *simply invariant*, often just called *invariant*, if $\mathfrak{M}$ contains strictly $\chi_\lambda\mathfrak{M}$ for any positive λ in Γ. For any simply invariant subspace $\mathfrak{M}$ of $L^2(\sigma)$, we define

$$\mathfrak{M}_+ = \bigcap_{\lambda<0} \chi_\lambda\mathfrak{M} \quad \text{and} \quad \mathfrak{M}_- = \text{ the closure of } \bigcup_{\lambda>0} \chi_\lambda\mathfrak{M}.$$

Then $\mathfrak{M}$ is called to be *normalized* if $\mathfrak{M} = \mathfrak{M}_+$. If φ lies in $L^2(\sigma)$, then we denote by $\mathfrak{M}[\varphi]$ the smallest invariant subspace containing φ, and φ is called a *single generator* of $\mathfrak{M}[\varphi]$. In order for $\mathfrak{M}[\varphi]$ to be simply invariant it is necessary and sufficient that

$$\int_{-\infty}^{\infty} \log|\varphi(x+e_t)|\frac{dt}{1+t^2} > -\infty \tag{1.1}$$

for σ-a.e. x in K.

A *cocycle* is a unitary Borel function $A(x,t)$ on $K \times R$ which satisfies the cocycle identity

$$A(x,t+u) = A(x,t)A(x+e_t,u) \tag{1.2}$$

for all x in K and t, u in R. A cocycle is a *coboundary* if it has the form $\overline{\psi(x)}\psi(x+e_t)$ for some unitary function ψ on K. Two cocycles are called *cohomologous* if one is a coboundary times the other. A one-to-one correspondence is established between normalized invariant subspaces and cocycles (see [**6**; Chapter 2]).

We denote by $H^\infty(dt/(1+t^2))$ the space of all boundary functions of bounded analytic functions in the upper half-plane $\mathscr{H}$. The closure of $H^\infty(dt/(1+t^2))$ in $L^p(dt/(1+t^2))$, $0 < p < \infty$, is denoted by $H^p(dt/(1+t^2))$, where we use the ordinary metric on $L^p(dt/(1+t^2))$ when $0 < p < 1$. The class $N(dt/(1+t^2))$ consists of all boundary functions of analytic functions on $\mathscr{H}$ which are the quotients of two bounded analytic functions.

A cocycle $A(x,t)$ on K is *analytic* if, by considering the restriction to $K_{2\pi} \times R$, the function of t, $A(y,t)$, lies in $H^\infty(dt/(1+t^2))$ for σ_1-a.e. y in $K_{2\pi}$. We say an analytic cocycle $A(x,t)$ is a *Blaschke* or a *singular* cocycle if the function of t, $A(y,t)$, is an inner function of that type for σ_1-a.e. y in $K_{2\pi}$. It follows from (1.2) that our definitions are equivalent to usual ones. There is a vague sense in which the Blaschke cocycles are generic among all cocycles. Surprisingly, it happens that every cocycle is cohomologous to a Blaschke cocycle (see [**6**; Theorem 26]).

Our objective in this paper is to characterize singly generated subspaces in terms of Blaschke cocycles. In the next section, we introduce a certain class of Blaschke cocycles and present some lemmas which we shall use. After preparing some lemmas, the main theorem, Theorem 3.1, is proved in §3. Applications to analyticity are presented in §4, and we close with some remarks in §5.

We refer the reader to [**6**] and [**2**; Chapter VII] for further details of analyticity on compact abelian groups and [**3**] for results about classical Hardy spaces.

The following lemma is a minor variation of known facts, so the proof is omitted.

LEMMA 1.1. *Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$, and let A be the cocycle of $\mathfrak{M}_+$. Then*

(i) *a function φ in $L^2(\sigma)$ lies in $\mathfrak{M}_+$ if and only if the function of t, $A(y,t)\varphi^\#(y,t)$, lies in $H^2(dt/(1+t^2))$ for σ_1-a.e. y in $K_{2\pi}$, and*

(ii) *a function φ in $\mathfrak{M}$ is a single generator of $\mathfrak{M}$ if and only if the function of t, $A(y,t)\varphi^\#(y,t)$, is outer in $H^2(dt/(1+t^2))$ for σ_1-a.e. y in $K_{2\pi}$.*

We see from (i) of Lemma 1.1 that A is analytic if and only if $\mathfrak{M}_+$ contains $H^2(\sigma)$. Equivalently, $\overline{A}$ is analytic if and only if $\mathfrak{M}_+$ is contained in $H^2(\sigma)$.

Let V be a function on $K_{2\pi} \times R$ such that the function of t, $V(y,t)$, lies in $H^1(dt/(1+t^2))$. Then we define

$$V(y,t+ir) = \frac{1}{\pi}\int_{-\infty}^{\infty} V(y,s)\frac{r}{(t-s)^2+r^2}\,ds \tag{1.3}$$

for each $r > 0$. We now derive some simple properties of cocycles by restricting them to $K_{2\pi} \times R$.

Lemma 1.2. *Let A be an analytic cocycle, and let $\mathfrak{M}$ be the normalized invariant subspace with cocycle $\overline{A}$. If $r > 0$, then we have*

(i) *$|A(y, t+ir)|$ is automorphic on $K_{2\pi} \times R$, so there is a function v on K with $0 \le v \le 1$ for which*

$$|A(y, t+ir)| = v^{\#}(y, t)$$

on $K_{2\pi} \times R$, and

(ii) *if we write $\varphi^{\#}(y, t) = A(y, t)V(y, t)$ for any φ in $\mathfrak{M}$, then there is a function ψ in $\mathfrak{M}$ such that*

$$\psi^{\#}(y, t) = A(y, t)V(y, t+ir)$$

on $K_{2\pi} \times R$.

Proof. (i) We see by (1.2) that the function $A(y, z+1)A(y+e_1, z)^{-1}$ on $K_{2\pi} \times \mathscr{H}$ is a unitary function only of y. This implies that

$$|A(y, t+ir+1)| = |A(y+e_1, t+ir)|$$

on $K_{2\pi} \times R$. Thus (i) follows from the definition of automorphic functions.

(ii) Observe that the function of t, $V(y, t)$, lies in $H^2(dt/(1+t^2))$ by (i) of Lemma 1.1. By the similar way as above we see that $A(y, t)V(y, t+ir)$ is also the automorphic extension of a function ψ on K. It follows from Lemma 1.1 that ψ lies in $\mathfrak{M}$ again.

The next elementary fact will be used later.

Lemma 1.3. *Let $B(z)$ be a Blaschke product on $\mathscr{H}$, and let $\{t_n + is_n\}_{n=1}^{\infty}$ be the zeros of $B(z)$, listed according to their multiplicities. If $\{s_n\}_{n=1}^{\infty}$ is bounded and bounded away from zero, then the infinite product*

$$u(z) = \prod_{n=1}^{\infty} \frac{(z-t_n)^2}{(z-t_n)^2 + s_n^2}$$

defines a meromorphic function in the complex plane C which has a pole at each point $t_n \pm is_n$. Furthermore, the function $u(t)B(t)$ on R is an outer function in $H^{\infty}(dt/(1+t^2))$.

