PACIFIC JOURNAL OF MATHEMATICS
Vol. 144, No. 1, 1990

A SHORT PROOF OF ISBELL'S ZIGZAG THEOREM

PETER M. HIGGINS

Isbell's Zigzag Theorem, which characterizes semigroup dominions (defined below) by means of equations, has several proofs. We give a short proof of the theorem from first principles.

The original proof Isbell [4] and that of Philip [6] are topological in flavour. The algebraic proofs of Howie [2] and Storrer [8] are based on work by Stenstrom [7] on tensor products of monoids. Yet another proof, using the geometric approach of regular diagrams, is due to David Jackson [5]. This latter approach also employs HNN extensions of semigroups to solve the problem. In this note we follow Jackson's lead in using what is essentially a HNN extension for our embedding (instead of the more intractable free product with amalgamation) to derive a short and direct proof of the Zigzag Theorem.

Following Howie and Isbell [3] we say that a subsemigroup U of a semigroup S *dominates* an element $d \in S$ if for every semigroup T and all morphisms $\phi_1 : S \to T$, $\phi_2 : S \to T$, $\phi_1|U = \phi_2|U$ implies that $d\phi_1 = d\phi_2$. The set of all elements in S dominated by U is called the *dominion* of U in S; it is obviously a subsemigroup of S containing U, and we denote it by $\mathrm{Dom}(U, S)$. Dominions are connected with epimorphisms (pre-cancellable morphisms) by the fact that a morphism $\alpha : S \to T$ is epi iff $\mathrm{Dom}(S\alpha, T) = T$.

ISBELL'S ZIGZAG THEOREM. *Let U be a subsemigroup of S. Then $d \in \mathrm{Dom}(U, S)$ if and only if $d \in U$ or there exists a sequence of factorizations of d as follows:*

$$d = u_0 y_1 = x_1 u_1 y_1 = x_1 u_2 y_2 = x_2 u_3 y_2 = \cdots = x_m u_{2m-1} y_m = x_m u_{2m},$$

where

$$u_i \in U, \quad x_i, y_i \in S, \quad u_0 = x_1 u_1, \quad u_{2i-1} y_i = u_{2i} y_{i+1},$$
$$x_i u_{2i} = x_{i+1} u_{2i+1} \quad (1 \le i \le m-1) \quad \text{and} \quad u_{2m-1} y_m = u_{2m}.$$

Such equations are known as a *zigzag* in S over U with *value* d, *length* m, and *spine* the list $u_0, u_1, \ldots, u_{2m}$. For a survey on epimorphisms and semigroup amalgams featuring applications of the Zigzag Theorem see Higgins [**1**].

We give a new proof of the forward implication in the theorem; the reverse implication follows by a straightforward manipulation of the zigzag.

Suppose that $d \in \mathrm{Dom}(U, S)\backslash U$. Form a semigroup H by adjoining a new element t to S subject to the relations $t^2 = 1$, $tu = ut$, $tut = u$ for all $u \in U$. Define the morphisms $\phi_1, \phi_2 \colon S \to H$ by $s\phi_1 = s$ and $s\phi_2 = tst$ (indeed ϕ_1 and ϕ_2 are embeddings). Clearly $\phi_1|U = \phi_2|U$ so that $tdt = d$, or what is the same, $td = dt$ in H. We prove that this latter equation implies that d is the value of some zigzag in S over U.

Since $td = dt$ there is a sequence of transitions of minimal length $I \colon td \to \cdots \to dt$ where each transition $pwq \to pw'q$ $(p, w, w', q \in H)$ is either a *t-transition*, i.e., involves a relation in which t occurs, or is a *refactorization*, i.e., $w = w'$ in S. We claim that no transition in I involves any of the relations $t^2 = 1$ or $tut = u$ $(u \in U)$. Suppose to the contrary that I has a transition $\alpha \colon pq \to pt^2q$ $(p, q \in H)$. Clearly α is not the final transition of I, so consider the next transition $\beta \colon pt^2q \to$. Suppose that the right-hand side of β has one of the forms

$$\text{(i) } pq; \quad \text{(ii) } p't^2q; \quad \text{(iii) } pt^2q'.$$

In the first case the two transitions cancel, while in cases (ii) and (iii) α and β can be performed in the opposite order without changing the net effect. If β does not have one of these forms then either (iv) the product p has the form $p = p'u$ or $p'tu$ $(u \in U)$ and the right side has the form $p'tutq$ or $p'utq$ or (v) a similar remark applies to q. In this case the pair of transitions α, β could be replaced by the single transition $p'uq \to p'tutq$ or $p'tuq \to p'utq$ (with a similar remark applying to case (v)). Therefore cases (i), (iv) and (v) contradict our minimum length assumption, whence it follows that all transitions of I of the form $pq \to pt^2q$ can be taken to appear at the end of I, and thus there are none.

