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# DEFORMING VARIETIES OF $k$-PLANES OF PROJECTIVE COMPLETE INTERSECTIONS

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**We consider the variety $F$ of $k$-dimensional linear projective subspaces lying on a generic projective complete intersection $S$. Under general assumptions involving $k$, the multidegree and the dimension of $S$, we prove that $F$ is connected, smooth, and its local deformations come from deformations of $S$.**

**Introduction.** Linear varieties lying on a projective variety have been considered in several contexts.

A classical instance, going back to Cayley [**6**], is that of a smooth cubic surface. There are twenty-seven lines on such a surface, and, as observed later, the incidence preserving permutations of this set of lines form a group isomorphic to the Weyl group of a root system of type $E_6$. It is also the monodromy group of the global family of smooth cubics and the Galois group of the corresponding enumerative problem (see [**12**]).

Similar results (involving the root system $D_{2k+3}$) hold for the $k$-planes contained in a smooth $2k$-dimensional intersection of two quadrics ([**14**, **16**]).

Beyond the enumerative level, and besides homogeneous-rational varieties such as Grassmannians or linear spaces lying on a smooth quadric, a first example should be the Fano surface of lines contained in a cubic threefold ([**11**]). The Abel-Jacobi map induces an isomorphism from the Albanese variety of the Fano surface to the intermediate Jacobian of the cubic threefold and one has a global Torelli theorem ([**7**, **19**]).

With planes instead of lines, but generically this time, the analogous statements hold true for cubic fivefolds ([**8**, **10**]).

Nor should cubic fourfolds be neglected here: their varieties of lines are irreducible symplectic projective fourfolds ([**3**]) which play an important role in the proof of the global Torelli theorem ([**20**]).

We also mention the variety of $k$-planes contained in a smooth $(2k+1)$-dimensional intersection of two quadrics: it is an Abelian variety isomorphic with the intermediate Jacobian of the given intersection of quadrics ([**9**, **16**]).

All these varieties may be realized as zero loci of sections of certain homogeneous vector bundles over Grassmannians ([**1**, **18**]). This circumstance makes the Schubert calculus relevant, for instance, in computing Chern numbers; it also reduces questions about connectivity, regularity, etc., as well as deformations to questions about the cohomology of homogeneous vector bundles.

Our main concern will be to set up a general framework for a calculus with weights, such that the theorem of Bott [**5**] become expressive in this context—a perspective we initially used in [**4**].

Specific computations enabled Wehler to deal with small deformations of Fano surfaces: he showed, namely, that all of them are induced by deformations of the corresponding cubic threefolds ([**21**]). This result is here extended to a large class (Theorem 5.3). Similarly (Theorem 4.1), we extend (and give an alternative proof for) the connectedness result of Barth and Van de Ven concerning lines on hypersurfaces ([**2**]).

**1. Varieties of $k$-planes.** We shall consider projective $k$-planes contained in a complete intersection $S = S_n(d)$ of dimension $n$ and multidegree $d = (d_1, \dots, d_r)$ in the projective space $P = P_{n+r}$ over the complex field $C$.

Let $\mathscr{O}_P(m)$ denote the $m$th tensor power of the hyperplane line bundle on $P$ and let $S$ be given as the variety of zeros $Z(s) = S$ of a section $s \in H^0(P, E)$, where $E = \bigoplus_{t=1}^r \mathscr{O}_P(d_t)$.

Denote by $G = G(k+1, n+r+1)$ the Grassmann variety of projective $k$-planes in $P$, i.e. $(k+1)$-planes in $C^{n+r+1}$, and let $\Gamma \subset P \times G$ be the subvariety defined by the incidence relation $\Gamma = \{(x, \pi) | x \in \pi\}$, with canonical projections:

$$\begin{array}{ccc} & \Gamma & \\ {}^{p}\swarrow & & \searrow^{q} \\ P & & G \end{array}$$

$p$ represents $\Gamma$ as a $G(k, n+r)$-bundle over $P$ and $q$ represents $\Gamma$ as a $P_k$-bundle over $G$. Accordingly, we have isomorphisms: $H^0(P, E) \xrightarrow{\sim} H^0(\Gamma, p^*E) \xrightarrow{\sim} H^0(G, q_*p^*E)$.