Proof. Recall that the Blaschke condition is given by

$$\sum_{n=1}^{\infty} \frac{s_n}{t_n^2 + s_n^2 + 1} < \infty.$$

The hypotheses imply that, on each compact subset of $C \setminus \{t_n \pm is_n\}$, there is a constant $M > 0$ such that

$$\left|1 - \frac{(z-t_n)^2}{(z-t_n)^2 + s_n^2}\right| = \frac{s_n^2}{(z-t_n)^2 + s_n^2} \leq M \frac{s_n}{t_n^2 + s_n^2 + 1}$$

for all n. It follows from [7; Theorem 15.6] that $u(z)$ converges uniformly on each compact subset of $C \setminus \{t_n \pm is_n\}$. It is also easy to see that $u(z)$ has a pole at each $t_n \pm is_n$; thus the first part is obtained.

For the second part, let u_N and B_N be the Nth partial products of u and B, respectively. Since $u_N(z)B_N(z)$ is analytic at $z = \infty$, we verify easily that $u_N(t)B_N(t)$ is outer in $H^\infty(dt/(1+t^2))$. Then

$$\begin{aligned}\log|u_N B_N(i)| &= \frac{1}{\pi}\int_{-\infty}^{\infty} \log|u_N B_N(t)| \frac{dt}{1+t^2} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \log|u_N(t)| \frac{dt}{1+t^2} > -\infty\end{aligned}$$

since $|B_N(t)| = 1$. Observe that $0 \leq u_{N+1} \leq u_N \leq 1$ on R and that

$$\lim_{N\to\infty} u_N B_N(i) = uB(i) \neq 0.$$

It follows from the monotone convergence theorem that

$$\log|uB(i)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \log|uB(t)| \frac{dt}{1+t^2} > -\infty;$$

thus $u(t)B(t)$ is an outer function in $H^\infty(dt/(1+t^2))$.

2. A certain class of Blaschke cocycles. Let q be a Borel function on $K_{2\pi}$ which takes nonnegative integral values. We call q a *multiplicity function* on $K_{2\pi}$ if q satisfies

$$\sum_{n\neq 0} q(y + e_n)\frac{1}{n^2} < \infty \tag{2.1}$$

for σ_1-a.e. y in $K_{2\pi}$. Obviously if q lies in $L^1(\sigma_1)$, then q is a multiplicity function on $K_{2\pi}$. For the remainder of this paper, we always fix an $\alpha > 0$. Let $\mathbb{E}$ be the Borel set in $K_{2\pi} \times \mathscr{H}$ of all $(y, \frac{1}{2} + n + i\alpha)$ for $n = 0, \pm 1, \pm 2, \ldots$. By using a multiplicity function q on $K_{2\pi}$, we define a Borel function $\tilde{q}$ on $\mathbb{E}$ by

$$\tilde{q}(y, \tfrac{1}{2} + n + i\alpha) = q(y + e_n).$$

Then $\mathbb{E}$ and $\tilde{q}$, interpreted as a zero set $\mathbb{E}$ with multiplicity function $\tilde{q}$, satisfy all properties arised from a Blaschke cocycle, that is,

$$\sum_{n=-\infty}^{\infty} \frac{\tilde{q}(y, \frac{1}{2}+n+i\alpha)}{(\frac{1}{2}+n)^2+\alpha^2+1} < \infty, \quad \text{and}$$

$$\tilde{q}(y, \tfrac{1}{2}+n+1+i\alpha) = \tilde{q}(y+e_1, \tfrac{1}{2}+n+i\alpha)$$

for σ_1-a.e. y in $K_{2\pi}$. Thus we can construct a Blaschke cocycle B_q^α whose zero set matches $\mathbb{E}$, and whose multiplicity function matches $\tilde{q}$ by [**5**; Theorem 1 and Remark in §5]. We say that B_q^α is *the Blaschke cocycle induced by a multiplicity function q on $K_{2\pi}$*. Of course, $B_q^\alpha = 1$ if $q = 0$. The structure of B_q^α is so simple that we can describe it easily: Let

$$g(z) = \frac{z - i\alpha}{z + i\alpha}. \tag{2.2}$$

Then B_q^α can be written as

$$B_q^\alpha(y,t) = p(y) \prod_{n=-\infty}^{\infty} \{\varepsilon_n g(t - \tfrac{1}{2} - n)\}^{q(y+e_n)} \tag{2.3}$$

where ε_n with $|\varepsilon_n| = 1$ is chosen so that $\varepsilon_n g(i-n) > 0$, and where $p(y)$ is the unitary function on $K_{2\pi}$ that makes $B_q^\alpha(y,0) = 1$. Recall that there is a canonical way of extending the restriction of a cocycle to $K_{2\pi} \times R$ to the cocycle on K (see [**2**; Chapter VII, §11]).

LEMMA 2.1. (i) *There is a multiplication function q on $K_{2\pi}$ which does not lie in $L^1(\sigma_1)$.*

(ii) *Let B_q^α be the Blaschke cocycle induced by such q, and let v be the function on K such that $|B_q^\alpha(y, t+ir)| = v^\#(y,t)$ on $K_{2\pi} \times R$ for $r > 0$ (see Lemma 1.2). Then $\log v$ cannot lie in $L^1(\sigma)$ for all $r > 0$.*

Proof. (i) It is well-known that there is a function w on K with $0 \le w \le 1$ such that $\log w$ does not lie in $L^1(\sigma)$, while w satisfies (1.1) with w in place of $|\varphi|$ (cf. [**2**; Chapter VII, Lemma 9.2]). Regarding w as a function on $K_{2\pi} \times [0,1)$, we define a nonnegative integral value function q on $K_{2\pi}$ by

$$q(y) = \left[\!\!\left[-\int_0^1 \log w(y,s)\, ds \right]\!\!\right].$$

Then it is easy to see that q does not lie in $L^1(\sigma_1)$ but has the property (2.1).

(ii) Let g be the function in (2.2). Then by (2.3) we see

$$\begin{aligned} |B_q^\alpha(y, t+ir)| &= \prod_{n=-\infty}^{\infty} |g(t - \tfrac{1}{2} - n + ir)|^{q(y+e_n)} \\ &\le |g(t - \tfrac{1}{2} + ir)|^{q(y)} \\ &= \left\{ \frac{(t-\frac{1}{2})^2 + (r-\alpha)^2}{(t-\frac{1}{2})^2 + (r+\alpha)^2} \right\}^{q(y)/2}, \end{aligned}$$

on $K_{2\pi} \times [0, 1)$ since $|g(z)| \le 1$ on $\mathscr{H}$. We then put

$$\int_0^1 \log \frac{(t-\frac{1}{2})^2 + (r-\alpha)^2}{(t-\frac{1}{2})^2 + (r+\alpha)^2}\, dt = a < 0.$$

Since we have

$$\begin{aligned} \int_K \log v(x)\, d\sigma(x) &= \int_{K_{2\pi}} \left\{ \int_0^1 \log v(y, t)\, dt \right\} d\sigma_1(y) \\ &\le \int_{K_{2\pi}} \tfrac{1}{2} a q(y)\, d\sigma_1(y) = -\infty, \end{aligned}$$

$\log v$ does not lie in $L^1(\sigma)$.

We now introduce nonnegative functions u on K for which the functions of t, $u(x + e_t)$, can be extended as meromorphic functions on $\mathscr{H}$. This utility depends on the fact that they can remove the zeros of $B_q^\alpha(x, z)$ by multiplying one of them.