Next suppose that α has the form $puq \to ptutq$, and once again consider the following transition β. If p has the form $p'v$ or $p'tv$ $(v \in U)$ then β could have the form $p'vtutq \to p'tvutq$ or $p'tvtutq \to p'vutq$; but in that case the pair α, β could be replaced by the single transition $p'vuq \to p'tvutq$ or $p'tvuq \to p'vutq$. A similar remark applies if q has the form vq' or vtq'. If p has the form $p't$ then β could have the form $p'ttutq \to p'utq$; but again it would then be possible to shorten I by replacing our pair α, β with the single transition $p'tuq \to p'utq$; and again a similar remark applies to q. Another possibility for β is $ptutq \to put^2q$ or $ptutq \to pt^2uq$, but here again α and β could be replaced by just one transition. The remaining possibilities for β (β cancels α, or β involves only the product p or only the product q) are disposed of as in the previous paragraph, thus establishing the claim.

Call a t-transition of the form $putq \to ptuq$ [$ptuq \to putq$] a *left* [*right*] transition, so that our sequence I consists entirely of refactorizations and left and right transitions with exactly one occurrence of the symbol t in each word of I. Suppose that ptq is a product occurring in I, and that the next t-transition in the sequence is a left transition. We claim that we may assume that this left transition occurs immediately, or is preceded by just one refactorization of the form $ptq \to p'utq$, for it is clear that any refactorization of p can be performed in one step, while any refactorization of q can be delayed until after the left transition. Next suppose that I contains two left transitions with no intervening right transition, which we may assume have the form $putq \to ptuq \to p'vtuq \to p'tvuq$ $(u, v \in U)$, or simply the form $p'vutq \to p'vtuq \to p'tvuq$. In the latter case the pair of transitions can be replaced by a single left transition, while the three transitions of the first case can be replaced by two: $putq \to p'vutq \to p'tvuq$. Coupling all this with similar arguments for right transitions allows us to conclude that I consists of alternate left and right transitions, separated by single refactorizations; furthermore the first t-transition is right and the final t-transition is right. The sequence I therefore implies equalities in H of the form:

$$\begin{aligned} td &= tu_0y_1 = u_0ty_1 = x_1u_1ty_1 = x_1tu_1y_1 = x_1tu_2y_2 = x_1u_2ty_2 \\ &= x_2u_3tu_2 = \cdots = x_{m-1}u_{2m-2}ty_m = x_mu_{2m-1}ty_m \\ &= x_mtu_{2m-1}y_m = x_mtu_{2m} = x_mu_{2m}t = dt, \end{aligned}$$

for some $m \geq 1$, $u_i \in U$ $(1 \leq i \leq 2m)$ $x_i, y_i \in S^1$, and $u_0 = x_1u_1$,

$$u_{2i-1}y_i = u_{2i}y_{i+1} \quad x_iu_{2i} = x_{i+1}u_{2i+1}, \quad (1 \leq i \leq m-1) \quad \text{and}$$
$$u_{2m-1}y_m = u_{2m}.$$

In fact $x_i, y_i \in S$ for if $x_i = 1$ then in S we have

$$d = u_0y_1 = x_1u_1y_1 = x_1u_2y_2 = \cdots = u_{2i}y_{i+1} = \cdots = x_mu_{2m};$$

and so I could be shortened by beginning with $td \to tu_{2i}y_{i+1}$, with a similar remark applying if some $y_i = 1$. Hence d is the value of a zigzag in S over U, thus completing the proof.

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Received July 17, 1989.

ROYAL MELBOURNE INSTITUTE OF TECHNOLOGY
GPOB 2476V
MELBOURNE, VIC. 3001, AUSTRALIA