If $0 \to \tau = \tau_{k+1} \to G \times C^{n+r+1} \to Q = Q_{n+r-k} \to 0$ denotes the canonical exact sequence of vector bundles over the Grassmannian $G$, we have a natural identification: $q_*p^*\mathscr{O}_P(m) = S^m(\tau^*) =$ the $m$th symmetric tensor power of the dual tautological bundle.

Put $\mathscr{E} = q_* p^* E$.

Let $\Phi$ be the isomorphism indicated above:

$$\Phi\colon H^0(P, E) \xrightarrow{\sim} H^0(G, \mathscr{E}) = \bigoplus_{t=1}^{r} H^0(G, S^{d_t}(\tau^*)).$$

To $s \in H^0(P, E)$, defining the variety $Z(s) = S$, we thus associate $\Phi(s) \in H^0(G, \mathscr{E})$, defining the variety of zeros $Z(\Phi(s)) = F_k(S) = F$, which consists of all $k$-planes contained in $S \subset P$.

REMARK 1.1. The rank of $\mathscr{E}$ is $\sum_{t=1}^{r} \binom{d_t+k}{k}$, and we expect $F$ to be non-empty for $\dim G - \operatorname{rk} \mathscr{E} \geq 0$, i.e. for

$$(A_0) \qquad (k+1)(n+r-k) - \sum_{t=1}^{r} \binom{d_t+k}{k} \geq 0.$$

This will presently be seen to be true, provided $S$ is not a quadric, in which case the assumption $n \geq 2k$ is needed. Note that, if $S$ is neither a quadric, nor a linear space, condition $(A_0)$ already implies $n > 2k$.

**2. Dimension and smoothness in the generic case.** Let $V = H^0(P, E)$ and consider the subvariety $I \subset G \times V$ defined by: $I = \{(s, \pi) | s|_\pi = 0\}$, with projections:

$$\begin{array}{ccc} & I & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ G & & V \end{array}$$

$\alpha$ represents $I$ as a sub-vector-bundle of $G \times V \to G$, which shows that $I$ is smooth, while $\beta$ is proper and the fibre over $s \in V$ is precisely $Z(\Phi(s))$.

Confirming our Remark 1.1, we have:

PROPOSITION 2.1. *If* $\dim G - \operatorname{rk} \mathscr{E} \geq 0$, $\beta$ *is onto, provided* $n \geq 2k$ *in the case of quadrics.*

*Proof*. If we find a $k$-plane $\pi$ in $S$, with $S$ smooth along $\pi$, and such that the normal bundle $N_{\pi/S}$ has $H^1(\pi, N_{\pi/S}) = 0$, the proposition will follow from Kodaira's criterion for stability of compact submanifolds **[15]**.

We consider the exact sequence:

$$(1) \qquad 0 \to N_{\pi/S} \to N_{\pi/P} \to N_{S/P}|_\pi \to 0.$$

We have:

$$N_{\pi/P} = \bigoplus^{n+r-k} \mathcal{O}_\pi(1) \quad \text{and} \quad N_{S/P}|\pi = \bigoplus_{t=1}^{r} \mathcal{O}_\pi(d_t).$$

Let $\pi$ be given by $x_{k+1} = \cdots = x_{n+r} = 0$, for homogeneous coordinates $(x_0 \colon \cdots \colon x_{n+r})$, so that $s \in H^0(P, E)$, $s|_\pi = 0$ will be given by $r$ homogeneous polynomials $(s_1, \ldots, s_r)$ of the form

$$s_t = \sum_{i=k+1}^{n+r} x_i \cdot p_t^{(i)} + r_t \tag{2}$$

where

$$p_t^{(i)} = \sum_\mu c_{t\mu}^{(i)} \cdot x^\mu , \tag{3}$$

$$\mu = (\mu_0, \ldots, \mu_k), \, x^\mu = x_0^{\mu} \cdots x_k^{\mu_k}, \, |\mu| = \mu_0 + \cdots + \mu_k = d_t - 1$$

and every monomial in $r_t$ contains a product $x_i x_j$ with $i \geq j > k$.