Let $f_\alpha(z)$ be the meromorphic function on C given by

$$f_\alpha(z) = \frac{z^2}{z^2 + \alpha^2}. \tag{2.4}$$

Notice that $0 \le f_\alpha(t) \le 1$ on R. It is convenient to calculate the equation

$$\int_{-\infty}^{\infty} \log f_\alpha(t)\, dt = -2\alpha\pi. \tag{2.5}$$

Let q be a multiplicity function on $K_{2\pi}$, and let

$$\Omega = C \setminus \{\tfrac{1}{2} + n \pm i\alpha; n = 0, \pm 1, \pm 2, \ldots\}.$$

It follows from Lemma 1.3 that the infinite product

$$U(y, z) = \prod_{-\infty}^{\infty} f_\alpha(z - \tfrac{1}{2} - n)^{q(y+e_n)} \tag{2.6}$$

converges uniformly on each compact subset of Ω for σ_1-a.e. y in $K_{2\pi}$. We also see that $U(y,z)$ has a pole of multiplicity $q(y+e_n)$ at $\frac{1}{2}+n\pm i\alpha$. Since $f_\alpha(t-\frac{1}{2}-n)^{p(y+e_n)}$ is a Borel function on $K_{2\pi}\times R$ for all n, so is $U(y,t)$. Observe that $U(y,t)$ is automorphic on $K_{2\pi}\times R$, that is, $U(y,t+1)=U(t+e_1,t)$ on $K_{2\pi}\times R$. So there is a function u_q on K for which $U(y,t)=u_q^{\#}(y,t)$. Thus we obtain a function u_q on K satisfying that $0\le u_q\le 1$ and that the function of t, $u_q^{\#}(y,t)$, can be extended as a meromorphic function $u_q^{\#}(y,z)$ which has no zeros on $\mathscr{H}$ and has a pole of multiplicity $q(y+e_n)$ at each $\frac{1}{2}+n+i\alpha$ for σ_1-a.e. y in $K_{2\pi}$. This u_q is called *the function on K induced by $(f_\alpha(t-\frac{1}{2}),q(y))$ via infinite product*. Of course, in this definition, we may replace f_α with another suitable function.

LEMMA 2.2. *Let q be a multiplicity function on $K_{2\pi}$, and let B_q^α and u_q be as in above. Suppose that $\mathfrak{M}$ is the normalized invariant subspace with cocycle B_q^α. Then*

(i) *if q lies in $L^1(\sigma_1)$, then B_q^α is a coboundary, equivalently $\mathfrak{M}$ is generated by a unitary function, and*

(ii) *if q does not lie in $L^1(\sigma_1)$, then* $\log u_q$ *does not lie in $L^1(\sigma)$ and $\mathfrak{M}_-$ is generated by u_q.*

Proof. By Lemma 1.3, the function of t, $B_q^\alpha(y,t)u_q^{\#}(y,t)$, is an outer function in $H^\infty(dt/(1+t^2))$ for σ_1-a.e. y in $K_{2\pi}$. Then we see from Lemma 1.1 that the cocycle of $\mathfrak{M}[u_q]_+$ is B_q^α. On the other hand, let f_α be the function in (2.4). Since $\log f_\alpha(t)\le 0$ on R, it follows from (2.5), (2.6) and Fubini's theorem that

$$
\begin{aligned}
(2.7)\ \int_K &\log u_q(x)\,d\sigma(x)\\
&=\int_{K_{2\pi}}\left\{\int_0^1\sum_{n=-\infty}^{\infty} q(y+e_n)\log f_\alpha(t-\tfrac{1}{2}-n)\,dt\right\}d\sigma_1(y)\\
&=\int_{-\infty}^{\infty}\log f_\alpha(t)\,dt\cdot\int_{K_{2\pi}} q(y)\,d\sigma_1(y)\\
&=-2\alpha\pi\int_{K_{2\pi}} q(y)\,d\sigma_1(y).
\end{aligned}
$$

(i) If q lies in $L^1(\sigma_1)$, then (2.7) above implies that $\log u_q$ lies in $L^1(\sigma)$. Hence there is a unitary function ψ on K such that

$$\mathfrak{M}[u_q]=\mathfrak{M}[u_q]_+=\psi H^2(\sigma)$$

by Szegö's theorem. This shows also that $B_q^\alpha(x,t) = \psi(x)\overline{\psi(x+e_t)}$ on $K \times R$.

(ii) Suppose q does not lie in $L^1(\sigma_1)$. Then we see by (2.7) that $\log u_q$ does not lie in $L^1(\sigma)$, so u_q belongs to $\mathfrak{M}_-$. It follows from the preceding remark that $\mathfrak{M}[u_q] = \mathfrak{M}_-$.

We remark that if $\mathfrak{M} \neq \mathfrak{M}_-$ in (ii), although we do not know if such a case occurs, then $\mathfrak{M} = \psi H^2(\sigma)$ for some unitary function ψ on K. Thus $\mathfrak{M}$ has also a single generator.

3. Singly generated subspaces. In this section, we show the converse to Lemma 2.2 essentially holds. This enables us to characterize single generated subspaces by means of Blaschke cocycles. The following theorems give the details.

THEOREM 3.1. *Let w be a function on K with $0 \leq w \leq 1$ satisfying* (1.1) *with w in place of $|\varphi|$, and let f_α be the function in* (2.4). *Define a multiplicity function q on $K_{2\pi}$ by*

$$q(y) = \left[\!\!\left[-\frac{1}{2\alpha\pi} \int_0^1 \log w(y,t)\, dt \right]\!\!\right]. \tag{3.1}$$

If u_q is the function on K induced by $(f_\alpha(t-\frac{1}{2}), q(y))$ via infinite product, then there is a unitary function ψ on K for which

$$\mathfrak{M}[w] = \psi \mathfrak{M}[u_q].$$

We can restate Theorem 3.1, together with Lemma 2.2, in terms of cocycles.

THEOREM 3.2. *Let $\mathfrak{M}$ be a simply invariant subspace, and let A be the cocycle of $\mathfrak{M}_+$. Then $\mathfrak{M}$ is generated by one of its elements if and only if A is cohomologous to the Blaschke cocycle B_q^α induced by some multiplicity function q on $K_{2\pi}$. In particular, $H_0^2(\sigma)$ has a single generator if and only if there exists a coboundary of the form B_q^α where q does not lie in $L^1(\sigma_1)$.*

We begin with adopting a wider definition of outer functions. Let φ be a Borel function on K. We call φ an *outer function on K in the wide sense* if the function of t, $\varphi(x+e_t)$, is an outer function in the class $N(dt/(1+t^2))$ for σ-a.e. x in K. It follows, of course,

$$\int_{-\infty}^{\infty} |\log|\varphi(x+e_t)|| \frac{dt}{1+t^2} < \infty, \tag{3.2}$$

although not only $\log|\varphi|$ but also φ may not belong to $L^1(\sigma)$.

LEMMA 3.3. *Let w be a nonnegative function on K satisfying* (3.2) *wtih w in place of $|\varphi|$. Define a function p on $K_{2\pi}$ by*

$$p(y) = \int_0^1 \log w(y,t)\,dt.$$

If p belongs to $L^1(\sigma_1)$, then there is an outer function φ on K in the wide sense for which $|\varphi| = w$.

Proof. Consider p as a function on K by $p(y,t) = p(y)$ for each (y,t) in $K_{2\pi} \times [0,1)$. Let $p = p^+ - p^-$ where p^+ and p^- are positive and negative parts of p. By Szegö's theorem we can find outer functions φ_1 and φ_2 in $H^\infty(\sigma)$ so that $|\varphi_1| = \exp(-p^+)$ and $|\varphi_2| = \exp(-p^-)$. If we put $\varphi_3 = \varphi_1^{-1}\varphi_2$, then φ_3 is an outer function in the wide sense for which $|\varphi_3| = e^p$. Thus, by replacing w with we^{-p}, we may assume $p = 0$.

Let us consider the Hilbert transform $V(y,t)$ of $\log w^\#(y,t)$ on $K_{2\pi} \times R$, explicitly

$$V(y,t) = \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{\varepsilon < |t-s| < 1/\varepsilon} \log w^\#(y,s) \frac{1}{t-s}\,ds.$$

By our assumption, we may replace $(t-s)^{-1}$ by $(t-s)^{-1} - (t - [\![s]\!])^{-1}$ in the above integral when $|t - [\![s]\!]| > 1$. Then we see easily that this integral converges for σ_1-a.e. y in $K_{2\pi}$. Observe that $V(y,t)$ is an automorphic Borel function on $K_{2\pi} \times R$. Therefore there is a function v on K for which $V(y,t) = v^\#(y,t)$ on $K_{2\pi} \times R$. This implies that the function of t, $v(x+e_t)$, is a conjugate function of $\log w(x+e_t)$ for σ-a.e. x in K. Thus

$$\varphi(x) = \exp\{\log w(x) + iv(x)\}$$

is the function with desired properties.

LEMMA 3.4. *Let w be a function as in Lemma* 3.3. *For each r in R, there is an outer function φ on K in the wide sense for which $|\varphi(x)| = w(x)w(x+e_r)^{-1}$.*

Proof. We notice that the function of t,

$$\log w(x+e_t) - \log w(x+e_t+e_r),$$

belongs to $L^1(dt/(1 + t^2))$ for σ-a.e. x in K. Consider the Hilbert transform $v(x)$ of it. Then we have

$$\begin{aligned} v(x) &= \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{\varepsilon<|s|<1/\varepsilon} \{\log w(x+e_s) - \log w(x+e_s+e_r)\} \frac{1}{-s}\, ds \\ &= \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{\varepsilon<|s|,|s-r|} \log w(x+e_s) \left\{ \frac{1}{s-r} - \frac{1}{s} \right\} ds. \end{aligned}$$

Since $(s-r)^{-1} - s^{-1} = O(s^{-2})$, as $|s| \to \infty$, the above integral converges. Thus the function

$$\varphi(x) = \exp\{\log w(x) - \log w(x+e_r) + iv(x)\}$$

satisfies the desired properties.

LEMMA 3.5. *Let w be a function as in Lemma 3.3, and let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence in R with the property that $|a_n| = O(n^{-2})$, as $|n| \to \infty$. Then the infinite product*

$$w_1(x) = \prod_{n=-\infty}^{\infty} w(x+e_n)^{a_n}$$

converges σ-a.e. x in K and satisfies (3.2) with w_1 in place of $|\varphi|$.

Proof. We may assume that $0 \le w \le 1$ and $a_n \ge 0$ for all n. Let $f(s) = 1/(1+s^2)$. Then we see the Fourier transform of f,

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{-its} f(s)\, ds,$$

is equal to $\pi e^{-|t|}$. Since the convolution $f * f$ of f and f satisfies that

$$(f * f)\hat{}\,(t) = \pi^2 e^{-|2t|} = (2\pi/(4+s^2))\hat{}\,(t),$$

it follows from the inversion theorem that $f * f(s) = 2\pi/(4+s^2)$.

On the other hand, let $h(t) = a_n$ on $[n, n+1)$ for all n. There is a constant $C > 0$ such that $h(t) \le Cf(t)$ and $f(s - [t]) \le Cf(s-t)$ for s, t in R. This yields that

$$\text{(3.3)} \qquad \begin{aligned} \sum_{n=-\infty}^{\infty} \frac{a_n}{1+(s-n)^2} &\le C \int_{-\infty}^{\infty} h(t) f(s-t)\, dt \\ &\le C^2 f * f(s) = \frac{2\pi C^2}{4+s^2}. \end{aligned}$$

Our assumption shows that $\log w(x+e_t) \leq 0$, so we have

$$\begin{aligned}\int_{-\infty}^{\infty} \log w_1(x+e_t)\frac{dt}{1+t^2} &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_n \log w(x+e_n+e_t)\frac{dt}{1+t^2} \\ &= \int_{-\infty}^{\infty} \log w(x+e_s) \sum_{n=-\infty}^{\infty} \frac{a_n}{1+(s-n)^2}\, ds \\ &\geq C^2 \int_{-\infty}^{\infty} \log w(x+e_s)\frac{2\pi}{4+s^2}\, ds > -\infty\end{aligned}$$

by (3.2). Simultaneously this assures the convergence of the product $w_1(x)$ since

$$\sum_{n=-\infty}^{\infty} a_n \log w(x+e_n) > -\infty$$

for σ-a.e. x in K, which completes the proof.

LEMMA 3.6. *Let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence in R such that*

$$a_n = O(n^{-4}), \quad \text{as } |n| \to \infty, \quad \text{and} \tag{3.4}$$

$$\sum_{n=-\infty}^{\infty} a_n = 1. \tag{3.5}$$

Let w and w_1 be as in Lemma 3.5. Then there is an outer function φ on K in the wide sense for which $|\varphi| = w_1 w^{-1}$. In particular, suppose that w is bounded and that $a_n \geq 0$. Then there is a unitary function ψ on K for which $\mathfrak{M}[w] = \psi\mathfrak{M}[w_1]$.

Proof. Observe that the automorphic extension $\log w^{\#}(y,t)$ lies in $L^1(dt/(1+t^2))$ as a function of t for σ_1-a.e. y in $K_{2\pi}$. Let $V(y,t)$ be the Hilbert transform of $\log w^{\#}(y,t)$ with the normalization $V(y,i)=0$, that is,

$$V(y,t) = \lim_{\varepsilon\to+0} \frac{1}{\pi}\int_{\varepsilon<|t-s|} \log w^{\#}(y,s)\left\{\frac{1}{t-s}+\frac{s}{1+s^2}\right\} ds.$$

Let $\frac{1}{2} \leq p < 1$. It then follows from Kolmogoroff's estimate [**3**; Chapter III, Theorem 2.1] that the function of t, $V(y,t)$, lies in $L^p(dt/(1+t^2))$ for σ_1-a.e. y in $K_{2\pi}$. We notice that the function of t, $\log w^{\#}(y,t) + iV(y,t)$, may be extended to $\mathscr{H}$ analytically.