Since we may suppose $n \geq 2k$, the condition that $S$ be smooth along $\pi$ is satisfied for generic $s$. (For example, the following matrix of partial derivatives

$$\left(\frac{\partial s_t}{\partial x_i}(x)\right)_{i \geq k+1}, \qquad x \in \pi$$

may be produced:

$$\left(\begin{matrix} x_0^{d_1-1} & \ldots & x_k^{d_1-1} & 0 & 0 & \ldots \\ 0 & x_0^{d_2-1} & \ldots & x_k^{d_2-1} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0\ldots 0 & x_0^{d_r-1} & \ldots & \ldots & x_k^{d_r-1} \ldots \end{matrix}\right)).$$

We represent a global section of $N_{\pi/P}$ by a matrix

$$a = (a_{ij})_{0 \leq j \leq k < i \leq n+r},$$

so that the map $H^0(N_{\pi/P}) \xrightarrow{\sigma} H^0(N_{S/P}|\pi)$ induced from (1) is described by

$$A \to \left(\sum_{j \leq k < i} a_{ij} \cdot p_t^{(i)} \cdot x_j\right)_{1 \leq t \leq r} \in H^0\left(\bigoplus_{t=1}^{r} \mathcal{O}_\pi(d_t)\right). \tag{4}$$

Looking at monomial coefficients in (4) and using (3), one obtains that $\sigma$ is a surjection if and only if the linear system (with indeterminates $a_{ij}$)

$$(5)\qquad \sum_{j\le k<i} a_{ij}\cdot c^{(i)}_{t,\nu(j)} = 0,$$

$$t=1,\dots,r,\ \nu=(\nu_0,\dots,\nu_k),\ |\nu| = d_t$$

where

$$\nu(j)=\nu-(0,\dots,1,0,\dots,0)\quad\text{and}\quad c^{(i)}_{t,\nu(j)}=0\quad\text{for } \nu(j) \text{ improper}$$

has maximal rank, namely $\sum_{t=1}^{r}\binom{d_t+k}{k} = \operatorname{rk}\mathscr{E} = R$.

For generic $s$, this is actually the case. To see it, consider the lexicographic order on the set of column-indices $\{(i,j)|0\le j\le k<i\le n+r\}$ and look at the $R\times R$ matrix given by the first $R$ columns. Its determinant is a polynomial in $c^{(i)}_{t,\mu}$, with $|\mu| = d_t-1$. It is not difficult to check that this polynomial is different from zero. Consider, for example, the lexicographic order on the set of indices $(i,t,\mu)$ affecting the coefficients $c^{(i)}_{t,\mu}$. Now order the monomials in the expression of the above determinant according to the rule: $m_1>m_2$ if the smallest index $(i,t,\mu)$ for which $c^{(i)}_{t,\mu}$ occurs in $m_1$ with exponent $p_1$ and in $m_2$ with exponent $p_2\ne p_1$, we have $p_1>p_2$. The greatest monomial in this ordering will have perforce coefficient 1 or $-1$, since in each row, the choice of $c^{(i)}_{t,\mu}$ entering this monomial is prescribed.

Thus, for generic $s$, $S$ is smooth along $\pi$ and $H^1(\pi, N_{\pi/S})=0$.

COROLLARY 2.2. *The projective $k$-planes contained in a generic complete intersection $S_n(d)$ of dimension $n$ and multidegree $d=(d_1,\dots,d_r)$ in $P_{n+r}$ define a smooth subvariety $F_k(S_n(d))$ of $G(k+1,n+r+1)$ of codimension $\sum_{t=1}^{r}\binom{d_t+k}{k}$, provided that $(k+1)(n+r-k)\ge\sum_{t=1}^{r}\binom{d_t+k}{k}$ and $S_n(d)$ is not quadric, in which last case $n\ge 2k$ is required.*

REMARK 2.3. The variety of lines $F_1(S_n(3))$ of a cubic hypersurface $S_n(3)\subset P_{n+1}$ is smooth if the cubic is smooth, but in general, the smoothness of $S_n(d)$ does not imply that of $F_k(S_n(d))$ (cf. **[12]**, **[18]**).