Since $\log w^{\#}(y,t+1) = \log w^{\#}(y+e_1,t)$ on $K_{2\pi}\times R$, we see that

$$V(y,t+1) - V(y+e_1,t) = V(y,i+1) \tag{3.6}$$

on $K_{2\pi} \times R$, a function only of y. We define a Borel function $U(y,t)$ on $K_{2\pi} \times R$ by

$$U(y,t) = \sum_{n=-\infty}^{\infty} a_n V(y, t+n). \tag{3.7}$$

Then we claim that the function of t, $U(y,t)$, also belongs to $L^p(dt/(1+t^2))$ for σ_1-a.e. y in $K_{2\pi}$. Indeed, recall that $L^p(dt/(1+t^2))$ is a complete metric space whose metric d is given by

$$d(f,g) = \int_{-\infty}^{\infty} |f(t) - g(t)|^p \frac{dt}{1+t^2}$$

for f, g in $L^p(dt/(1+t^2))$. Since $|a_n|^p = O(n^{-2})$, as $|n| \to \infty$, by (3.4), it follows from (3.3) with $|a_n|^p$ in place of a_n that

$$\begin{aligned}\int_{-\infty}^{\infty} |U(y,t)|^p \frac{dt}{1+t^2} &\le \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} |a_n V(y,t+n)|^p \frac{dt}{1+t^2} \\ &= \int_{-\infty}^{\infty} |V(y,s)|^p \sum_{n=-\infty}^{\infty} \frac{|a_n|^p}{1+(s-n)^2} ds \\ &\le C^2 \int_{-\infty}^{\infty} |V(y,s)|^p \frac{2\pi}{4+s^2} ds < \infty.\end{aligned}$$

By Lemma 3.5, we see that the function of t, $\log w_1^{\#}(y,t)$, belongs to $L^1(dt/(1+t^2))$ for σ_1-a.e. y in $K_{2\pi}$. From the definition (3.7) of $U(y,t)$, it follows that the function of t, $U(y,t)$, is a conjugate function of $\log w_1^{\#}(y,t)$, and $\log w_1^{\#}(y,t) + iU(y,t)$ belongs to $H^p(dt/(1+t^2))$ for σ_1-a.e. y in $K_{2\pi}$.

On the other hand, by (3.5) and (3.6), we obtain that

$$U(y,t+1) - U(y+e_1,t) = V(y,i+1)$$

on $K_{2\pi} \times R$. Together with (3.6), this yields

$$U(y,t+1) - V(y,t+1) = U(y+e_1,t) - V(y+e_1,t)$$

on $K_{2\pi} \times R$, that is, $U(y,t) - V(y,t)$ is automorphic. So we may find a function u on K for which

$$u^{\#}(y,t) = U(y,t) - V(y,t)$$

on $K_{2\pi} \times R$. Therefore the function of t, $u(x+e_t)$, is a conjugate function of $\log w_1(x+e_t) - \log w(x+e_t)$ for σ-a.e. x in K. Thus if we put

$$\varphi(x) = \exp\{\log w_1(x) - \log w(x) + iu(x)\},$$

then φ is the outer function on K in the wide sense such that $|\varphi| = w_1 w^{-1}$.

Suppose that w is bounded and $a_n \geq 0$. Then w_1 is also bounded and Lemma 3.5 assures that $\mathfrak{M}[w_1]$ is simply invariant. Since

$$\begin{aligned} e^{iu(x)} w_1(x) &= \exp\{\log w_1(x) - \log w(x) + iu(x)\} w(x) \\ &= \varphi(x) w(x) \end{aligned}$$

on K, it follows from Lemma 1.1 that the cocycle of $\mathfrak{M}[w]_+$ coincides with the one of $e^{iu}\mathfrak{M}[w_1]_+$. Observe that $\log w_1$ lies in $L^1(\sigma)$ if and only if so does $\log w$. Thus by Szegö's theorem we conclude $\mathfrak{M}[w] = e^{iu}\mathfrak{M}[w_1]$.

Now we may offer a proof of our main result stated at the beginning of this section.

Proof of Theorem 3.1. If $\log w$ lies in $L^1(\sigma)$, then the function q in (3.1) lies in $L^1(\sigma)$. So it follows from Szegö's theorem and (i) of Lemma 2.2 that $\mathfrak{M}[w] = \theta_1 H^2(\sigma)$ and $\mathfrak{M}[u_q] = \theta_2 H^2(\sigma)$ for some unitary functions θ_1 and θ_2 on K. Thus we may assume that $\log w$ does not lie in $L^1(\sigma)$. We then notice that $\mathfrak{M}[w] = \mathfrak{M}[w]_-$ and $\mathfrak{M}[u_q] = \mathfrak{M}[u_q]_-$ by (ii) of Lemma 2.2.

If we define a function p_1 on $K_{2\pi}$ by

$$p_1(y) = -\int_0^1 \log w(y, t)\, dt,$$

then it follows from Lemma 3.3 that there is an outer function φ on K in the wide sense for which $|\varphi| = w \exp(p_1)$, where p_1 is regarded as a function on $K = K_{2\pi} \times [0, 1)$. Hence we can choose a unitary function ψ_1 on K such that

$$\mathfrak{M}[w] = \psi_1 \mathfrak{M}[\exp(-p_1)]. \tag{3.8}$$

Define a meromorphic function g_α on $\mathscr{H}$ by

$$g_\alpha(z) = \frac{z^4}{z^4 + 4\alpha^4}.$$

Since $z^4 + 4\alpha^4 = \{(t-\alpha)^2 + \alpha^2\}\{(t+\alpha)^2 + \alpha^2\}$, $g_\alpha(z)$ has a pole of multiplicity 1 at $z = \alpha(\pm 1 + i)$ in $\mathscr{H}$. Easy calculation shows that

$$\int_{-\infty}^{\infty} \log g_\alpha(t)\, dt = -4\alpha\pi.$$

We then define a multiplicity function q_1 on $K_{2\pi}$ by

$$q_1(y) = \left[\!\left[\frac{1}{4\alpha\pi} p_1(y) \right]\!\right].$$

Since $4\alpha\pi q_1 \leq p_1 < 4\alpha\pi(q_1+1)$ and q_1 does not lie in $L^1(\sigma)$ as a function on K, it follows from Lemma 3.3 that there is a unitary function ψ_2 on K such that

$$\mathfrak{M}[\exp(-p_1)] = \psi_2 \mathfrak{M}[\exp(-4\alpha\pi q_1)]. \tag{3.9}$$

We next put

$$a_n = -\frac{1}{4\alpha\pi}\int_{-n}^{-n+1} \log g_\alpha(t-\tfrac{1}{2})\,dt.$$

Then we see easily the sequence $\{a_n\}_{n=-\infty}^{\infty}$ with $a_n > 0$ satisfies the conditions (3.4) and (3.5) in Lemma 3.6. Since $\sum_{n=-\infty}^{\infty} a_n q_1(x+e_n)$ does not belong to $L^1(\sigma)$, there is a unitary function ψ_3 on K for which

$$\mathfrak{M}[\exp(-4\alpha\pi q_1)] = \psi_3 \mathfrak{M}\left[\exp\left(-4\alpha\pi \sum_{n=-\infty}^{\infty} a_n q_1(x+e_n)\right)\right]. \tag{3.10}$$

Let v be the function on K induced by $(g_\alpha(t-\frac{1}{2}), q_1(y))$ via infinite product. Then we see that $0 \leq v \leq 1$ on K and that the function of t, $v^\#(y,t)$, may be extended to $\mathscr{H}$ as a meromorphic function $v^\#(y,z)$, which has a pole of multiplicity $q_1(y+e_n)$ at $z = (\frac{1}{2} \pm \alpha) + n + i\alpha$ and has no zeros on $\mathscr{H}$ for σ_1-a.e. y in $K_{2\pi}$. Let B be the Blaschke cocycle determined by the property that the function of z, $B(y,z)$, in $\mathscr{H}$ has a zero of multiplicity $q(y+e_n)$ at $z = (\frac{1}{2} \pm \alpha) + n + i\alpha$ and has no zeros elsewhere. Then it can be seen by Lemmas 1.1 and 1.3 that the cocycle of $\mathfrak{M}[v]_+$ is B. On the other hand, it follows from the definition of v that

$$\begin{aligned}\int_0^1 \log v(y,t)\,dt &= \sum_{n=-\infty}^{\infty} q_1(y+e_n)\int_0^1 \log g_\alpha(t-\tfrac{1}{2}-n)\,dt \\ &= -4\alpha\pi \sum_{n=-\infty}^{\infty} a_n q_1(y+e_n).\end{aligned}$$

From this fact we see also $\log v$ does not lie in $L^1(\sigma)$. Therefore it follows from Lemma 3.3 that there is a unitary function ψ_4 on K for which

$$\mathfrak{M}\left[\exp\left(-4\alpha\pi \sum_{n=-\infty}^{\infty} a_n q_1(x+e_n)\right)\right] = \psi_4 \mathfrak{M}[v]. \tag{3.11}$$

Let f_α be the function in (2.4), and let u_1 be the function on K induced by $(f_\alpha(t-\frac{1}{2}-\alpha)f_\alpha(t-\frac{1}{2}+\alpha), q_1(y))$ via infinite product.

Then we see easily that $0 \le u_1 \le 1$ on K and $\log u_1$ does not lie in $L^1(\sigma)$ by the same way as the proof of Lemma 2.2. Since $\mathfrak{M}[u_1]_+$ has the same Blaschke cocycle as $\mathfrak{M}[v]_+$ has, we thus obtain

$$\mathfrak{M}[v] = \mathfrak{M}[u_1]. \tag{3.12}$$

Let u be the function on K induced by $(f_\alpha(t - \frac{1}{2}), q_1(y))$ via infinite product. It follows from Lemma 3.4 that there are outer functions φ_1 and φ_2 in the wide sense so that $|\varphi_1(x)| = u(x)u(x + e_\alpha)^{-1}$ and $|\varphi_2(x)| = u(x)u(x - e_\alpha)^{-1}$ on K. Observe that

$$u_1(x) = u(x + e_\alpha)u(x - e_\alpha),$$

and $\log u$ does not lie in $L^1(\sigma)$. Then we see that there is a unitary function ψ_5 on K such that

$$\mathfrak{M}[u_1] = \psi_5 \mathfrak{M}[u^2]. \tag{3.13}$$

It is easy to see that the cocycle of $\mathfrak{M}[u^2]_+$ is $B^\alpha_{2q_1}$. Let q be the multiplicity function on K given by (3.1). Then we have

$$2q_1(y) \le q(y) \le 2q_1(y) + 1,$$

from the definition of q_1. So if $q_2 = q - 2q_1$, then q_2 becomes a multiplicity function on $K_{2\pi}$. By (i) of Lemma 2.2, $B^\alpha_{q_2}$ is a coboundary. If u_q is the function on K induced by $(f_\alpha(t - \frac{1}{2}),\ q(y))$ via infinite product, then the cocycle of $\mathfrak{M}[u_q]_+$ is B^α_q by (ii) of Lemma 2.2. Thus we may choose a unitary function ψ_6 on K for which

$$\mathfrak{M}[u^2] = \psi_6 \mathfrak{M}[u_q]. \tag{3.14}$$

Define the unitary function ψ on K by

$$\psi = \psi_1\psi_2\psi_3\psi_4\psi_5\psi_6.$$

It then follows from the equalities from (3.8) to (3.14) that $\mathfrak{M}[w] = \psi\mathfrak{M}[u_q]$. This completes the proof.

Proof of Theorem 3.2. Suppose that $\mathfrak{M}$ has a single generator φ. Then we may assume that $0 \le \varphi \le 1$ by Szegö's theorem. It follows from Theorem 3.1 that there are a unitary function ψ on K and a multiplicity function q on $K_{2\pi}$ so that $\mathfrak{M} = \psi\mathfrak{M}[u_q]$ where u_q denotes the function on K induced by $(f_\alpha(t - \frac{1}{2}), q(y))$ via infinite product. Thus Lemma 2.2 shows that the cocycle A of $\mathfrak{M}_+$ is cohomologous to B^α_q. Converse is a consequence of Lemma 2.2, so the proof is complete.

4. Applications. We first ask under what conditions a Blaschke cocycle B is cohomologous to the one B^α_q induced by a multiplicity function q on $K_{2\pi}$.

THEOREM 4.1. *Let B be a Blaschke cocycle which has no zeros on $K_{2\pi} \times \{\operatorname{Im} z > r\}$ for some $r > 0$. Then there is a multiplicity function q on $K_{2\pi}$ such that B is cohomologous to B_q^α. Consequently, the invariant subspace with cocycle B is singly generated.*

Proof. We let $F(y,t)$ be a function on $K_{2\pi} \times R$ defined by

$$F(y,t) = B(y,t+ir),$$

where $B(y,t+ir)$ is given by (1.3) with B in place of V. Then the hypothesis implies that the function of t, $F(y,t)$, is an outer function in $H^\infty(dt/(1+t^2))$ for σ_1-a.e. y in $K_{2\pi}$. Observe that $\overline{B(y,t)}F(y,t)$ is automorphic on $K_{2\pi} \times R$. Then, together with (i) of Lemma 1.2, there are two functions w and v on K whose automorphic extensions $w^\#$ and $v^\#$ satisfy $|F(y,t)| = w^\#(y,t)$ and $\overline{B(y,t)}F(y,t) = v^\#(y,t)$ on $K_{2\pi} \times R$. Since $0 \le w \le 1$ on K, it follows from Theorem 3.1 and Lemma 2.2 that there is a unitary function ψ_1 on K such that the cocycle of $\psi_1 \mathfrak{M}[w]_+ = \mathfrak{M}[\psi_1 w]_+$ is B_q^α for some multiplicity function q on $K_{2\pi}$. By (ii) of Lemma 1.1 we see that the function of t, $B_q^\alpha(y,t)(\psi_1 w)^\#(y,t)$, is an outer function in $H^\infty(dt/(1+t^2))$ for σ_1-a.e. y in $K_{2\pi}$. Then the zeros of $F(y,z)\{(\psi_1 w)^\#(y,z)\}^{-1}$ exactly match zeros of $B_q^\alpha(y,z)$ on $K_{2\pi} \times \mathscr{H}$.

On the other hand, since $w = |v|$, if we put $\psi = v(\psi_1 w)^{-1}$, then ψ is a unitary function on K. Since

$$\begin{aligned} B(y,t)\psi^\#(y,t) &= B(y,t)v^\#(y,t)\{(\psi_1 w)^\#(y,t)\}^{-1} \\ &= B(y,t)\overline{B(y,t)}F(y,t)\{(\psi_1 w)^\#(y,t)\}^{-1} \\ &= F(y,t)\{(\psi_1 w)^\#(y,t)\}^{-1} \end{aligned}$$

on $K_{2\pi} \times R$. Thus we have $B_q^\alpha(x,t) = \overline{\psi(x)}\psi(x+e_t)B(y,t)$ on $K \times R$.
The last assertion follows from Theorem 3.2.