**3. Weights.** In what follows, we take $\dim G\ge\operatorname{rk}\mathscr{E}$ (and $n\ge 2k$ for quadrics), and assume the complete intersection $S=S_n(d)$ to be such that the codimension of $F = F_k(S)$ in $G = G(k+1, n+r+1)$ be precisely $\operatorname{rk} \mathscr{E}$. Generically, this is the case (Corollary 2.2).

Let $J_F$ denote the sheaf of ideals defining $F$ on $G$.

The Koszul complex of (the section of $\mathscr{E} = q_* p^* E$ defining) $J_F$ gives, for any holomorphic vector bundle $M$ on $G$, spectral sequences:

$$(6) \qquad H^p\left(G, M \otimes \bigwedge^q \mathscr{E}^*\right) \Rightarrow H^{p-q}(F, M|_F),$$

$$H^p\left(G, M \otimes \bigwedge^{q+1} \mathscr{E}^*\right) \Rightarrow H^{p-q}(G, M \otimes J_F), \qquad q \geq 0.$$

If $M$ is a homogeneous vector bundle, we may use the theorem of Bott [**5**, Th. IV′] for dealing with the groups on the left. To this purpose, we use the following description of the Grassmann manifold $G(k+1, n+r+1)$:

$\mathrm{SL}(n+r+1, C)$, which is the universal cover of $\operatorname{Aut}(P_{n+r}) = \mathrm{PGL}(n+r+1, C)$, has Lie algebra $\mathrm{sl}(n+r+1, C) = \{A = (a_{ij}) | \operatorname{tr} A = 0\}$. Take as Cartan subalgebra $h = \{A | a_{ij} = 0 \text{ for } i \neq j\}$. This gives root spaces $L_{ij} = C \cdot E_{ij}$ $(i \neq j)$ where $E_{ij}$ has zeros everywhere except the $(i, j)$ entry.

The Killing form identifies the corresponding roots $\alpha_{ij}$ with $E_{ii} - E_{jj}$ $(i \neq j)$ so that the root system $A_{n+r}$ may be viewed as embedded in a euclidean space with orthonormal basis $e_i = E_{ii}$, $i = 1, \dots, n+r+1$, the roots being represented by vectors $\alpha$ orthogonal to $e_1 + \cdots + e_{n+r+1}$ and of square-norm $(\alpha, \alpha) = 2$ (cf. [**36**, p. 64]).

Put $\alpha_s = \alpha_{s+1,s} = e_{s+1} - e_s$. $\{\alpha_s | s = 1, \dots, n+r\}$ gives a basis of the root system $A_{n+r}$.

If $U_{k+1}$ denotes the subgroup of $\mathrm{SL}(n+r+1, C)$ consisting of the transformations which preserve the linear space $\{x_{k+2} = \cdots = x_{n+r+1} = 0\} \subset C^{n+r+1}$ with coordinates $(x_1, \dots, x_{n+r+1})$, the Lie algebra $u_{k+1}$ of $U_{k+1}$ will contain $h$, all the negative roots $(\alpha_{ij}, i < j)$ and all positive roots not involving $\alpha_{k+1}$ when expressed in terms of the given basis.

We have $G(k+1, n+r+1) = \mathrm{SL}(n+r+1, C)/U_{k+1}$, which is the description we shall use.

Let us now investigate the weights associated to various homogeneous vector bundles over $G = G(k+1, n+r+1)$.

Such a bundle is defined by a holomorphic representation $\rho \colon U_{k+1} \to \mathrm{GL}(N, C)$ and the weights are taken with respect to $h$.

(a) Consider first the tautological bundle $\tau$ over $G$. It corresponds to the natural representation of $U_{k+1}$ on the invariant subspace $\{x_{k+2} = \cdots = x_{n+r+1} = 0\}$.

Let $\beta_s$ denote the weight characterized by

$$(\beta_s, \alpha_t) = 0 \text{ for } t \neq s \quad \text{and} \quad (\beta_s, \alpha_s) = \tfrac{1}{2}(\alpha_s, \alpha_s) = 1.$$

An elementary computation then gives the weights of

$$\tau_{k+1}: t_1 = -\beta_1, t_2 = \beta_1 - \beta_2, \dots, t_{k+1} = \beta_k - \beta_{k+1}.$$

(b) The line bundle $\det(\tau_{k+1}^*)$, which gives the Plücker embedding of $G(k+1, n+r+1)$, has therefore associated weight: $\beta_{k+1}$.

(c) The tangent bundle of $G$: $\theta_G$ is given by the adjoint representation of $U_{k+1}$ on $\mathrm{sl}(n+r+1, C)/u_{k+1}$. Consequently, its weights are precisely the positive roots involving $\alpha_{k+1}$ in their expression, namely $\alpha_{ij}$, $i > k+1 \geq j$.

(d) $\mathscr{E}^* = \bigoplus_{m=1}^{r} S^{d_m}(\tau_{k+1}^*)^*$ and (a) immediately gives that its weights are of the form:

$$\sum_{i=1}^{k+1} a_i t_i = (a_2 - a_1)\beta_1 + (a_3 - a_2)\beta_2 + \cdots + (a_{k+1} - a_k)\beta_k - a_{k+1}\beta_{k+1}$$

with $a_i \in N$, $\sum_{i=1}^{k+1} a_i = d_m$ for some $m \leq r$.

We now draw up a table of scalar products of positive roots and various weights, which will be relevant in estimating indices of weights.

$\delta$ is half the sum of all positive roots.

$\omega = \sum_{i=1}^{k+1} a_i t_i$, $a_i \in Z$ (motivated by (d) above and the spectral sequences (6)).

$1 \leq m \leq k$.

We anticipate here the type of reasoning to be used in the sequel. Given a homogeneous vector bundle over $G$, defined by a representation $U_{k+1} \to \mathrm{GL}(N, C)$, we first produce a filtration with consecutive quotients corresponding to irreducible representations of $U_{k+1}$. Such an irreducible representation determines a highest weight, say $\rho$. This $\rho$ has to be one of the weights of the original representation and further satisfy $(\rho, \alpha_s) \geq 0$ for all $s \neq k+1$.

In our computations $\rho$ will be either of type $\omega$ or $\omega + \alpha_{n+r+1,m}$ $(m \leq k+1)$.

In order to obtain the vanishing of $H^s(G, \rho)$, it will suffice either to ascertain the singularity of the weight $\rho + \delta$ or to prove: $s < \mathrm{index}(\rho + \delta)$.

In this context, the main feature of our table of products is that $(\alpha_{t,m}, \rho + \delta)$ increases by 1 when $t$ increases by 1, except the last step for $\rho = \omega + \alpha_{n+r+1,m}$ $(m \leq k+1)$.

TABLE 1

| | Conditions | $\delta$ | $\omega$ | $\alpha_{n+r+1,m}$ | $\alpha_{n+r+1,k+1}$ |
|---|---|---|---|---|---|
| $\alpha_p$ | $p \neq m-1, m$<br>$p \leq k$ | 1 | $a_{p+1} - a_p$ | 0 | $p < k$ 0<br>$p = k$ $-1$ |
| $\alpha_{m-1}$ | | 1 | $a_m - a_{m-1}$ | $-1$ | 0 |
| $\alpha_m$ | | 1 | $a_{m+1} - a_m$ | 1 | $m < k$ 0<br>$m = k - 1$ |
| $\alpha_q$ | $k+1 \leq q \leq n+r$ | 1 | 0 | $q < n+r$ 0<br>$q = n+r$ 1 | 0<br>1 |
| $\alpha_{t,k+1}$ | $t > k+1$ | $t-k-1$ | $-a_{k+1}$ | $t < n+r+1$ 0<br>$t = n+r+1$ 1 | 1<br>2 |
| $\alpha_{t,m}$ | $t > k+1$ | $t-m$ | $-a_m$ | $t < n+r+1$ 1<br>$t = n+r+1$ 2 | 0<br>1 |
| $\alpha_{t,p}$ | $t > k+1 > p$<br>$p \neq m$ | $t-p$ | $-a_p$ | $t < n+r+1$ 0<br>$t = n+r+1$ 1 | 0<br>1 |