We can strengthen the conclusion of [**6**; Theorem 26] which is one of the most important features of cocycles.

THEOREM 4.2. *Every cocycle is cohomologous to a Blaschke cocycle B with the property that the function of z, $B(x,z)$, on $\mathscr{H}$ has no zeros on $\{0 < \operatorname{Im} z < \alpha\}$, so $B(x,z)$ may be extended to $\{-\alpha < \operatorname{Im} z\}$, analytically, for σ-a.e. x in K.*

Proof. It follows from [**6**; Theorem 26] that every cocycle is cohomologous to some Blaschke cocycle B_1. By restricting B_1 to $K_{2\pi} \times R$, we denote by $\mathbb{E}_1$ the set of all zeros of $B_1(y, z)$ in $K_{2\pi} \times R$, and $\tilde{q}_1(y, z)$ denotes the multiplicity of zero at (y, z) in $\mathbb{E}_1$. Define

$$\mathbb{E}_2 = \{(y, z) \in \mathbb{E}_1; 0 < \operatorname{Im} z < \alpha\},$$

and let $\tilde{q}_2(y, x)$ be the restriction of $\tilde{q}_1$ to $\mathbb{E}_2$. Then by [**5**; Theorem 1 and §5] there is a Blaschke cocycle B_2 whose zero set and multiplicity match $\mathbb{E}_2$ and $\tilde{q}_2$. Observe that $B_3 = B_1\overline{B}_2$ is also a Blaschke cocycle which has no zeros on $K_{2\pi} \times \{0 < \operatorname{Im} z < \alpha\}$. On the other hand, it follows from Theorem 4.1 that B_2 is cohomologous to B_q^α for some multiplicity function q on $K_{2\pi}$, which has zeros only on line $\{\operatorname{Im} z = \alpha\}$. Since B_1 is cohomologous to $B_q^\alpha B_3$, the Blaschke cocycle $B = B_q^\alpha B_3$ is the desired one.

Let φ be a function in $H^1(\sigma)$. For each $r > 0$, we define

$$\varphi_r(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi(x + e_t)\frac{r}{t^2 + r^2}\, dt,$$

which is an analogue of (1.3). We notice that φ_r also lies in $H^1(\sigma)$ by Lemma 1.1. Recall that an inner function $f(z)$ on the unit disc is a Blaschke product if and only if

$$\lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\, d\theta = 0$$

(see [**3**; Chapter II, Theorem 2.4]). Similar characterization also holds in the case of $\mathscr{H}$ ([**1**]). Strange to say, suchlike does not hold in the almost periodic setting.

THEOREM 4.3. *There is an inner function ψ in $H^\infty(\sigma)$ which has the following properties*:

(i) *the function of z, $\psi(x + e_z)$, is a Blaschke product on $\mathscr{H}$ for σ-a.e. x in K, and*

(ii) *for all $r > 0$,*

$$\int_K \log|\psi_r(x)|\, d\sigma(x) = -\infty.$$

Proof. Let q be a multiplicity function on $K_{2\pi}$ which does not lie in $L^1(\sigma_1)$, and let B_q^α be the Blaschke cocycle induced by q. If $\mathfrak{M}$ is the invariant subspace with cocycle $\overline{B_q^\alpha}$, then $\mathfrak{M}$ is contained in $H^2(\sigma)$ by Lemma 1.1. Choose and fix a bounded function φ in $\mathfrak{M}$. We may assume φ has no weight at infinity, that is, $\chi_\lambda\varphi$ does not lie in $H^2(\sigma)$ for each negative λ in Γ. Then we have

$$\varphi^\#(y, t) = B_q^\alpha(y, t)V(y, t)$$

on $K_{2\pi} \times R$, where the function of t, $V(y,t)$, lies in $H^\infty(dt/(1+t^2))$ for σ_1-a.e. y in $K_{2\pi}$. Let $V(y, t+i)$ be the function defined by (1.3) with $r = 1$. It then follows from (ii) of Lemma 1.2 that there is a function θ im $\mathfrak{M}$ such that

$$\theta^\#(y,t) = B_q^\alpha(y,t)V(y,t+i)$$

on $K_{2\pi} \times R$. Notice that the inner part of the function of z, $\theta^\#(y,z)$, is a Blaschke product on $\mathscr{H}$. By Theorem 3.2, the cocycle of $\mathfrak{M}[\theta]_+$ is cohomologous to the Blaschke cocycle $B_{q_1}^\alpha$ for some multiplicity function q_1 on $K_{2\pi}$. This implies that there is a unitary function ψ on K so that $B_{q_1}^\alpha(x,t)\overline{\psi(x+e_t)}\psi(x)$ is the cocycle of $\mathfrak{M}[\theta]_+$ which is the conjugate of some Blaschke cocycle. From this fact we see easily ψ is an inner function satisfying the property (i).

On the other hand, since the function of t, $\overline{B_q^\alpha(y,t)}\psi^\#(y,t)$, is inner in $H^\infty(dt/(1+t^2))$ for σ_1-a.e. y in $K_{2\pi}$, we have $|(\overline{B}_q^\alpha\psi^\#)(y,t+ir)| \leq 1$ on $K_{2\pi} \times R$, especially on $K_{2\pi} \times [0,1)$. Since

$$\psi_r^\#(y,t) = B_q^\alpha(y,t+ir)(\overline{B}_q^\alpha\psi^\#)(y,t+ir),$$

it follows from (ii) of Lemma 2.1 that

$$\begin{aligned}\int_K \log|\psi_r(x)|\,d\sigma(x) &= \int_{K_{2\pi}}\int_0^1 \log|\psi^\#(y,t+ir)|\,d\sigma_1(y)\,dt \\ &\leq \int_{K_{2\pi}}\int_0^1 \log|B_q^\alpha(y,t+ir)|\,d\sigma_1(y)\,dt = -\infty,\end{aligned}$$

this completes the proof.

We finally show that every invariant subspace contains a singly generated one as close as we please.

THEOREM 4.4. *Suppose that K is separable, and that $\mathfrak{M}$ is a simply invariant subspace of $L^2(\sigma)$. Then there is a sequence $\{\varphi_n\}_{n=1}^\infty$ of bounded functions in $\mathfrak{M}$ with same arguments such that*

(i) $|\varphi_1| \geq |\varphi_2| \geq |\varphi_3| \geq \cdots$,
(ii) $\mathfrak{M}[\varphi_1] \subset \mathfrak{M}[\varphi_2] \subset \mathfrak{M}[\varphi_3] \subset \cdots \subset \mathfrak{M}$, *and*
(iii) $\mathfrak{M} =$ *the closure of* $\lim_{n\to\infty}\mathfrak{M}[\varphi_n]$.