Note also that for $1 \leq p \leq k+1$, $(\alpha_{k+2,p}, \rho+\delta) < (\alpha_{k+2,p-1}, \rho+\delta)$ since $(\alpha_{p-1}, \rho) \geq 0$.

**4. Connectedness.** Suppose

$$(\mathrm{A}_1) \qquad \dim F = \dim G - \operatorname{rk} \mathscr{E} \geq 1.$$

$F$ is connected if and only if $H^0(\mathscr{O}_F) = C$.

We have $H^s(G, \bigwedge^s \mathscr{E}^*) \Rightarrow H^0(\mathscr{O}_F)$; therefore the vanishing of $H^s(G, \bigwedge^s \mathscr{E}^*)$ for $s > 0$ will imply the connectedness of $F$.

According to our method, described at the end of §3, we examine $H^s(G, \rho)$, with $\rho$ an irreducible representation of $U_{k+1}$ with highest weight (again denoted $\rho$) among the weights of $\bigwedge^s \mathscr{E}^*$. Thus $\rho = \omega = \sum_{i=1}^{k+1} a_i t_i$ and we know (see Table 1):

(1) $a_{k+1} \geq a_k \geq \cdots \geq a_q \geq 0$;

(2) $\rho + \delta$ is either singular or of index $u(n+r-k)$, $1 \leq u \leq k$ ($u = k+1$ is excluded because $\operatorname{rk} \mathscr{E} < \dim G$).

Suppose therefore $s = u(n+r-k)$.

For $\rho + \delta$ to have index $s$, we must have $(\alpha_{t,p}, \rho + \delta) > 0$ for $p = 1, \ldots, k+1-u$; in particular: $a_{k+1-u} \leq u$.

Now remember that $\rho$ is a weight of $\bigwedge^s \mathscr{E}^*$, thus a sum of $s$ weights of $\mathscr{E}^*$, each weight counted at most as many times as the dimension of its eigenspace. There are (multiplicities included) $\sum_{m=1}^{r}\binom{d_m+u-1}{u-1}$ weights involving only $t_i$, $i > k+1-u$. Adding any other weight increases some $a_j$, $j \leq k+1-u$; thus we must not add more than $u(k+1-u)$ such weights. This will be clearly impossible if $n$ satisfies the following conditions:

$$(\mathrm{C}_u) \qquad \sum_{m=1}^{r}\binom{d_m+u-1}{u-1} + u(k+1-u) < u(n+r-k) = s$$

with $u$ running from 1 to $k$.

Now, use (repeatedly) the formula:

$$(7) \qquad \frac{1}{q+1}\binom{d_m+q}{q} - \frac{1}{q}\binom{d_m+q-1}{q-1} = \frac{d_m-1}{q(q+1)}\binom{d_m+q-1}{q-1}$$

to show that if some $d_m \geq 3$, or at least two degrees in $d$ are $\geq 2$, then $(\mathrm{C}_u)$, $1 \leq u \leq k$, is a consequence of our assumption $(\mathrm{A}_1)$. Note that $(\mathrm{C}_1)$ reads: $n > 2k$.

We have therefore:

THEOREM 4.1. *Let $S = S_n(d_1, \ldots, d_r)$ be a complete intersection in $P_{n+r}$ and $F = F_k(S)$ its variety of projective $k$-planes. Suppose*

$$\dim F = (k+1)(n+r-k) - \sum_{m=1}^{r}\binom{d_m+k}{k} \geq 1,$$

*or, in case $S$ is a quadric, suppose $n > 2k$.*

*Then $F$ is connected.*

REMARK 4.2. For a smooth quadric $S = S_{2k}(2)$, $F_k(S)$ consists of two isomorphic (hermitian symmetric) connected components.

This should rather be viewed as the exception which confirms the rule: $S_{2k}(2)$ is a homogeneous (hermitian symmetric) space (of rank one) in its own right, and the generating $k$-planes of the two families in $F_k(S)$ correspond to Schubert cycles which are not homologically equivalent.

REMARK 4.3. There is a simple formula for the canonical bundle of $F = F_k(S_n(d))$, when smooth.

Let $\mathcal{O}_G(1)$ denote the positive generator of $\mathrm{Pic}(G)$, restricting to $\mathcal{O}_F(1)$ on $F$.

Set

$$K = \sum_{m=1}^{r}\binom{d_m+k}{k+1} - (n+r+1).$$

Then $K_F = \mathcal{O}_F(K)$.

**5. Deformations.** In this section we assume that $F = F_k(S_n(d))$ has the "right" codimension and dimension at least two:

$$(\mathrm{A}_2) \qquad \dim F = \dim G - \operatorname{rk} \mathscr{E} \geq 2.$$

Our purpose is to produce conditions on $(n, d, k)$ which ensure the completeness of the natural deformation of $F$, parametrized by a neighborhood of the section $\Phi(s) \in H^0(G, \mathscr{E})$ defining $F$. Notice that the family of complete intersections to which $S_n(d)$ belongs (parametrized by a neighbourhood of $s \in H^0(P, E) \cong H^0(G, \mathscr{E})$, i.e. the "same" base) is itself complete (see **[4]**, **[17]**, **[21]**).

A sufficient condition for completeness is the vanishing of $H^1(G, \mathscr{E} \otimes J_F)$ and $H^1(F, \theta_G|_F)$. This is a general result for varieties defined by sections in a vector bundle (see **[21]**).

We look therefore at the spectral sequences (6) abutting to the above two groups.

(5.1) Take first $H^s(G, \mathscr{E} \otimes \bigwedge^s \mathscr{E}^*)$, $s \geq 1$.

We obtain vanishing conditions for these groups as we did for $H^s(G, \bigwedge^s \mathscr{E}^*)$ in §4.

Let $D = \max_{1 \leq m \leq r}(d_m)$. Filtering and taking highest weights will produce as above weights $\rho = \omega = \sum_{i=1}^{k+1} a_i t_i$, with $(\alpha_p, \rho) \geq 0$ for $p \leq k$.

Since $\rho$ is the sum of a weight $\omega'$ of $\mathscr{E}$ and a weight $\omega''$ of $\bigwedge^s \mathscr{E}^*$, adding $\omega'$ to $\omega'' = \sum_{i=1}^{k+1} a_i'' t_i$ decreases some of its coefficients $a_i''$, diminishing their sum by at most $D$.

This means that our sufficient conditions $(\mathrm{C}_u)$, $1 \leq u \leq k$, for the vanishing of $H^s(G, \bigwedge^s \mathscr{E}^*)$, $s \geq 1$, become, by the same type of reasoning, sufficient conditions $(\mathrm{C}_u^D)$, $1 \leq u \leq k$, for the vanishing of $H^s(G, \mathscr{E} \otimes \bigwedge^s \mathscr{E}^*)$, once we add $D$ to the left hand side of each inequality:

$$(\mathrm{C}_u^D) \qquad \sum_{m=1}^{r} \binom{d_m + u - 1}{u - 1} + u(k + 1 - u) + D < u(n + r - k).$$

(5.2) Consider now $H^{s+1}(G, \theta_G \otimes \bigwedge^s \mathscr{E}^*)$, $s \geq 0$. For $s = 0$, we have $H^1(G, \theta_G) = 0$, because $G$ is rigid **[5]**. Suppose $s \geq 1$.