Proof. If $\mathfrak{M} \neq \mathfrak{M}_-$, then there is nothing to prove since $\mathfrak{M} = \psi H^2(\sigma)$ for some unitary function ψ on K; thus we assume that $\mathfrak{M} = \mathfrak{M}_-$. By

Theorem 4.2, we may also assume the cocycle of $\mathfrak{M}_+$ is $\overline{B}$, where B is a Blaschke cocycle whose zeros lie in $K \times \{\operatorname{Im} z \geq \alpha\}$. Perhaps, B might be 1, so $\mathfrak{M} = H_0^2(\sigma)$. We let $\mathbb{E}$ be the set of all zeros of $B(y, z)$ in $K_{2\pi} \times \mathscr{H}$, and $\tilde{q}(y, z)$ denotes the multiplicity of zero at (y, z) in $\mathbb{E}$.

Observe that $\mathfrak{M}$ is contained in $H_0^2(\sigma)$ and that some bounded function φ_1 in $\mathfrak{M}$ has no weight at infinity. We denote by $\mathbb{E}_1$ and $\tilde{q}_1$ the set of all zeros of $\varphi_1(y, z)$ in $K_{2\pi} \times \{\operatorname{Im} z \geq \alpha\}$ and the multiplicity of zero at (y, z) in $\mathbb{E}_1$, respectively. Since φ_1 lies in $\mathfrak{M}$, $\mathbb{E}_1$ contains $\mathbb{E}$ and $\tilde{q}_1 \geq \tilde{q}$ on $\mathbb{E}$. Since $\mathbb{E}_1$ and $\tilde{q}_1$ satisfy the properties arisen from a Blaschke cocycle, it follows from [**5**; Theorem 1] that there is a Blaschke cocycle B_1 whose zeros, together with their multiplicities, match $\mathbb{E}_1$ and $\tilde{q}_1$. We put $\varphi_1^{\#}(y, t) = B_1(y, t)V(y, t)$ on $K_{2\pi} \times R$. Then the function of t, $V(y, t)$, lies in $H^\infty(dt/(1 + t^2))$ for σ_1-a.e. y in $K_{2\pi}$. By (ii) of Lemma 1.2, we may choose a bounded function φ in $\mathfrak{M}$ for which

$$\varphi^{\#}(y, t) = B_1(y, t)V(y, t + i\alpha)$$

on $K_{2\pi} \times R$. Observe that the function of z, $\varphi^{\#}(y, z)$, has no zeros on $\{0 \leq \operatorname{Im} z < \alpha\}$ for σ_1-a.e. y in $K_{2\pi}$.

We next define

$$\mathbb{F}_n = \{(y, z) \in \mathbb{E}_1; \alpha n \leq \operatorname{Im} z < \alpha(n + 1)\} \setminus \mathbb{E},$$

for $n = 1, 2, 3, \ldots$. We then write

$$\mathbb{F}_n(y) = \{(y, t_j + is_j); j = 1, 2, 3, \ldots\},$$

listed according to their multiplicities $\tilde{q}_1 - \tilde{q}$. Since $\{s_j\}$ is bounded and bounded away from zero, it follows from Lemma 1.3 that the product

$$U_n(y, t) = \prod_{j=1}^{\infty} \frac{(t - t_j)^2}{(t - t_j)^2 + s_j^2}$$

converges for σ_1-a.e. y in $K_{2\pi}$. We, of course, consider $U_n(y, t) = 1$ if $\mathbb{F}_n(y)$ is empty. Furthermore, since K is separable, similarly as in the proof of [**5**; Lemma], we see that $U_n(y, t)$ is measurable on $K_{2\pi} \times R$. Since $U_n(y, t)$ is automorphic, we can find a function u_n on K for which $U_n(y, t) = u_n^{\#}(y, t)$ on $K_{2\pi} \times R$.

Define analytic functions φ_n on K by

$$\varphi_n = u_1 u_2 \cdots u_n \varphi.$$

Then since $0 \leq u_n \leq 1$ on K, $\{\varphi_n\}$ is a sequence of bounded functions with the same arguments and satisfies the property (i).

Let $\overline{B}_n$ be the cocycle of $\mathfrak{M}[\varphi_n]_+$. Then the conjugate cocycle B_n of $\overline{B}_n$ is a Blaschke cocycle with the property that the zero set of $B_n(y, z)$ in $K_{2\pi} \times \{\alpha \leq \operatorname{Im} z < \alpha n\}$ and their multiplicities match the restrictions of $\mathbb{E}$ and $\tilde{q}$ to $K_{2\pi} \times \{\alpha \leq \operatorname{Im} z < \alpha n\}$. We then see that $B_n \overline{B}_{n+1}$ and $B_n \overline{B}$ are analytic for all n. Hence the property (ii) follows.

On the other hand, it can be easily seen that the normalization of the closure of $\lim_{n\to\infty} \mathfrak{M}[\varphi_n]$ has the cocycle $\overline{B}$, the cocycle of $\mathfrak{M}_+$. Since $\mathfrak{M} = \mathfrak{M}_-$, we obtain the property (iii), this completes the proof.

5. Remarks. Let q be a multiplicity function on $K_{2\pi}$. We then denote by B_q^α the Blaschke cocycle induced by q as usual.

(a) The following question is interesting and probably difficult: *Is every cocycle cohomologous to some B_q^α?* By virtue of Theorem 3.2, this is equivalent to the old problem of whether every simple invariant subspace is generated by one of its elements (see [**6**; Chapter 5, §4]). Experimental evidence seems to indicate that the answer would be negative.

(b) Let φ be a nonnull function in $H^\infty(\sigma)$. Then the cocycle of $\mathfrak{M}[\varphi]_+$ is cohomologous to some B_q^α by Theorem 3.2. This assures the existence of an inner function which has exactly the zeros of φ and B_q^α together. In other words, by adding zeros on the line $\{\operatorname{Im} z = \alpha\}$, the zero set of any analytic function becomes the one of an inner function. This observation as well as Theorem 4.1 implies information to a problem posed by Helson:

When does the zeros of a Blaschke cocycle coincide with the zeros of some analytic function?

(c) Similarly as in the proof of Theorem 4.3, we can show the following

PROPOSITION 5.1. *Let $\mathfrak{M}$ be the simply invariant subspace with cocycle $\overline{B}_q^\alpha$. Suppose that q does not lie in $L^1(\sigma_1)$. Then, for every φ in $\mathfrak{M}$, we have*

$$\int_K \log|\varphi_r(x)|\, d\sigma(x) = -\infty$$

for all $r > 0$.

We remark that $\mathfrak{M}$ contains many unitary functions (see [**6**; Theorem 16]). If φ is continuous on K, so is φ_r. It then follows from Arens' theorem [**2**; Chapter VII, Theorem 9.4] that $\mathfrak{M}$ has no continuous functions other than the null function.

(c) The next proposition is an analogue of Theorem 4.4, and two proofs are quite similar.

PROPOSITION 5.2. *Let K and $\mathfrak{M}$ be as in Theorem* 4.4. *Then there is a sequence* $\{\psi_n\}_{n=1}^{\infty}$ *of unitary functions in* $\mathfrak{M}$ *such that*

(i) $\psi_1 H^2(\sigma) \subset \psi_2 H^2(\sigma) \subset \psi_3 H^2(\sigma) \subset \cdots \subset \mathfrak{M}$, *and*

(ii) $\mathfrak{M} =$ *the closure of* $\lim_{n\to\infty} \psi_n H^2(\sigma)$.

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