Again, using a filtration (actually, the representations we are dealing with are all completely reducible) and successive quotients corresponding to irreducible representations of $U_{k+1}$, we find that the highest weight $\rho$ associated to such a representation is necessarily of the form $\rho = \omega + \alpha_{t,m}$, with $\omega = \sum_{i=1}^{k+1} a_i t_i$ a weight of $\bigwedge^s \mathscr{E}^*$,

$t > k+1 \geq m$ (cf. §3 (c)), and further conditions: $(\rho, \alpha_q) \geq 0$ for all $q \neq k+1$, which imply in particular $t = n+r+1$.

Take therefore $\rho = \omega + \alpha_{n+r+1,m}$ $(m \leq k+1)$ and consider the series of integers: $(\rho+\delta, \alpha_{t,p})$ with $p \leq k+1$ fixed and $t$ increasing from $k+2$ to $n+r+1$. If $\rho+\delta$ is non-singular, this series of non-zero integers will keep the same sign, except possibly at the last step $t = n+r+1$, when it might "jump" precisely over zero (see Table 1).

Now let $p$ decrease from $k+1$ to $1$ and notice the relations of the starting values in each series:

$$(\rho+\delta, \alpha_{k+2,k+1}) < (\rho+\delta, \alpha_{k+2,k}) < \cdots < (\rho+\delta, \alpha_{k+2,1}).$$

This means that we might encounter non-vanishing cohomology $H^{s+1}(G, \rho)$ at most for $s+1$ or $s$ a multiple of $n+r-k$, say $u(n+r-k)$ $(u < k+1$ by our assumption $\operatorname{rk} \mathscr{E} \leq \dim G - 2)$.

For the coefficients $a_i$ in $\omega = \sum_{i=1}^{k+1} a_i t_i$, we have either:

(1) $a_{k+1} > a_k \geq \cdots \geq a_1$ for $m = k+1$, or

(2) $a_{k+1} \geq \cdots \geq a_{m+1}$; $a_{m+1} + 1 \geq a_m > a_{m-1} \geq \cdots \geq a_1$ for $m \leq k$.

Since $\omega$ is a weight of $\bigwedge^s \mathscr{E}^*$, it appears that $(\mathrm{C}_u^2)$ above is a sufficient condition for the vanishing of $H^{s+1}(G, \rho)$.

Now, one may verify that the combination of $(\mathrm{A}_2)$ and $(\mathrm{C}_1^D)$ above implies $(\mathrm{C}_u^D)$ for $1 \leq u \leq k$.

First, suppose $d_m \geq 2$, which is no restriction of generality. Making use of the identity (7) in §4 and the fact that the right hand side in (7) clearly increases with $q$, the following implications obtain:

(i) If $k \geq 2$, $(\mathrm{A}_2) \Rightarrow (\mathrm{C}_k^D)$ as soon as $\sum_{m=1}^r (d_m^2 - 1) > 3D + 2$, i.e. $d \neq (2), (2, 2), (3), (2, 3)$; and for $n > 6$ also for $d = (2, 3)$.

(ii) If $u > 1$, $(\mathrm{C}_{u+1}^D) \Rightarrow (\mathrm{C}_u^D)$ for $\sum_{m=1}^r (d_m^2 - 1) \geq D + 6$, i.e. $d \neq (2), (2, 2), (3)$.

Finally, for $d = (2), (2, 2), (3)$ or $(2, 3)$, a direct check shows that $(\mathrm{A}_2)\&(\mathrm{C}_1^D) \Rightarrow (\mathrm{C}_u^D)$.

Summing-up, we obtain:

THEOREM 5.3. *Let $S = S_n(d_1, \ldots, d_r)$ be a complete intersection in $P_{n+r}$ and suppose that its variety of k-planes $F = F_k(S)$ satisfies*

$$(\mathrm{A}_2) \qquad \dim F = (k+1)(n+r-k) - \sum_{m=1}^{r} \binom{d_m + k}{k} \geq 2.$$

*If $n > 2k + D$, where $D = \max_{1 \leq m \leq r}(d_m)$, then every small deformation of $F$ is induced by a (small) deformation of $S$.*

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